Imperial College
London

[MP2 2009]

B.Sc. and M.Sci. EXAMINATIONS 2009

SECOND YEAR STUDENTS OF PHYSICS

MATHEMATICS - M.PHYS 2

Date **Wednesday 3rd June 2009** **10.00 am - 12.00 pm**

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Do not attempt more than FOUR questions.

A mathematical formulae sheet is provided

[Before starting, please make sure that the paper is complete; there should be 4 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Show that $\tanh(z)$ is periodic along the imaginary axis and determine the periodicity.

   (ii) Consider

$$f(z) = U(x, y) + iV(x, y).$$

   Show that if $f(z)$ is a differentiable function, then the contours curves of U and V are orthogonal in the xy-plane.

Consider the functions $h(x)$ and $g(x)$ defined by

$$\begin{aligned} h(x) &= e^{iAx}(x - 2i) \\ g(x) &= (x^2 - 4 - 4ix)(x + i)^2, \end{aligned}$$

where $A \in \mathbb{R}$ is a constant.

   (iii) Find the singular points of

$$f(z) = \frac{h(z)}{g(z)}$$

   and determine their nature.

   (iv) For all values of $A \in \mathbb{R}$ except $A = 0$ determine the value of

$$I(A) = \int_{-\infty}^{\infty} f(x)dx.$$

2. Consider $g, f \in S = C^0([0, 1], \mathbb{R})$.

   (i) Show that

$$\langle g, f\rangle = \int_0^1 dx\, g(x) f(x)$$

   defines a scalar product on S.

   (ii) Let $h(x) = x$ for $x \in [0, 1]$. Is $h \in S$?

   (iii) Find an example of a function $g \in S$, which is orthogonal to h with respect to the above scalar product.

Use the scalar product in (i) to define a norm on S. Consider the sequence $f_n \in S$ given by

$$f_n(x) = x^n.$$

   (iv) Show that f_n is a Cauchy sequence in S with respect to the above norm.

   (v) Show that with respect to the above norm f_n is strongly convergent towards

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \end{cases}$$

   (vi) Explain why S is not a complete space.

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Question 3
Consider three functions $E(x)$, $G(x)$ and $h(x)$ related according to

$$E(x) = \int_{-\infty}^{\infty} dx' G(x - x') h(x').$$

(i) Derive the relation between the Fourier transforms $\hat{E}(k)$, $\hat{G}(k)$ and $\hat{h}(k)$.
Assume that the function $G(x)$ is the solution of

$$-\frac{d^2 G(x)}{dx^2} + M^2 G(x) = \delta(x),$$

where $M \in \mathbb{R}$ is a constant.

(ii) Find the Fourier transform $\hat{G}(k)$.
(iii) For $x > 0$ and $x < 0$ express $E(x)$ in terms of $\hat{h}$ and the poles of $\hat{G}$.
(iv) Assume

$$h(x) = \frac{1}{\sqrt{\pi}\alpha} e^{-(\frac{x}{\alpha})^2}.$$

Find $E(x)$.
[You may use without proof the result

$$\int_{-\infty}^{\infty} dt e^{-(t+ia)^2} = \int_{-\infty}^{\infty} dt e^{-t^2} = \sqrt{\pi}]$$

Question 4
The three orthogonal basis vectors $\mathbf{e}'_i$ (with $i = 1, 2, 3$) of the coordinate system S' and the three orthogonal basis vectors $\mathbf{e}_i$ of coordinate systems S are related according to

$$\mathbf{e}'_i = a_{ij}\mathbf{e}_j,$$

where we have assumed Einstein's convention of summation over repeated indices.

(i) Let the vector $\mathbf{v}$ have the coordinates v_i with respect to S and v'_i with respect to S'. Express the relation between v'_i and v_i in terms of the matrix $\mathbf{A} = \{a_{ij}\}$.
(ii) Show that $\mathbf{A}^T = \mathbf{A}^{-1}$.

