SUPERSIMPLE ω-CATEGORICAL GROUPS AND THEORIES

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Abstract. An ω-categorical supersimple group is finite-by-abelian-by-finite, and has finite SU-rank. Every definable subgroup is commensurable with an $\mathrm{acl}(\emptyset)$-definable subgroup. Every finitely based regular type in a CM-trivial ω-categorical simple theory is non-orthogonal to a type of SU-rank 1. In particular, a supersimple ω-categorical CM-trivial theory has finite SU-rank.

Introduction

It is well-known that an ω-categorical superstable theory is one-based and of finite Morley rank [3]. In the case of a supersimple ω-categorical theory, much less is known except that things do not turn out that nicely. Recently, Hrushovski [7] has constructed a simple ω-categorical theory of SU-rank 1 whose geometry is not (locally) modular, and Pourmahdian [13] has fused Hrushovski's example with an elementary abelian 2-group, and obtained an ω-categorical group of SU-rank 1 which is not one-based. Moreover, it is unknown whether a supersimple ω-categorical theory can have infinite rank. Nevertheless, we show that at least in the group case some results from the superstable case do generalize. Recall [1] that an ω-categorical superstable group is abelian-by-finite. It is easy to construct a supersimple ω-categorical group which is not abelian-by-finite (one can be interpreted in the SU-rank 1 structure consisting of a vector space over a finite field, equipped with a non-degenerate bilinear form). Thus the following is best-possible.

Theorem 1. *An ω-categorical supersimple group is finite-by-abelian-by-finite, and has finite SU-rank.*

The first step in the proof is to appeal to a theorem of Macpherson [10] that an ω-categorical group without the strict order property is nilpotent-by-finite. Then the proof proceeds in two stages. We first show that an ω-categorical group of finite SU-rank is finite-by-abelian-by-finite. For this we need some information about definable subgroups of an ω-categorical group of finite SU-rank. It turns out that as in the superstable case, every definable subgroup is commensurable with one definable over $\mathrm{acl}(\emptyset)$. Then we use the information obtained for ω-categorical groups of finite SU-rank to show that ω-categorical supersimple groups of infinite rank cannot exist.

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Although the ω-categorical simple structures produced by Hrushovski's methods are not modular, they nevertheless satisfy a weaker geometric property known as *CM-triviality*, and at the time of writing, it is unknown whether there exists a (super)simple ω-categorical structure which is not CM-trivial. In the second section of the paper, we show that a supersimple ω-categorical structure which is CM-trivial is of finite SU-rank. More precisely, we show that a finitely based CM-trivial regular type in a simple ω-categorical theory is non-orthogonal to a type of SU-rank 1.

We shall assume familiarity with the basic notions of simplicity and stability theory. For further background on simple theories, the reader could consult the survey [9]. By *conjugate* we shall usually mean model-theoretic rather than group theoretic conjugate, that is, a translate by an automorphism of the ambient model. Definability is always with parameters; we say A-definable if we only allow parameters from A. Recall also that a group G is *finite-by-abelian-by-finite* if it has a normal subgroup N of finite index such that the commutator subgroup N' is finite. Note that the product of two such normal subgroups of G is also of finite index and has finite commutator subgroup, thus if there exists such a subgroup N, we may take its index to be as small as possible, in which case it is characteristic in G. We say that G is *finite-by-abelian* if $N = G$, and *abelian-by-finite* if $N' = \{1\}$.

Finally we say a few words about credits. Evans noticed that Poizat's analysis of definable subgroups of an ω-categorical group of finite Morley rank in [12] could be adapted to the finite SU-rank context (to give Theorem 5 in the abelian case), and (following an argument from [5]) proved Theorem 6 for groups of nilpotency class 2. The general case of Theorem 6 is due to Wagner, as is the result on finiteness of SU-rank for ω-categorical supersimple groups. The observations in the second section are mainly due to Evans.

