Mathematical Methods **Spring Term 2021**

Revision Problem Sheet

1. The motion of a free relativistic particle moving in one dimension is de-scribed by the Lagrangian

$$L = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}},$$

where m is the particle mass and c is the speed of light.

i) Compute the momentum $p = \partial L/\partial \dot{x}$ and show that $\dot{x}$ is a constant of the motion.

ii) A Lagrangian for a charged particle in a constant electric field E is[1]

$$L = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}} + qEx,$$

where q is the charge. Solve the Euler-Lagrange equation assuming the particle is at rest at $t = 0$.

Hint: determine $p(t)$ and use this to find $\dot{x}(t)$ which can be integrated to obtain $x(t)$.

2. Find the poles and associated residues of the meromorphic functions

$$i)\quad f(z) = \frac{e^{iz}}{1+z^2}, \quad ii)\quad f(z) = \frac{1}{(z+1)(z+2)(z+3)}$$

3. Compute

$$\mathrm{P}\oint_C \frac{dz}{z}$$

where C is the square contour with vertices at 0, 1, $1+i$ and i (take the orientation anti-clockwise).

Hint: Does the half residue rule apply to this contour?

4. i) Express $x^2 e^{-\frac{1}{2}x^2}$ as a Fourier integral (results from previous problem sheets may be useful).

ii) Find a particular solution to the ODE

$$\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = t^2 e^{-\frac{1}{2}t^2}.$$

[1] A relativistic Lagrangian for a charged particle in an arbitrary electromagnetic field is $L = -mc^2\sqrt{1-\dot{\mathbf{r}}^2/c^2} + q\mathbf{A}\cdot\dot{\mathbf{r}} - q\phi$.

5. Find the Fourier transform of

$$f(x) = \textbf{sign}(x) - \frac{2}{\pi}\tan^{-1} x.$$

6. In quantum mechanics a particle is described by a complex wave function $\psi(x)$. Alternatively, a particle can be described by a complex wave function, $\tilde{\psi}(p)$, depending on momentum p rather than position x. The two wave functions are related through the Fourier integral formula

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\psi(x)\ dx.$$

i) Show that if $\psi(x)$ is normalised then so is $\tilde{\psi}(p)$, that is

$$\int_{-\infty}^{\infty} \tilde{\psi}^*(p)\tilde{\psi}(p)\ dp = 1.$$

ii) In three dimensions the momentum-space wave function is

$$\tilde{\psi}(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\int e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\psi(\mathbf{r})\ d^3r.$$

The ground state wave function for a Hydrogen atom is $\psi(\mathbf{r}) = e^{-|\mathbf{r}|/a}$ where a is the Bohr radius. Compute $\tilde{\psi}(\mathbf{p})$ for this state.

Hints: Exploit the spherical symmetry of the wave function - as $\psi(\mathbf{r})$ depends on $r = |\mathbf{r}|$ the momentum-space wave function $\tilde{\psi}(\mathbf{p})$ is a function of $|\mathbf{p}|$ only[2]. Set $\mathbf{p} = p\mathbf{k}$ and compute $\tilde{\psi}$ using spherical polar coordinates.

7. A scalar (or rank 0 tensor) is unaffected by an orthogonal transformation $x'_i = R_{ij}\,x_j$. A *pseudo-scalar* has the transformation prop-erty $\phi' = \det R\ \phi$. If magnetic monopoles exist, the following modified Maxwell equations might apply

$$\nabla\cdot\mathbf{B} = \rho_m \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} - \mathbf{j}_m.$$

where ρ_m is the magnetic charge density and $\mathbf{j}_m$ is the magnetic current density. Show that ρ_m is a pseudo-scalar and $\mathbf{j}_m$ is an axial vector. Can you write these modified Maxwell equations in tensor form?

[2]More formally, spherical symmetry is preserved by the Fourier transform.