Imperial College
London

[MP2 2012]

B.Sc. and M.Sci. EXAMINATIONS 2012

SECOND YEAR STUDENTS OF PHYSICS

MATHEMATICS - M.PHYS 2

Date Wednesday 6th June 2012 2 - 4 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Do not attempt more than FOUR questions.

A mathematical formulae sheet is provided

[Before starting, please make sure that the paper is complete; there should be 4 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Define $\cos(z)$ and $\cosh(z)$ for a complex variable $z = x + iy \in \mathbb{C}$.

   (ii) Show that $\cos(iz) = \cosh(z)$.

   Let

$$f(z) = \frac{1}{e^z - 1} .$$

   (iii) Identify all the poles of $f(z)$ and determine their nature.

   (iv) Determine the first three terms of the Laurent series of $f(z)$ about the point $z = 0$.

2. (i) Let

$$f(z) = \sum_{n=-\infty}^{\infty} K_n\, z^n ,$$

   where $K_n \in \mathbb{C}$ are constants.

   Show by direct calculation that the integral

$$I = \int_C f(z)\, dz ,$$

   around the unit circle C with centre at the origin and positive orientation, is given by

$$I = 2\pi i K_{-1} .$$

   (ii) Let $f(z)$ be a complex function. Consider the integral

$$I(R) = \int_\cap e^{ikz} f(z)\, dz$$

   around the semicircle in the upper half-plane from $(R, 0)$ to $(-R, 0)$. Assume $k > 0$ and that $|f(z)| \propto R^\alpha$ for $|z| = R$.

   Show that $\lim_{R\to\infty} I(R) = 0$ when $\alpha < 0$.

   You may use, without proof, Jordan's lemma $\displaystyle\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta < \frac{\pi}{2R}$.

   (iii) Use contour integration to compute the integral

$$J = \int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\, dx .$$

PLEASE TURN OVER

3. Dirac's δ-function is defined as follows

$$\forall f \in C^0([a,\,b],\,\mathbb{C}) \wedge \forall x_0 \in (a,\,b) : \int_a^b \delta(x-x_0)\,f(x)\,dx = f(x_0)\,.$$

(i) Show that

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\,.$$

(ii) Calculate the Fourier transform of $\sin x$.

(iii) Show that if the Fourier transforms satisfy

$$\widehat{f}_1(k) = \widehat{f}_2(k)\widehat{f}_3(k)$$

then the functions $f_1(x)$, $f_2(x)$ and $f_3(x)$ are related as follows

$$f_1(x) = \int_{-\infty}^{\infty} f_2(x-y)\,f_3(y)\,dy\,.$$

(iv) Assume the function $f(x)$ satisfies the equation

$$\frac{d^2 f}{dx^2} + mf(x) + \int_{-\infty}^{\infty} dy\, e^{-|x-y|}\,f(y) = \delta(x)\,.$$

Express the Fourier transform of $f(x)$ as a rational function of k.

4. (i) Show that any strongly convergent sequence is also a Cauchy sequence.

(ii) Let $f_n(x) = \tanh(nx) \in C^0(\mathbb{R})$ with $n \in \mathbb{N}$.

Use the l_2 norm to show that

$$\lim_{n\to\infty} \|f_n - \phi\| = 0$$

where

$$\phi(x) = \begin{cases} 1 & \text{for } x \geq 0\,, \\ -1 & \text{for } x < 0\,. \end{cases}$$

Hint: Consider the behaviour of $f_n(x)$ for n large. Consider the two cases $x > 0$ and $x < 0$ separately.

(iii) Explain why the result in (ii) proves that $C^0(\mathbb{R})$ with the l_2 norm is not a complete space.

(iv) Let $\langle x,\,y\rangle$ denote a scalar product on a complex vector space S.
Show that $\|x\| \equiv \langle x,\,x\rangle^{1/2}$ defines a norm on S.

5. (i) Use calculus of variations to find the shortest distance between two points in the plane.