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1. Groups

Recall that two subgroups F_1 and F_2 of a group G are *commensurable* if their intersection has finite index in both F_1 and in F_2; the *commensurability class* of a subgroup F of G is the set of all subgroups of G commensurable with F. One of our main tools is the following Theorem due to Schlichting [14], and rediscovered by Bergman and Lenstra [2]:

Fact 2. *Let G be a group and $\mathfrak{F}$ a family of subgroups such that there is a finite bound on the indices $|F_1 : F_1 \cap F_2|$ for $F_1, F_2 \in \mathfrak{F}$. Then there is a subgroup F^c, invariant under all automorphisms of G which stabilize $\mathfrak{F}$ setwise, and commensurable with all $F \in \mathfrak{F}$.*

In fact, it is easy to see that if all the elements of $\mathfrak{F}$ are definable, then F^c is definable as a finite extension of a finite intersection of groups in $\mathfrak{F}$. (In our case definability also follows immediately from ω-categoricity.)

Corollary 3. *Let F be a definable subgroup of an ω-categorical group G. Then there is a definable subgroup F^c of G, commensurable with F, such that F^c is stabilized under any automorphism of G stabilizing the commensurability class of F.*

Proof: By ω-categoricity, there is a bound on the indices $|F : F \cap F'|$, for F' a commensurable conjugate of F. Now apply Fact 2. $\square$

In fact, the same conclusion holds for any relatively definable subgroup of a group type-definable in a simple theory, see [15, Corollary 1.8]. Clearly, F^c is defined over the same parameters as the original F.

We shall first look at abelian groups of finite SU-rank. The following definition differs from the stable case.

Definition 1. A definable group is *minimal* if it is infinite and every definable subgroup of infinite index is finite.

Note that in a supersimple theory every definable group contains a definable minimal subgroup (any infinite definable subgroup of minimal possible SU-rank will do).

Definition 2. Let A and B be definable abelian minimal groups. A *virtual isogeny* f between A and B is a definable isomorphism $f : D/K \to I/C$, where D has finite index in A, I has finite index in B, and K and C are both finite. Two virtual isogenies f_1 and f_2 are *equivalent* if the derived maps from $D_1 \cap D_2$ to $(I_1 + I_2)/(C_1 + C_2)$ agree on a subgroup of finite index in A.

Equivalence of virtual isogenies is a congruence with respect to addition and composition (whenever composition makes sense). Note that if f is defined over some parameters $\bar{a}$, say $f = f_{\bar{a}}$, then the equivalence relation on $\mathrm{tp}(\bar{a})$ given by equivalence of $f_{\bar{x}}$ and $f_{\bar{y}}$ is definable by ω-categoricity. Thus the equivalence class $[f]$ of f is an imaginary element.

Lemma 4. *Let A and B be $\emptyset$-definable abelian minimal groups in an ω-categorical theory. Then the equivalence class of every virtual isogeny from A to B is algebraic over $\emptyset$.*

Proof: It is standard that in a minimal group A, the family of virtual isogenies from A to A modulo equivalence, with addition and composition as operations, forms the set of invertible elements of a division ring R. If A is ω-categorical, R must be locally finite, whence a locally finite field, and every element of R is algebraic over $\emptyset$. Furthermore, if f and f' are two virtual isogenies between A and B, then f^{-1} is a virtual isogeny between B and A, and $f^{-1}f'$ is a virtual isogeny of A. Clearly f' is algebraic over $f^{-1}f'$ and f; as $[f^{-1}f']$ is algebraic over $\emptyset$, we see that $[f']$ is algebraic over $[f]$.

As f and f' were arbitrary virtual isogenies between A and B, we see that $[f''] \in \mathrm{acl}([f])$ for any $[f''] \models \mathrm{tp}([f'])$. Hence $\mathrm{tp}([f'])$ has only boundedly, whence finitely, many realizations, and $[f'] \in \mathrm{acl}(\emptyset)$. This proves the lemma. $\square$

Theorem 5. *Let G be an ω-categorical finite-by-abelian-by-finite group of finite SU-rank. Then any definable subgroup of G is commensurable with one definable over* $\mathrm{acl}(\emptyset)$.

Proof: Suppose N is a normal subgroup of finite index in G and with finite commutator subgroup N'. By the remarks in the introduction we may assume that N is characteristic and hence $\emptyset$-definable, as is N'. Replacing a definable subgroup H of G by the definable subgroup $(H \cap N)N'/N'$ of N/N', we can reduce the problem to the case when G is abelian.