(iii) Show that the following relations are fulfilled

$$\begin{aligned} a_{ik}a_{jk} &= \delta_{ij} \\ a_{ki}a_{kj} &= \delta_{ij}. \end{aligned}$$

(iv) Assume that T_{ijk} is a tensor of rank 3. Show that T_{ikk} is a tensor of rank 1, i.e. a vector.
(v) Let $\mathbf{c} = \mathbf{d} \times \mathbf{b}$, where $\mathbf{d}$ and $\mathbf{b}$ are vectors. Show that $\mathbf{c}$ is a pseudovector.
[You may uses without proof the result

$$\epsilon_{ijk}a_{ip}a_{jq}a_{km} = \det\mathbf{A}\epsilon_{pqm}.]$$

5. Consider a closed contour in the plane described by polar co-ordinates

$$\mathbf{r} = (\theta, r(\theta)).$$

We want to determine the contour that encloses the largest area for a given fixed length of the contour.

(i) Use a Lagrange multiplier λ and show that the relevant functional for this extremal problem is

$$J = \int_0^{2\pi} d\theta [\frac{r^2}{2} + \lambda \sqrt{r^2 + (\frac{dr}{d\theta})^2}].$$

(ii) Derive the Euler-Lagrange equation for the problem.

(iii) Show that a circle satisfies the Euler-Lagrange equation.

(iv) If you were to solve the Euler-Lagrange equation, how would you determine λ?

6. Consider the integral $I = \int_a^b dx\, f(x)$.

(i) Derive Simpson's Rule.

Consider the differential equation

$$\frac{dy}{dx} = f(x, y).$$

(ii) Derive the Runge-Kutta iteration scheme

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2) \ , \\ k_1 &= f(x_n, y_n) \ , \\ k_2 &= f(x_{n+1}, y_n + hk_1) \ . \end{aligned}$$

(iii) Use the Runge-Kutta scheme with $h = 0.01$ to find the value of y at $x = 0.03$ given that

$$\frac{dy}{dx} = xe^y,$$

and $y(0) = 0$.

END OF PAPER

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2008-09	Course: Mphys 2
Question 1		Marks & seen/unseen

Parts

(i) $\tanh(z) = \dfrac{e^z - e^{-z}}{e^z + e^{-z}} = \dfrac{e^x e^{iy} - e^{-x} e^{-iy}}{e^x e^{iy} + e^{-x} e^{-iy}}$ — 2

and $e^{\pm iy} = \cos y \pm i \sin y$ is periodic with period 2π. $= \dfrac{e^x e^{i2y} - e^{-x}}{e^x e^{i2y} + e^{-x}}$ — 2

Hence $\tanh(z)$ period π.

SEEN

(ii) A complex differentiable function satisfies the Cauchy-Riemann conditions:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}.$$ — 2

The tangents of the contour curves for U an V are orthogonal to the gradients ∇U and ∇V.

$$\nabla U \cdot \nabla V = \frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y}$$

$$= \frac{\partial U}{\partial x}\left(-\frac{\partial U}{\partial y}\right) + \frac{\partial U}{\partial y}\frac{\partial U}{\partial x} = 0$$ — 2

i.e. $\nabla U \perp \nabla V$.

SEEN

(iii) Note $g(x) = (x-2i)^2(x+i)^2$, hence

$$f(z) = \frac{e^{iAz}}{(z-2i)(z+i)^2}.$$

Since e^{iAz} is analytic and non-zero we have a single pole at $z = 2i$ and a pole of order 2 at $z = -i$ — 2

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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2008-09

Course: Mphys 2

Question: 1

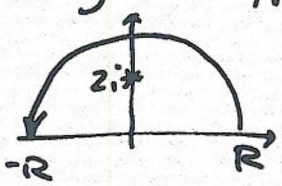
Marks & seen/unseen

Parts

(iv)

$$I(A) = \int_{-\infty}^{\infty} dx \frac{e^{iAx}}{(x-2i)(x+i)^2}$$

For $A>0$: $\left| e^{iA(x+iy)} \right| = e^{-Ay}$ exponential decay in upper half plane. Use contour 2



Pick up residue from pole at $z = 2i$.