(ii) Consider a function $y(x)$ which satisfies

$$y(x) \;>\; 0 \;\text{ for }\; -a < x < a\,,$$

$$y(x) \;=\; 0 \;\text{ for }\; |x| \geq a\,.$$

Assume also that for a given specified length of the curve between $(-a,\,0)$ and $(a,\,0)$ the function $y(x)$ maximizes the area between the x-axis and the graph of $y(x)$. Use calculus of variation under the constraint to derive an equation for $y(x)$ and thereby show that for $|x| \leq a$ the function $y(x)$ is part of a circle through $(-a,\,0)$ and $(a,\,0)$.

6. (i) Derive the Newton-Raphson algorithm.

(ii) Find to two significant digits the roots of the equation

$$\cosh(x) \;+\; 2x \;=\; \cos(x)\;.$$

(iii) Derive the trapezium rule.

(iv) Derive the first order Runge-Kutta iteration scheme for the equation

$$\frac{dy}{dx} \;=\; f(x,\,y\,(x))\;.$$

END OF PAPER

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012	Course MPhys II
Question 1	TOPIC Complex functions	Marks & seen/unseen
Parts (i)	Defined through their power series representation $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$ $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$ $\cosh(z) = \frac{e^z + e^{-z}}{2}$ Or (similar) through $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$	2 1 3
(ii)	$\cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$ and $\cos(iz) = 1 - \frac{(iz)^2}{2!} + \frac{(iz)^4}{4!} + \dots$ $= 1 + \frac{z^2}{2} + \frac{z^4}{4!} + \dots$ i.e. $\cos(iz) = \cosh(z)$	2 1 3 seen similar
(iii)	Pole at z when $e^z = 1$ $\Rightarrow e^x e^{iy} = 1 \Rightarrow \lvert e^x e^{iy} \rvert = 1 \Rightarrow e^x = 1$ $\Rightarrow x = 0.$ and hence $e^{iy} = 1 \Rightarrow \cos y + i \sin y = 1 \Rightarrow \cos y = 1$, $\sin y = 0$ $\Rightarrow y = 2\pi n,\ n \in \mathbb{Z}.$ I.e. poles at $z = 2n\pi i,\ n \in \mathbb{Z}$	2 2 4
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		Page number 1

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012	Course MPhys II
Question 1	TOPIC Complex functions	Marks & seen/unseen

Parts

Nature of the pole:

$f(z) = \frac{1}{g(z)}$, $g(z) = e^{z} - 1$

$g(z) = 0$ at poles $z_0 = 2n\pi i$

$g'(z) = e^{z} \rightarrow g'(z_0) = e^{2n\pi i} = 1^{n} \neq 0$

thus:

$$f(z) = \frac{1}{g'(z_0)(z-z_0) + \mathcal{O}(z-z_0)^2}$$

$$= \frac{1}{g'(z_0)(z-z_0)} \quad \frac{1}{1 + \mathcal{O}(z-z_0)^2}$$

$$= \frac{1}{g'(z_0)(z-z_0)} \left(1 - \mathcal{O}(z-z_0)^2\right)$$

$$= \frac{1}{g'(z_0)(z-z_0)} - \text{const.} + \mathcal{O}(z-z_0)$$

All poles are of first order.

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seen similar

(iv)

$$\frac{1}{e^{z}-1} = \left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right)^{-1}$$ 2

$$= \frac{1}{z}\left(1 + \frac{z}{2} + \frac{z^2}{6} + \mathcal{O}(z^3)\right)^{-1}$$

$$= \frac{1}{z}\left[1 - \left(\frac{z}{2} + \frac{z^2}{6} + \mathcal{O}(z^3)\right) + \left(\frac{z}{2} + \frac{z^2}{6} + \mathcal{O}(z^3)\right)^2 - \ldots\right]$$ 2

$$= \frac{1}{z}\left[1 - \frac{z}{2} - \frac{z^2}{6} + \frac{z^2}{4} + \mathcal{O}(z^3) + \cdots\right]$$

$$= \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(z^2)$$ 2

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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course MPhys IV
Question 2	TOPIC Contour integration	Marks & seen/unseen