For abelian G, consider first a minimal subgroup A of G. By the finiteness of rank, there are only finitely many conjugates of A, say $(A_i : i < n)$, such that every conjugate of A intersects $A^0 := \sum_{i<n} A_i$ in a subgroup of finite index. We may choose the A_i almost linearly independent, i.e. $A_i \cap \sum_{j \neq i} A_j$ is finite for all $i < n$.

Now consider another conjugate A' of A. Since $A' \cap A^0$ is infinite by maximality of n, there must be some $i = i(A') < n$ such that $A_i \cap (A' + \sum_{k \neq i} A_k)$ is infinite; as A and therefore A_i are both minimal, $|A_i : A_i \cap (A' + \sum_{k \neq i} A_k)|$ is finite. For every $j \neq i$ with $A_j \cap (A' + \sum_{k \neq j} A_k)$ infinite we define a virtual isogeny $r(A', j)$ from A_i to A_j via: $r(A', j)(x) := \{y \in A_j : x - y \in A' + \sum_{k \neq i,j} A_k\} = A_j \cap (x + A' + \sum_{k \neq i,j} A_k)$ (it is easy to check from minimality that this is indeed a virtual isogeny). If $A_j \cap (A' + \sum_{k \neq j} A_k)$ is finite, we put $r(A', j) = 0$. Suppose now A'' is such that $i(A'') = i(A')$, and $r(A', j)$ and $r(A'', j)$ are equivalent virtual isogenies for all $j \neq i(A')$ with $r(A', j) \neq 0$ or $r(A'', j) \neq 0$. One can check that then A' and A'' must be commensurable. It follows that there are only finitely many commensurability classes among the conjugates of A.

Consider the group A^c given by Corollary 3. It has only finitely many conjugates, corresponding to the commensurability classes of conjugates of A, and must therefore be definable over $\mathrm{acl}(\emptyset)$. This proves the assertion for minimal groups.

If $H \leq G$ is not minimal, then by supersimplicity it contains a minimal subgroup A which is commensurable with some $\mathrm{acl}(\emptyset)$-definable A^c. But HA^c/A^c is a subgroup of G/A^c of smaller SU-rank; by induction it is commensurable with an $\mathrm{acl}(\emptyset)$-definable group H_c/A^c, whose preimage H^c in G is as required. $\square$

Theorem 6. *An ω-categorical group of finite SU-rank is finite-by-abelian-by-finite.*

Proof: Suppose not, and let G be a counterexample. By ω-categoricity, we may assume that G has no proper $\emptyset$-definable subgroups of finite index.

Claim: *We may assume that $G'/(G' \cap Z(G))$ is finite.*

Proof of Claim: By simplicity, G does not have the strict order property; by [10, Theorem 1.2] it is nilpotent. By ω-categoricity, the subgroups $\gamma_m(G)$ in the lower central series are $\emptyset$-definable for all $m \geq 1$, and there is some minimal $n > 2$ such that $G'/\gamma_n(G)$ is infinite. (Note that G/G' must be infinite since G has no characteristic subgroup of finite index, and G' is infinite since G is not finite-by-abelian.)

We claim that $G/\gamma_n(G)$ is not finite-by-abelian-by-finite. So suppose otherwise, and let N be a normal subgroup of G of finite index containing $\gamma_n(G)$ such that $N'\gamma_n(G)/\gamma_n(G)$ is finite. By the remarks in the introduction we may assume that N is characteristic in G. But since G has no proper characteristic subgroup of finite index, $N = G$, and $G'/\gamma_n(G)$ is finite, a contradiction.

$G'/\gamma_{n-1}(G)$ is finite by minimality of n, so $\gamma_{n-1}(G)/\gamma_n(G)$ is infinite; since $\gamma_{n-1}(G)/\gamma_n(G)$ is contained in $Z(G/\gamma_n(G))$, we may replace G by $G/\gamma_n(G)$ and are done. $\square$