$$\mathrm{Res}(2i) = \frac{e^{iA2i}}{(2i+i)^2} = \frac{e^{-2A}}{-9}.$$

I.e.

$$A>0 : \quad I(A) = \frac{-2\pi i}{9} e^{-2A}.$$ 2

For $A<0$ close in lower half plane. Since pole at $z=-i$ is of 2nd order we need to note that Laurent series is of the form

$$\frac{e^{iAz}}{(z-2i)(z+i)^2} = \frac{f(-i) + f'(-i)(z+i) + \frac{1}{2} f''(-i)(z+i)^2 + \cdots}{(z+i)^2}$$

$$= \frac{f(-i)}{(z+i)^2} + \frac{f'(-i)}{z+i} + \frac{1}{2} f''(-i) + \mathcal{O}(z+i)$$

where $f(z) = \dfrac{e^{iAz}}{(z-2i)}$

$$f'(z) = \frac{iAe^{iAz}(z-2i) - e^{iAz}}{(z-2i)^2}$$ 2

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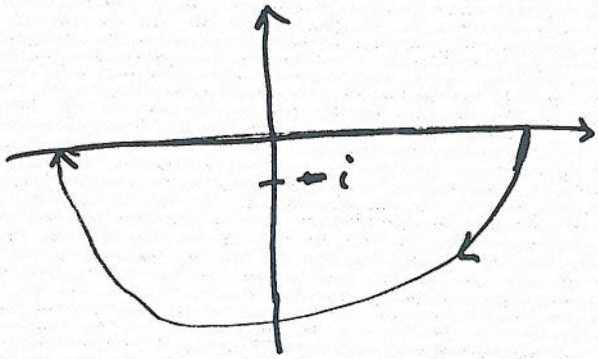
	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2008-09	Course Mphys 2
Question 1		Marks & seen/unseen

Parts

$$f'(-i) = \frac{3A-1}{-9}\, e^{+A}$$ 2

Note: pole at $z=-i$ is now encircled in the negative direction.



Thus for $A<0$:

$$I(A) = -2\pi i\, f'(-i)$$

$$= 2\pi i\, \frac{3A-1}{9}\, e^{+A}$$ 2

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	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2008-09	Course Mphys 2
Question 2		Marks & seen/unseen

Parts

(i) Scalar product must satisfy

1) $\langle x, (\mu_1 y_1 + \mu_2 y_2)\rangle = \mu_1 \langle x, y_1\rangle + \mu_2 \langle x, y_2\rangle$

2) $\langle x, y\rangle^* = \langle y, x\rangle$ — 2

3) $\langle x, x\rangle \geq 0$ with $\langle x, x\rangle = 0 \Leftrightarrow x = 0$

where $x, y \in S$ and μ_1, μ_2 are scalars.

1) Is satisfied since $\int_0^1 dx\, g(x) f(x)$ is linear in both g and f.

2) Is satisfied since $g, f \in \mathbb{R}$.

3) $\langle x, x\rangle = \int_0^1 dx\, g^2(x) \geq 0$. — 2

If $\int_0^1 dx\, g^2(x) = 0$ it follows, since g is continuous, that $g(x) = 0$. — 2

SEEN

(ii) Yes, $h \in S$ since $h(x) = x$ is continuous. — 2

(iii) We want $\langle g, h\rangle = \int_0^1 dx\, x\, g(x) = 0$

A choice could be

$$g(x) = \begin{cases} \frac{\sin(2\pi x)}{x} & 0 < x \leq 1 \\ 2\pi & x = 0. \end{cases}$$ — 2

g is continuous and

$$\int_0^1 dx\, x\, g(x) = \int_0^1 dx\, \sin(2\pi x) = 0$$

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EXAMINATION QUESTIONS/SOLUTIONS 2008-09

Course: Mphys2

Question: 2

Marks & seen/unseen

Parts

(iv) $\| f_n - f_m \|^2 = \int_0^1 dx \, (x^n - x^m)^2$ assume $n<m$

$= \int_0^1 dx \; x^{2n} (1 - x^{m-n})^2$

$\leq \int_0^1 dx \; x^{2n} = \frac{1}{2n+1}$. — 2

Given $\varepsilon > 0$ we can choose $n_0 \geq \frac{1}{2}\left(\frac{1}{\varepsilon} - 1\right)$ and conclude that for

$n, m \geq n_0 \Rightarrow \| f_n - f_m \| < \varepsilon$. — 2

I.e. f_n is a Cauchy sequence.