Parts

(i)

$$I = \int_C f(z)\,dz = \sum_{n=-\infty}^{\infty} K_n \int_C dz\, z^n$$

sub $z = e^{i\theta} \Rightarrow dz = i e^{i\theta} d\theta =$

$$I = \sum_{n=-\infty}^{\infty} K_n\, i \int_0^{2\pi} d\theta\, e^{i(n+1)\theta}$$ 2

Note

$$\int_0^{2\pi} d\theta\, e^{i(n+1)\theta} = \begin{cases} 2\pi & \text{if } n=-1 \\ 0 & \text{if } n \neq 1 \end{cases}$$ 2

$\Downarrow$

$$I = 2\pi i K_{-1}$$ 1

5

(ii)

$$\left| \int_\cap e^{ikz} f(z)\, dz \right|$$

$$\leq \int_\cap |e^{ikz} f(z)|\,|dz| = \int_\cap e^{-ky} |e^{ikx}|\,|f(z)|\,|dz|$$ 2

$$= \int_0^{\pi} e^{-R\sin\theta} |f(z)|\, ds$$ ← arch length

$$\propto \int_0^{\pi} e^{-R\sin\theta} R^{\alpha} R\, d\theta$$ use symmetry of $\sin\theta$ about y-axis 2

$$= R^{\alpha+1}\, 2 \int_0^{\pi/2} e^{-R\sin\theta}\, d\theta$$

$$\leq 2 R^{\alpha+1} \frac{\pi}{2R} = \pi R^{\alpha} \to 0 \text{ if } \alpha < 0$$ 2

when $R \to \infty$

6

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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course MPhys II
Question 2	TOPIC Contour integration	Marks & seen/unseen
Parts (iii)	Note $f(z) = \frac{z^2}{(z^2+1)(z^2+4)} \to |f(z)| \sim \frac{1}{|z|^2}$ for $|z| \gg 1$. I.e. $\alpha < 0$ and we may neglect contribution from semicircle. We close in UHP (close in LHP also OK).	2
	Poles at $z = \pm i$ and $z = \pm 2i$	2
	$\text{Res}(i) = \frac{i^2}{2i(3i)(-i)} = \frac{i}{6}$ Note: All poles are simple	2
	$\text{Res}(2i) = \frac{(2i)^2}{(2i-i)(2i+i)4i} = -\frac{i}{3}$	6
	Now $J = \frac{1}{2}\int_{-\infty}^{\infty} dx \frac{x^2}{(x^2+1)(x^2+4)}$ which can be closed by semicircle $= \frac{1}{2} 2\pi i \{\text{Res}(i) + \text{Res}(2i)\}$ $= \pi i \{\frac{i}{6} - \frac{i}{3}\} = \frac{\pi}{6}$	3 3
		seen similar 20
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	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012	Course MPhys II
Question 3	TOPIC Fourier transform	Marks & seen/unseen

Parts

(i) The Fourier transform of the δ-func.:

$$\hat{\delta}(k) = \int_{-\infty}^{\infty} dx\, \delta(x) e^{-ikx} = e^{-ik0} = 1$$ 2

the inversion formula gives

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{\delta}(k) e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}$$ 2

4 seen

(ii) $$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$ 2

therefore

$$\mathcal{F}\{\sin(x)\}(k) = \int_{-\infty}^{\infty} dx\, \frac{e^{ix} - e^{-ix}}{2i} e^{-ikx}$$

$$= \frac{1}{2i}\left[\int_{-\infty}^{\infty} dx\, e^{i(1-k)x} - \int_{-\infty}^{\infty} dx\, e^{-i(1+k)x}\right]$$ 1

$$= \frac{1}{2i} 2\pi \left[\delta(1-k) - \delta(1+k)\right]$$

$$= \frac{\pi}{i}\left[\delta(k-1) - \delta(k+1)\right]$$ 2

5 seen similar

(iii) $$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}_2(k)\hat{f}_3(k) e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} f_1(k) e^{ikx} = f_1(x)$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} f_2(k)\left[\int_{-\infty}^{\infty} dk'\, \delta(k'-k) f_3(k')\right] e^{ikx}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dk'\, \delta(k'-k)\, f_2(k) f_3(k') e^{ikx}$$ 2

