Imperial College London
BSci/MSci EXAMINATION May 2016

MPH2 MATHEMATICAL METHODS

For 2nd year Physics students

Tuesday, May 31, 2016: 14:00 to 16:00

Answer ALL of Question 1 and TWO questions out of Questions 2, 3, 4 and 5. A mathematical formula sheet is provided.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Complete the front cover of each of the 3 answer books provided, entering the number of the question attempted in the box on the front cover of the corresponding answer book.

Hand in 3 answer books even if they have not all been used.

Examiners attach great importance to legibility, accuracy and clarity of expression.

$$\text{Fourier transform} \quad \hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$\text{Fourier integral} \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk.$$

$$\text{Convolution} \quad (f \star g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t)\, dt.$$

1. (i) Show that an analytic function is harmonic.

(ii) The Lagrangian

$$L = \frac{ml^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right) + \frac{\mathtt{I}\dot{\phi}^2}{2} + mgl\cos\theta,$$

describes the motion of a simple pendulum of length l mounted on a freely rotating turntable with moment of inertia $\mathtt{I}$. Here θ and ϕ are generalised coordinates and g, $\mathtt{I}$, l and m are constants. Obtain the equations of motion. Are any of the generalised coordinates cyclic?

(iii) Compute the contour integral

$$\oint_C \frac{dz}{e^z + e^{-z}},$$

where C is a circle of unit radius centred at $z = i$ (take the orientation anti-clockwise).

(iv) Use the identity

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-inx} = \sum_{m=-\infty}^{\infty} \delta(x - 2\pi m),$$

to derive Poisson's summation formula

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) = \sum_{m=-\infty}^{\infty} f(2\pi m).$$

A discussion of the convergence of the two infinite series is not required.

(v) The functions f and g are defined by $f(x) = \theta(x)$ and $g(x) = xe^{-x^2}$. Find $(f * g)(x)$. Here θ is the Heaviside function defined by $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$.

(vi) The functions

$$a) \quad \tanh z \qquad b) \quad \frac{e^z}{e^z + 2} \qquad c) \quad \cosh(z^2 - 4)$$

are expanded as Taylor series about $z = 0$. In each case give the radius of convergence.

(vii) Define what is meant by polar and axial vectors. What kind of vector is the cross product of a polar and an axial vector? Briefly justify your answer.

(viii) Obtain a rational approximation to $\sqrt{11}$ by applying the Newton-Raphson method to $f(x) = x^2 - 11$ (start with the initial guess $x_0 = 3$ and compute x_1 and x_2).

[Total 40 marks]

2. (i) The area of the surface of revolution obtained by rotating a curve $y = y(x) > 0$ for $a \leq x \leq b$ about the x-axis is

$$A = 2\pi \int_a^b y(x)\sqrt{1 + (y'(x))^2}\ dx$$

Show that if A is stationary then

$$\frac{y^2}{1 + y'^2} = \text{constant.}$$

(ii) The volume, V, enclosed by the surface of revolution considered in part (i) is given by the integral

$$V = \pi \int_a^b (y(x))^2\ dx.$$

Assuming that $y(a) = 0$, what shape should the surface be so that V is stationary for fixed A?

[Hint: As in part (i) find a relation between y' and y. This relation includes constants. Use $y(a) = 0$ to fix one of these constants and then solve to find y as a function of x.]

[Total 30 marks]

3. (i) Let $f(z) = \ln z$. Show that $f(z) = u(x, y) + iv(x, y)$ satisfies the Cauchy-Riemann equations (except at $z = 0$ and any branch cut).

(ii) Show that

$$\mathsf{P}\int_{-\infty}^{\infty} \frac{dx}{x^2 - 1} = 0.$$

Is it true that

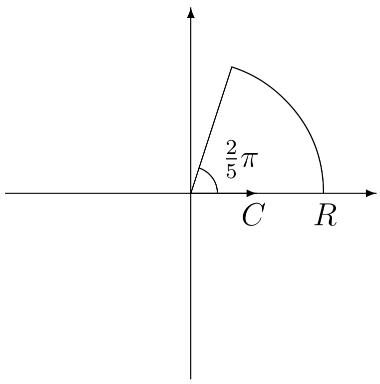
$$\mathsf{P}\int_{-\infty}^{\infty} \frac{dx}{(x - a_1)(x - a_2)...(x - a_n)} = 0?$$

Here a_1, a_2, ..., a_n are distinct ($a_i \neq a_j$ for $i \neq j$) real constants.

(iii) Compute the integral

$$\int_0^{\infty} \frac{dx}{1 + x^5}.$$

Hint: integrate $f(z) = 1/(z^5 + 1)$ over the contour C in the diagram below:



[Total 30 marks]

4. (i) Using contour integration, or otherwise, show that the Fourier transform of

$$f(x) = \frac{1}{x + ia}, \quad (a \text{ real and positive})$$

is $\hat{f}(k) = -ie^{-ka}\theta(k)$.

(ii) Obtain in the form of a Fourier integral a particular solution to the complex ODE

$$\ddot{x}(t) + \gamma\dot{x}(t) + \Omega^2 x(t) = \frac{1}{t + ia},$$

where a, γ and Ω^2 are positive constants. Find a particular solution to the real ODE

$$\ddot{x}(t) + \gamma\dot{x}(t) + \Omega^2 x(t) = \frac{t}{t^2 + a^2}.$$

(iii) Find a solution, $\phi(x, t)$, to the diffusion equation

$$\frac{\partial^2 \phi}{\partial x^2} = D\frac{\partial \phi}{\partial t}, \quad (D > 0)$$

subject to the initial condition $\phi(x, t = 0) = e^{-ax^2}$ where a is a positive constant
Hint: write $\phi(x, t)$ in the Fourier integral form

$$\phi(x, t) = \int_{-\infty}^{\infty} \hat{\phi}(k, t)e^{ikx}dk.$$

Find $\hat{\phi}(k, t)$ and evaluate the Fourier integral explicitly.
The following standard integral may be quoted without proof

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right) \quad (a > 0,\ b \in \mathbb{C}).$$

[Total 30 marks]

5. (i) Prove the identity

$$\epsilon_{ijp}\epsilon_{klp} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

and use the result to show that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

(ii) The Lorentz force law for a charged particle in a magnetic field $\mathbf{B}$ is

$$\frac{d\mathbf{p}}{dt} = q\mathbf{v} \times \mathbf{B}.$$

Show that this can be written in the form

$$\dot{p}_i = q(\partial_i A_j - \partial_j A_i)\dot{x}_j,$$

where $\mathbf{A}$ is the vector potential defined through $\mathbf{B} = \nabla \times \mathbf{A}$.

(iii) Simpson's rule is based on the approximation

$$\int_a^b f(x)\, dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\tfrac{1}{2}(a+b)\right) + f(b)\right].$$

Show that this is exact for the quadratic $f(x) = px^2 + qx + r$ where p, q and r are constants. Apply the approximation to the integral

$$\int_0^1 \frac{dx}{1+x^2}$$

and compare with the exact result.

[Total 30 marks]