(v) $\| f_n - f \| = \int_0^1 dx \, (f_n(x) - f(x))^2 = \int_0^1 dx \, f_n(x)^2$

$= \int_0^1 dx \; x^{2n} = \frac{1}{2n+1}$. — 2

Again. Given $\varepsilon > 0$, choose $n_0 = \frac{1}{2}\left(\frac{1}{\varepsilon} - 1\right)$ then for

$n > n_0 \Rightarrow \| f_n - f \| < \varepsilon$. — 2

I.e. f_n strongly convergent towards f.

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(vi) The sequence f_n is Cauchy and strongly convergent towards the discontinuous function f. I.e. $\lim_{n\to\infty} f_n = f \notin S$. — 2

Hence, S in complete.

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	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2008-09	Course Mphys 2
Question 3		Marks & seen/unseen

Parts

(i) $\hat{E}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} E(x)$

$= \int_{-\infty}^{\infty} dx\, e^{-ikx} \int_{-\infty}^{\infty} dx'\, G(x-x')h(x')$

$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, G(x-x') e^{-ik(x-x')} h(x') e^{-ikx'}$ 3

For fixed x' sub $u = x - x'$

$= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} du\, G(u) e^{-iku} h(x') e^{-ikx'}$

$= \int_{-\infty}^{\infty} du\, G(u) e^{-iku} \int_{-\infty}^{\infty} dx'\, h(x') e^{-ikx'}$

$= \hat{G}(k)\, \hat{h}(k)$ 2

SEEN

(ii) $G(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{G}(k) e^{ikx}$

$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}$ 3

Transf. of eq:

$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} [-(ik)^2 \hat{G}(k) + M^2 \hat{G}(k)]$

$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}$

$\Downarrow$

$\hat{G}(k) = \frac{1}{k^2 + M^2}$ 2

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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2008-09

Course: Mphys 2

Question 3

Marks & seen/unseen

Parts

(iii)
$$E(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{E}(k)\, e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{G}(k)\hat{h}(k) e^{ikx}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\hat{h}(k) e^{ikx}}{k^2 + M^2} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\hat{h}(k) e^{ikx}}{(k - iM)(k + iM)}$$ 2.

Simple poles at $k = \pm iM$.

For $x > 0$: $|e^{i(k'+ik'')x}| = e^{-k''x}$ decays exponentially in upper half plane.

I.e. close contour in UHP pick up residue at $k = iM$: 2

$$E(x) = 2\pi i \frac{\hat{h}(iM) e^{-Mx}}{2\pi \; 2iM} = \frac{\hat{h}(iM) e^{-Mx}}{2M}$$

For $x < 0$ close in LHP: remember sign from negative orientation of contour

$$E(x) = -2\pi i \frac{\hat{h}(-iM) e^{Mx}}{2\pi(-2iM)} = \frac{\hat{h}(-iM) e^{Mx}}{2M}$$ 2

(iv)
$$\hat{h}(k) = \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{\pi}\,\alpha} e^{-\left(\frac{x}{\alpha}\right)^2} e^{-ikx}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} du\, e^{-[u^2 + i\alpha k u]} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} du\, e^{-[(u + \frac{\alpha k}{2} i)^2 + (\frac{\alpha k}{2})^2]}$$ 2

$$= \frac{1}{\sqrt{\pi}} e^{-\left(\frac{\alpha k}{2}\right)^2} \int_{-\infty}^{\infty} du\, e^{-(u + \frac{\alpha k}{2} i)^2} = e^{-\left(\frac{\alpha k}{2}\right)^2}.$$

use result in (iii) and conclude

$$E(x) = \exp\left\{-M|x| + \left(\frac{\alpha M}{2}\right)^2\right\} \frac{1}{2M}$$ 2 (20)