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Parts (iii) cont.

$$= \int_{-\infty}^{\infty} \frac{dt}{2\pi} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} \frac{dy}{2\pi}\, e^{iy(t'-t)} f_3(t') e^{itx}$$

$$= \int_{-\infty}^{\infty} dy \int \frac{dt}{2\pi} \int \frac{dt'}{2\pi}\, f_2(t)\, e^{it(x-y)}\, f_3(t') e^{it'y}$$ 2

$$= \int_{-\infty}^{\infty} dy \left[\int \frac{dt}{2\pi} f_2(t) e^{it(x-y)}\right]\left[\int \frac{dt'}{2\pi} f_3(t') e^{it'y}\right]$$

$$= \int_{-\infty}^{\infty} dy\, f_2(x-y)\, f_3(y)$$ 2

6 unseen

(iv) Sub $f(x) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} \hat{f}(t) e^{+itx}$

and make use of (iii), namely

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} dy\, e^{-|x-y|} f(y)\right\} = \mathcal{F}\{e^{-|x-y|}\}(t)\hat{f}(t)$$

to obtain

$$(it)^2 \hat{f}(t) + m\hat{f}(t) + \mathcal{F}\{e^{-|x|}\}(t)\hat{f}(t) = 1$$ 2

$$\mathcal{F}\{e^{-|x|}\}(t) = \int_{-\infty}^{\infty} dx\, e^{-|x|} e^{-itx}$$

$$= \int_{-\infty}^{0} dx\, e^{x(1-it)} + \int_{0}^{\infty} dx\, e^{-x(1+ti)}$$

$$= \left[\frac{e^{x(1-it)}}{1-it}\right]_{-\infty}^{0} + \left[\frac{e^{-(x+it)}}{-(1+ti)}\right]_0^{\infty} = \frac{2}{1+t^2}$$ 2

5 Seen similar

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012	Course MPhys II
Question 3	TOPIC Fourier transform	Marks & seen/unseen

Parts

(iv) cont.

$$\Downarrow$$

$$-t^2 \tilde{f} + m\tilde{f} + \frac{2\tilde{f}}{1+t^2} = 1$$

$$\Downarrow$$

$$\tilde{f}(t) = \frac{1+t^2}{(1+t^2)(m-t^2)+2}$$

1 seen similar

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	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012	Course MPhys II
Question 4	TOPIC Linear vector spaces	Marks & seen/unseen

Parts

(i) Assume x_n strongly convergent towards x. Then

$$\|x_n - x_m\| \le \|x_n - x + x - x_m\| \le \|x_n - x\| + \|x - x_m\|$$ 2

For abitrary $\varepsilon > 0$ we can chose n_0 such that if $n, m > n_0$

$\|x_n - x\| < \varepsilon/2$ and $\|x_m - x\| < \varepsilon/2$

and therefore also

$$\|x_n - x_m\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ 1

3 seen

(ii) For $x > 0$

$$\tanh(nx) = \frac{e^{nx} - e^{-nx}}{e^{nx} + e^{-nx}} = \frac{1 - e^{-2nx}}{1 + e^{-2nx}}$$

$$\simeq (1 - e^{-2nx})^2 \quad \text{for } xn \gg 1$$

$$= 1 + e^{-4nx} - 2e^{-2nx}$$ 2

For $x < 0$

$$\tanh(nx) = \frac{e^{2nx} - 1}{e^{2nx} + 1} \simeq -(1 - e^{2nx})^2$$

$$= -1 + 2e^{2nx} - e^{-4nx}$$ 2

Hence

$$\tanh(nx) - \varphi(x) \simeq \begin{cases} -2e^{-2nx} & \text{for } x \gg \frac{1}{n} \\ +2e^{-2nx} & \text{for } x \ll -\frac{1}{n} \end{cases}$$

