Mathematical Methods **Spring Term 2021**

Problem Sheet 7

1. *Lorentz Force*

   Show that the Lagrangian

   $$L = \frac{m}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t) - q\phi(\mathbf{r}, t),$$

   leads to the Lorentz force law for a (non-relativistic) charged particle

   $$m\ddot{\mathbf{r}} = q\mathbf{E}(\mathbf{r}, t) + q\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}, t).$$

   The vector and scalar potentials, $\mathbf{A}(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$, are related to the electric and magnetic fields, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, through

   $$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

   Hint: Check that the Euler-Lagrange equation

   $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

   yields the x-component of the Lorentz force law. Here ϕ, A_x, A_y and A_z are arbitrary functions of x, y, z and t.

2. A bead of mass m moves without friction on a helical wire. The helix can be parametrized as follows

   x = R cos u, y = R sin u z = αu,

   where u is a real parameter. Here α and R are constants. The acceleration due to gravity is the constant g (the gravitational force is pointing in the negative z direction). Obtain a Lagrangian *L(z,z')* for the bead and solve the equation of motion.

3. *A Version of Noether's Theorem*

i) Suppose the Lagrangian $L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t)$ is invariant [1] under the *time-independent* transformation

$$q_i' = q_i + \epsilon Q_i(q_1, q_2, ..., q_n) \quad i = 1, ..., n$$

where ϵ is 'small'. Show that

$$\sum_{i=1}^{n} p_i Q_i(q_1, q_2, ..., q_n)$$

is a constant of the motion.

ii) A particle moving in two dimensions is described by the Lagrangian $L(x, y, \dot{x}, \dot{y}, t)$. The Lagrangian is invariant under the transformation

$$x' = x + \epsilon y, \qquad y' = y - \epsilon x.$$

What do you conclude? Give an example of a Lagrangian with this property.

4. *Rotating Pendulum Problem*

A simple pendulum is mounted on a rotating turntable with constant angular velocity Ω. The pivot is on the axis of rotation.

i) Show that the kinetic energy of the pendulum bob is

$$T = \frac{ml^2}{2}\left(\dot{\theta}^2 + \Omega^2 \sin^2\theta\right).$$

Here m is the mass of the bob, l is the length of the rod (assumed to be massless) and θ is the angle between the rod and the vertical. The potential energy is the same as for a non-rotating pendulum, ie. $V = -mgl\cos\theta$.

Hint: what is the kinetic energy of a particle in spherical polar coordinates?

ii) Obtain the equation of motion.

iii) Find all solutions of the form θ =constant.

iv) Find a conserved quantity and show that $T + V$ is not a constant of the motion (unless θ is constant).

v) Determine the frequency of small oscillations about the constant θ solutions from part iii).

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[1] Here 'invariant' means $L(q_1', q_2', ..., q_n', \dot{q}_1', \dot{q}_2', ..., \dot{q}_n', t) = L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t)$ up to terms of order ϵ^2.