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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2008-09

Course: Mphys 2

Question 4

Marks & seen/unseen

Parts

(i) $\underline{v} = v_p \underline{e}_p = v_q' \underline{e}_q'$

and

$\underline{e}_i' = a_{ik} \underline{e}_k \rightarrow a_{ij} = \underline{e}_i' \cdot \underline{e}_j$

$\underline{e}_j = b_{jk} \underline{e}_k' \rightarrow b_{ji} = \underline{e}_j \cdot \underline{e}_i' = a_{ij}$ 3

thus

$\underline{e}_j = a_{kj} \underline{e}_k'$ and therefore

$v_p a_{kp} \underline{e}_k' = v_q' \underline{e}_q' \Rightarrow v_i' = a_{ip} v_p$

$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \underline{\underline{A}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ 3

(ii) From $\underline{v}' = \underline{\underline{A}}\, \underline{v}$ and $\underline{v} = \underline{\underline{B}}\, \underline{v}'$

with $\underline{\underline{B}} = \{b_{ij}\}$ we conclude

$\underline{v} = \underline{\underline{B}}\,\underline{\underline{A}}\,\underline{v} \Rightarrow \underline{\underline{B}}\,\underline{\underline{A}} = \underline{\underline{I}}$ Thus $\underline{\underline{B}} = \underline{\underline{A}}^{-1}$ 2

In (i) we saw that $\underline{\underline{B}} = \underline{\underline{A}}^T$, i.e.,

$\underline{\underline{A}}^T = \underline{\underline{A}}^{-1}$

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(iii) $\underline{\underline{I}} = \underline{\underline{A}}\,\underline{\underline{A}}^T \rightarrow \delta_{ij} = \underline{\underline{A}}\,\underline{\underline{A}}^T = a_{ik} a^T_{kj} = a_{ik} a_{jk}$ 2

$\underline{\underline{I}} = \underline{\underline{A}}^T\underline{\underline{A}} \rightarrow \delta_{ij} = \underline{\underline{A}}^T \underline{\underline{A}} = a^T_{ik} a_{kj} = a_{ki} a_{kj}$ 2

(iv) $T_{ikk}' = a_{in} \underbrace{a_{km} a_{kl}}_{\delta_{ml} \text{ from (iii)}} T_{nml}$

$= a_{in} \delta_{ml} T_{nml} = a_{in} T_{nmm}$ 2

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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2008-09

Course: Mphys2

Question 4

Marks & seen/unseen

Parts (v)

$$c_i = \varepsilon_{ijk} d_j b_k$$

$$c'_i = \varepsilon'_{ijk} d'_j b'_k = \varepsilon_{ijk} d'_j b'_k$$

$$= \varepsilon_{ijk} a_{jp} a_{kq} d_p b_q$$

$$= \delta_{im} \varepsilon_{mjk} a_{jp} a_{kq} d_p b_q$$

$$= a_{is} a_{ms} \varepsilon_{mjk} a_{jp} a_{kq} d_p b_q$$

$$= \varepsilon_{mjk} (a_{ms} a_{jp} a_{kq}) a_{is} d_p b_q$$

$$= \det \underline{\underline{A}} \; \varepsilon_{spq} a_{is} d_p b_q \qquad 3$$

$$= \det \underline{\underline{A}} \; a_{is} c_s \qquad 3$$

the det $\underline{\underline{A}}$ factor means that $\underline{c}$ transforms as a pseudovector.