unseen

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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course MPhys II
Question 4	TOPIC Linear vector spaces	Marks & seen/unseen
Parts (ii)	$(\tanh(nx) - \varphi(x))^2 \simeq 4e^{-4n\lvert x\rvert}$ for $\lvert x\rvert \gg \frac{1}{n}$ and therefore $\lVert f_n - \varphi \rVert^2 \simeq 2\int_0^\infty 4e^{-4nx}\,dx$ $= 8\left[\frac{e^{-4nx}}{-4n}\right]_0^\infty = \frac{2}{n} \to 0 \quad n \to \infty.$	2 1 7
(iii)	$C^0(\mathbb{R})$ with l_2 norm cannot be complete since $\tanh(nx) \in C^0(\mathbb{R})$ but the sequence converges strongly in l_2 towards $\varphi \notin C^0(\mathbb{R})$. And since $\tanh(nx)$ is strongly convergent it is also a Cauchy sequence within $C^0(\mathbb{R})$. But if $C^0(\mathbb{R})$ with l_2 norm was a complete space any Cauchy sequence should converge strongly within the space $C^0(\mathbb{R})$ with l_2-norm.	3 3 seen similar
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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012

Course: MPhys II

Question | TOPIC: Linear vector spaces | Marks & seen/unseen

Parts

(iv) A norm ~~must~~ satisfy:

1) $\|\mu x\| = |\mu| \|x\|$

2) $\|x\| \geq 0$ with $\|x\| = 0 \Leftrightarrow x = 0$

3) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$

$\|\mu x\| = [\langle \mu x, \mu x\rangle]^{1/2} = [\mu^* \mu \langle x, x\rangle]^{1/2}$

$= |\mu| \langle x, x\rangle^{1/2} = |\mu| \|x\|$ 1

$\|x\|^2 = \langle x, x\rangle \geq 0$ per definition of scalar prod.

$\|x\| = 0 \Leftrightarrow \langle x, x\rangle = 0 \Leftrightarrow x = 0$ again per def. of scalar prod. 1

$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle$

$= \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle$

$\Downarrow$

$\|x+y\|^2 = |\, \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle |$ 2

$\leq \|x\|^2 + \|y\|^2 + 2|\langle x,y\rangle|$

$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$ $\leftarrow$ by Schwartz 2

$= (\|x\| + \|y\|)^2$

and ~~therefore~~

$\|x+y\| \leq \|x\| + \|y\|$ 1

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EXAMINATION QUESTIONS/SOLUTIONS 2011-2012

Course: MPhys II

Question: 5

TOPIC: Calculus of variation

Marks & seen/unseen

Parts

(i) Curve between point $A=(0,0)$ and $B=(x_0,y_0)$ given by $y=y(x)$.

length of curve

$$\ell = \int_0^{x_0} d\ell = \int_0^{x_0} dx \underbrace{\sqrt{1+y'^2}}_{f(y,y',x)}$$ 2

Euler-Lagrange eq:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$ 2

where

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}},$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{y''}{(1+y'^2)^{3/2}}$$

I.e. E-L eq: $y''=0$

$$\Downarrow \quad \underline{\underline{y = ax+b}} \quad \text{straight line}$$ 2

6 — Seen similar

(ii) length $\ell = \int_{-a}^{a} dx \sqrt{1+y'^2}$ ← <u>constraint</u> 2

area $A = \int_{-a}^{a} dx\, y$

Introduce Lagrange multiplier and consider

$J = A + \lambda \ell = \int_{-a}^{a} dx\, \tilde{f}(y,y',x)$ with 2

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EXAMINATION QUESTIONS/SOLUTIONS 2011-2012

Course: MPhys II

Question 5

TOPIC: Calculus of variation

Marks & seen/unseen

Parts (ii)

$$\tilde{f}(y, y', x) = y + \lambda \sqrt{1+y'^2}$$ 2

$$\frac{\partial \tilde{f}}{\partial y} = 1, \quad \frac{\partial \tilde{f}}{\partial y'} = \lambda \frac{y'}{\sqrt{1+y'^2}}$$