Since $G'/(G' \cap Z(G))$ is finite, $G_g := C_{G/Z(G)}(gZ(G))$ has finite index in $G/Z(G)$ for every $g \in G$. Put $H_g := \{(hZ(G), [h,g]) : hZ(G) \in G_g\}$, a subgroup of $G/Z(G) \times Z(G)$ (note that the commutator $[h,g]$ does not depend on the choice of h in $hZ(G)$). Since $G/Z(G) \times Z(G)$ is finite-by-abelian, H_g is commensurable with an $\mathrm{acl}(\emptyset)$-definable group by Theorem 5; it follows by compactness that there are only finitely many commensurability classes in the family $\{H_g : g \in G\}$. If π_1 denotes the projection to the first co-ordinate, then $[h,g] = [h,g']$ for any $hZ(G) \in \pi_1(H_g \cap H_{g'})$, whence $[h, g'g^{-1}] = 1$. However, we may choose g and g' to be two independent generic elements such that H_g and $H_{g'}$ are commensurable. Then $\pi_1(H_g \cap H_{g'})$ is a subgroup of finite index in $G/Z(G)$, and $g^{-1}g'$ is a generic element of G with $|G : C_G(g^{-1}g')|$ finite.

The set of all $g \in G$ such that $C_G(g)$ has finite index in G is a subgroup of G, which is $\emptyset$-definable by ω-categoricity; since it contains a generic element, it has finite index in G. So at this stage we could conclude by appealing to a result of B. H. Neumann that a group in which all conjugacy classes are boundedly finite has finite derived group. However, we can give a short, direct proof using supersimplicity.

As G has no $\emptyset$-definable subgroups of finite index by assumption, $C_G(g)$ has finite index in G for all $g \in G$. But then $[g, G]$ is finite for all $g \in G$. If $g \mathop{\smile\!\!\!\!\!\mid}\, g'$, then $[g, g'] \in \mathrm{acl}(g) \cap \mathrm{acl}(g') = \mathrm{acl}(\emptyset)$. Since every element g' of G can be written as $g' = g_1 g_2$ with $g \mathop{\smile\!\!\!\!\!\mid}\, g_1$ and $g \mathop{\smile\!\!\!\!\!\mid}\, g_2$, we obtain $[g, g'] = [g, g_1 g_2] = [g, g_2][g, g_1][g, g_1, g_2]$. Then $[g, g_1] \in \mathrm{acl}(\emptyset)$ is independent of g_2, and $[g, g_1, g_2] \in \mathrm{acl}(\emptyset)$ as well. Thus there are only finitely many commutators and G' is finite, a contradiction. $\square$

Theorem 7. *An ω-categorical supersimple group has finite SU-rank.*

Proof: Let G be an ω-categorical supersimple group of infinite rank, and let p be a 1-type of SU-rank ω over a finite set, which we may assume to be $\emptyset$. By ω-categoricity, p is isolated, say by some formula $\varphi(x)$. As $|S_1(\emptyset)|$ is finite, there is a bound $n < \omega$ for the SU-rank of a 1-type over $\emptyset$ of finite SU-rank.

Take a forking extension q of p of SU-rank greater than n, and a Morley sequence $(a_i : i < \omega)$ in q. Let $\bar{a} = (a_i : i < k)$ be a maximal independent initial segement of $(a_i : i < \omega)$. Then $a_k \not\mathop{\smile\!\!\!\!\!\mid}\, \bar{a}$, so

$$\omega = SU(p) = SU(a_k) > SU(a_k/\bar{a}) \geq SU(q) > n.$$

Let $\bar{p}$ be the type of $\bar{a}$, and let m be the maximal SU-rank of an extension of p of finite SU-rank over $2k$ elements; clearly $m > n$.

Claim: *There is a set X of parameters and an extension $\bar{q} \in S_k(X)$ of $\bar{p}$ of finite SU-rank, such that the SU-rank of every co-ordinate of a realization of $\bar{q}$ over $X \cup ($the other co-ordinates$)$ is greater than m.*