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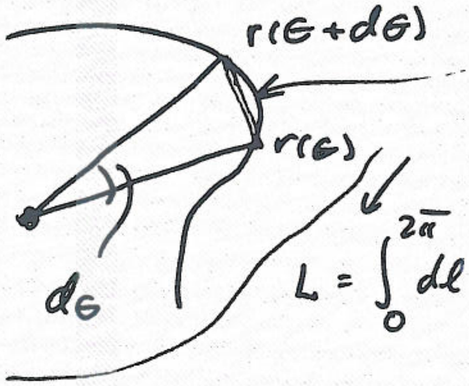
	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2008-09	Course Mphys 2
Question 5		Marks & seen/unseen

Parts

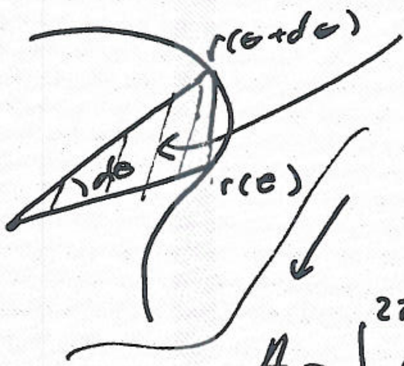
(i) The perimeter, L, and area enclosed, A, of curve $\underline{r} = (\theta, r(\theta))$, $\theta \in [0, 2\pi]$.



$$dl = \sqrt{(r\,d\theta)^2 + dr^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$L = \int_0^{2\pi} dl = \int_0^{2\pi} d\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$ 3

<u>Area</u>



$$dA = \frac{1}{2} r(\theta) r(\theta + d\theta) \sin d\theta$$

$$\simeq \frac{1}{2} r(\theta)(r(\theta) + r'(\theta)\, d\theta)\, d\theta$$

$$\simeq \frac{1}{2} r^2(\theta)\, d\theta$$

$$A = \int_0^{2\pi} dA = \int_0^{2\pi} \frac{r^2}{2}\, d\theta$$ 3

So maximising A under constraint L can be done by considering:

$$J = A + \lambda L = \int_0^{2\pi} d\theta \left[\frac{r^2}{2} + \lambda\sqrt{r^2 + r'^2}\right]$$ 2

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	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2008-09	Course Mphys2
Question 5		Marks & seen/unseen
Parts (ii)	E-L: $\frac{\partial J}{\partial r} - \frac{d}{d\theta}\frac{\partial J}{\partial r'} = 0$ in general	2
	$\frac{\partial J}{\partial r} = r + \lambda \frac{r}{\sqrt{r^2+r'^2}}$	
	$\frac{\partial J}{\partial r'} = \lambda \frac{r'}{\sqrt{r^2+r'^2}}$	
	$\frac{d}{d\theta}\frac{\partial J}{\partial r'} = \lambda \frac{r''\sqrt{r^2+r'^2} - r'\frac{rr'+r'r''}{\sqrt{r^2+r'^2}}}{r^2+r'^2}$	2
	$= \frac{r^2 r'' - r r'^2}{(r^2+r'^2)^{3/2}}$	
	E-L: $r(r^2+r'^2)^{3/2} + \lambda r(r^2+r'^2) - r^2 r'' + r r'^2 = 0$	1
(iii)	For circle $r(\theta) = R$ indep. of θ $\Rightarrow r' = 0,\ r'' = 0.$	2
	$\Rightarrow$ E-L: $r^4 + \lambda r^3 = 0 \Rightarrow r = -\lambda$, satisfies E-L eq.	2
(iv)	Determine λ from given L: $L = 2\pi R = 2\pi(-\lambda) \Rightarrow \lambda = \frac{-L}{2\pi}$	3
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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2008-09

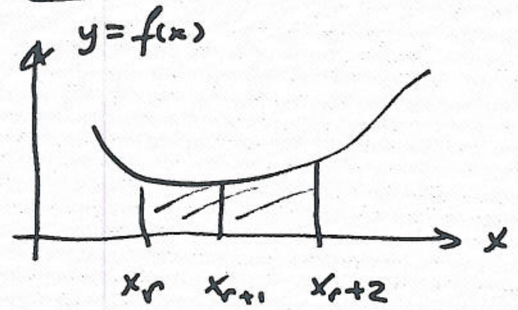
Course: Mphys 2

Question 6

Marks & seen/unseen

Parts

(i) <u>Simpson's rule</u>



Divide $[a,b]$ into <u>even</u> number N of equal sized strips.