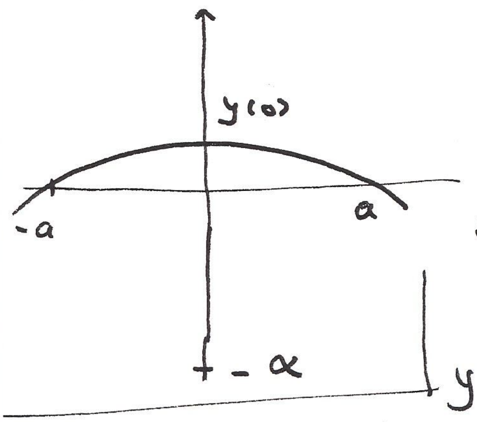
$$\frac{d}{dx}\frac{\partial \tilde{f}}{\partial y'} = \lambda \frac{y''}{(1+y'^2)^{3/2}}$$

E-L eq: $$1 - \lambda \frac{y''}{(1+y'^2)^{3/2}} = 0$$ 2 — 8

$$\Downarrow$$

$$(*) \quad (1+y'^2)^{3/2} = \lambda y''$$

Proof the section of circle is a solution:



$$R = y(0) + \alpha$$

$$x^2 + (y+\alpha)^2 = R^2$$

$$\Downarrow$$

$$y = \sqrt{R^2 - x^2} - \alpha$$ 2

$$y' = \frac{-x}{\sqrt{R^2-x^2}}, \quad y'' = \frac{-R^2}{(R^2-x^2)^{3/2}}$$ — 6

LHS of (*) $$: \left[1 + \frac{x^2}{R^2-x^2}\right]^{3/2} = \frac{R^3}{(R^2-x^2)^{3/2}}$$ 2

RHS of (*) $$= \lambda \frac{-R^2}{(R^2-x^2)^{3/2}}$$

I.e solution for $\lambda = -R$. 2 — 20

see similar

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	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012	Course MPhys II

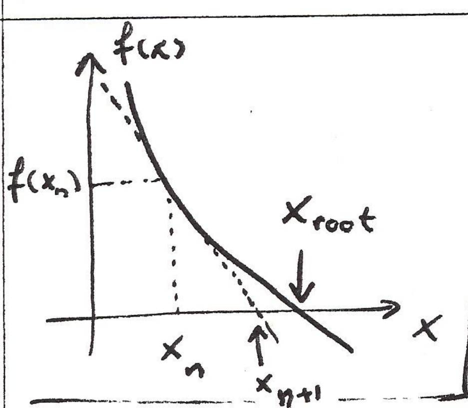
Question 6

TOPIC Numerical analysis

Marks & seen/unseen

Parts

(i)



Estimate xroot from zero of tangent

$y = f(x_n) + f'(x_n)(x - x_n)$ 2

I.e.

$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n)$ 2

$\Downarrow$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ 1

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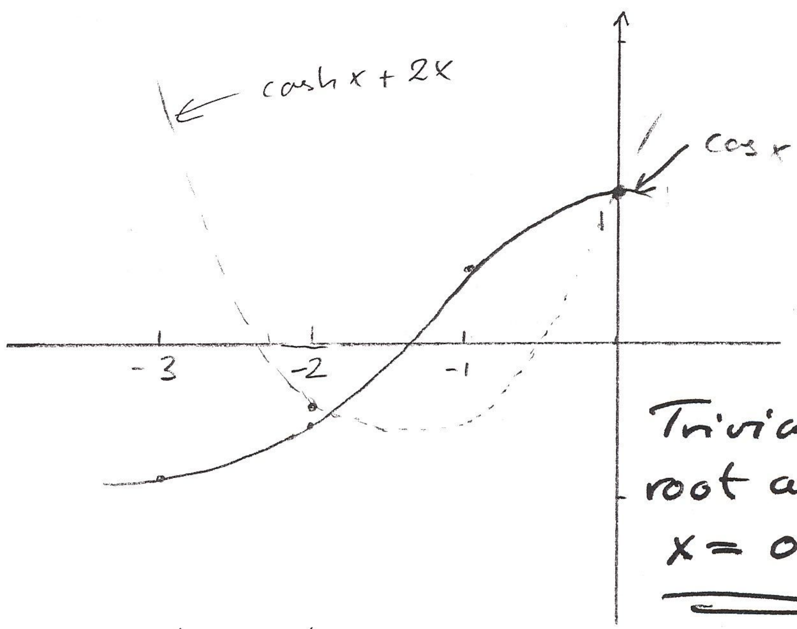
(ii)