Consider strips pair wise:

3

$$dA = \int_{x_r}^{x_{r+2}} f(x)\,dx$$

Notation: $c = x_{r+1}$, $t = x - c$, $h = x_{r+1} - x_r$

$$dA = \int_{-h}^{h} f(c+t)\,dt.$$

Taylor expand

$$f(c+t) = f(c) + f'(c)t + \tfrac{1}{2}f''(c)t^2 + \dots$$

$\Downarrow$

$$dA \simeq \int_{-h}^{h} \left[f(c) + f'(c)t + \tfrac{1}{2}f''(c)t^2\right] dt$$

$$= 2h f(c) + 0 + \tfrac{1}{2}f''(c)\left[\frac{t^3}{3}\right]_{-h}^{h}$$

$$= 2h f(c) + \frac{f''(c)}{3!} h^2$$

Eliminate $f''(c)$ from

$$f(c+h) = f(c) + f'(c)h + \tfrac{1}{2}f''(h)h^2$$

$$f(c-h) = f(c) - f'(c)h + \tfrac{1}{2}f''(h)h^2$$

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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2008-09

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Question: 6

Marks & seen/unseen

Parts

$$f''(c) = \frac{1}{h^2}\left(f(c+h) + f(c-h) - 2f(c) \right)$$

Sub. into expression for dA:

$$dA = 2h\left(f(c) + \frac{h^2}{3!}\frac{1}{h^2}\left(f(c+h) + f(c-h) - 2f(c) \right)\right)$$ 2

$$= \frac{h}{3}\left(f(c-h) + 4f(c) + f(c+h) \right)$$

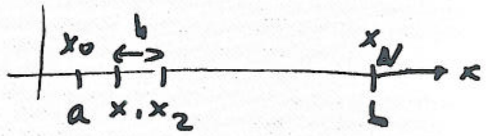
Add together all the pairs of strips and use notation $y_r = f(x_r)$

$$A = \sum dA =$$

$$\frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \dots + y_{N-1}) + 2(y_2 + y_4 + \dots y_{N-2}) + y_N \right]$$ 1

seen

(ii) $\frac{dy}{dx} = f(x, y(x))$



$$y_1 = y_0 + \int_{x_0}^{x_1} \frac{dy}{dx}\,dx$$

$$= y_0 + \int_{x_0}^{x_1} f(x, y(x))\,dx$$ 2

$$\simeq y_0 + \frac{h}{2}\left[f(x_0, y(x_0)) + f(x_1, y(x_1)) \right]$$ (*)

trapezium approx.

$$y_1 = y(x_1) = y(x_0 + h) = y(x_0) + y'(x_0)h$$ 2

$$= y_0 + f(x_0, y_0)h$$

seen

Setter's initials: HJJ

Checker's initials: AJP

Page number: 13

Question 6

Marks & seen/unseen

Parts

Sub. this expression for y_1 into right hand side of (*):

$$y_1 = y_0 + \frac{h}{2}\Big[\underbrace{f(x_0, y_0)}_{t_1} + \underbrace{f(x_1, y_0 + f(x_0, y_0)h)}_{t_2}\Big]$$ 2

(iii) $f(x,y) = x e^{y}$

x_1	y_1	t_1	t_2	y_2
0	0	0	0.01	$5.0 \cdot 10^{-4}$
0.01	$5 \cdot 10^{-5}$	0.010	0.020	$2.0 \cdot 10^{-4}$
0.02	$2 \cdot 10^{-4}$	0.020	0.030	$4.5 \cdot 10^{-4}$
0.03	$4.5 \cdot 10^{-4}$	0.030	0.040	$8.0 \cdot 10^{-4}$

3

UNSEEN

we used

$$y_{n+1} = y_n + \frac{h}{2}(t_1 + t_2)$$

$$t_1 = x_n e^{y_n}$$

$$t_2 = (x_n + h)\exp\{y_n + h x_n e^{y_n}\}$$ 2

20

Setter's initials: HJJ

Checker's initials: AP