Trivial root at $x = 0$. 1

seen similar

x	0	-1	-2	-3
$\cos x$	1	0.54	-0.42	-0.99
$\cosh x$	1	1.54	3.76	10.1
$\cosh x + 2x$	1	-0.45	-0.24	4.1

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Question 6 | TOPIC *Numerical analysis* | Marks & seen/unseen

Parts

(ii) cont

sketch suggest to look for 2nd root near $x = -2$. — 1

Newton-Raphson

$$f(x) = \cosh x + 2x - \cos x$$

$$f'(x) = \sinh x + 2 + \sin x$$ — 2

$$x_{n+1} = x_n - \frac{\cosh x_n + 2x_n - \cos x_n}{\sinh x_n + 2 + \sin x_n}$$

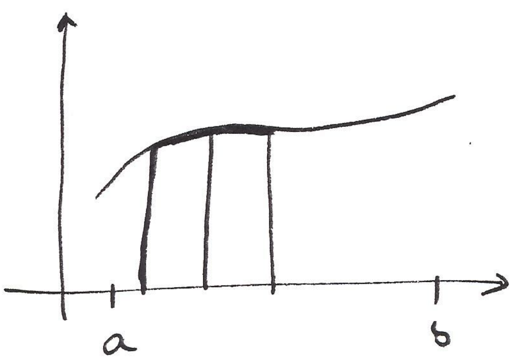
x_n	x_{n+1}
-2	-1.92968
-1.92968	-1.92616
-1.92616	-1.92615

$x_{root} \simeq -1.93$ — 2

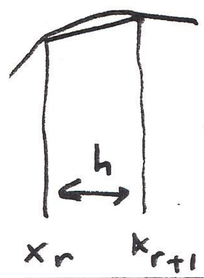
6

seen similar

(iii)



Approximate graph by linear sections:



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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012

Course: MPhys II

Question 6

TOPIC: Numerical analysis

Marks & seen/unseen

Parts

Sub-area = area of trapezium

$$= \frac{h}{2}(f(x_r) + f(x_{r+1})) = \frac{h}{2}(y_r + y_{r+1})$$ 1

Sum up:

$$\int_a^b f(x)\,dx \simeq \sum_{r=0}^{N-1} \frac{h}{2}(y_r + y_{r+1})$$

$$= \frac{h}{2}(y_0 + 2(y_1 + \cdots + y_{N-1}) + y_N)$$ 1

4

(iv) Runge-Kutta

$$\frac{dy}{dx} = f(x, y(x))$$

$$y_1 = y_0 + \int_{x_0}^{x_1} \frac{dy}{dx}\,dx = y_0 + \int_{x_0}^{x_1} f(x, y(x))\,dx$$ 2

$$= y_0 + \frac{h}{2}\{f(x_0, y(x_0)) + f(x_1, y(x_1))\}$$

trapezium approx.

Taylor expand $y(x_1) = y(x_0)\, y'(x_0)(x_1 - x_0) = y(x_0) + y'(x_0)h$

$$= y_0 + f(x_0, y_0)h$$ 2

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_0 + f(x_0, y_0)h)\}$$

or $y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$

$k_1 = f(x_n, y_n)$ 1

$k_2 = f(x_{n+1}, y_n + hk_1)$

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seen similar

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Setter's initials: H.I.I. Checker's initials: ADP

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