Proof of Claim: $SU(a_i) = SU(p) = \omega$ for all $i < k$, and $\{a_i : i < k\}$ is an independent set. For every $i < k$ let A_i be a set of parameters with $\omega > SU(a_i/A_i) > m$ and choose inductively $A_i \mathop{\smile\!\!\!\!\!\mid}_{a_i} \bar{a} \cup \{A_j : j < i\}$. Now if $\{x_i : i < k\}$ is an independent set and $x_0' \mathop{\smile\!\!\!\!\!\mid}_{x_0} \{x_i : i > 0\}$, then $x_0x_0' \mathop{\smile\!\!\!\!\!\mid} \{x_i : i > 0\}$ and $\{x_0x_0', x_1, x_2, \ldots, x_{k-1}\}$ is an independent set. It follows inductively (putting $x_0 = a_i$ and $x_0' = A_i$) that $\{A_0a_0, \ldots, A_ia_i, a_{i+1}, \ldots, a_{k-1}\}$ is an independent set for all $i < k$; in particular $\{A_ia_i : i < k\}$ is independent. Hence $A_ia_i \mathop{\smile\!\!\!\!\!\mid} \{A_ja_j : j \neq i\}$ for all $i < k$, whence $a_i \mathop{\smile\!\!\!\!\!\mid}_{A_i} \{A_ja_j : j \neq i\}$, and $\mathrm{tp}(\bar{a}/A_i : i < k)$ is as required. □

Consider a Morley sequence $I := (\bar{a}^s : s < \omega)$ in $\bar{q}$, and consider the type of $\bar{a}^0\bar{a}^1$ not over the domain of $\bar{q}$, but over $\emptyset$. For every $i < k$

$$SU(a_i^1/\bar{a}^0, a_j^1 : j \neq i) \geq SU(a_i^1/X, \bar{a}^0, \bar{a}_j^1 : j \neq i) = SU(a_i^1/X, a_j^1 : j \neq i) > m,$$

so by assumption on m we get $SU(a_i^1/\bar{a}_0, a_j^1 : j \neq i) = \omega$. Similarly $SU(a_i^0/\bar{a}^1, a_j^0 : j \neq i) = \omega$, and $\bar{a}^0\bar{a}^1$ is a tuple of $2k$ independent realizations of p. In particular $\bar{a}^0 \mathop{\smile\!\!\!\!\!\mid} \bar{a}^1$. Choose a finite initial segment $A = (\bar{a}^i : i < i_0)$ of I such that $SU(\bar{a}^{i_0}/A) = SU(\bar{q}) < \omega$. Let $H := \{g \in G : SU(g/\bar{a}^{i_0}) < \omega\}$ and $K := \{g \in G : SU(g/A) < \omega\}$; note that $H \subseteq K$. By ω-categoricity, H is an $\bar{a}^{i_0}$-definable subgroup and K is an A-definable subgroup of G, with $H \leq K$ and $SU(K) < \omega$. Note that $SU(H) \geq SU(a_k/\bar{a}) > n$, since $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{a}^{i_0})$.

As I is a Morley sequence, for every $j \geq i_0$ there is an automorphism σ_j which fixes A and maps $\bar{a}^{i_0}$ to $\bar{a}^j$. So σ_j stabilizes K setwise. By Theorem 5 there are only finitely many commensurability classes among K-conjugates of H, so there is $i_0 \leq i < j$ such that $\sigma_i(H)$ and $\sigma_j(H)$ are commensurable, whence $\sigma_i(H^c) = \sigma_j(H^c)$.

Now H^c is definable over $\bar{a}^{i_0}$, so $\sigma_i(H^c)$ is definable over $\mathrm{dcl}(\bar{a}^i) \cap \mathrm{dcl}(\bar{a}^j)$. But $\bar{a}^0 \mathop{\smile\!\!\!\!\!\mid} \bar{a}^1$ implies $\bar{a}^i \mathop{\smile\!\!\!\!\!\mid} \bar{a}^j$ by indiscernibility, and $\mathrm{dcl}(\bar{a}^i) \cap \mathrm{dcl}(\bar{a}^j) \subseteq \mathrm{acl}(\emptyset)$. Thus $\sigma_i(H^c)$ is a group of finite SU-rank definable over $\mathrm{acl}(\emptyset)$. If $r \in S_1(\mathrm{acl}(\emptyset))$ is a generic type for $\sigma_i(H^c)$, then $SU(r) = SU(\sigma_i(H^c)) = SU(H^c) = SU(H) > n$. This contradicts the maximal choice of n, as r does not fork over $\emptyset$. □

2. CM-triviality

We work throughout in a large, saturated model $\mathbb{C}$ of a simple, ω-categorical theory T; acl will denote algebraic closure in $\mathbb{C}^{eq}$. By a theorem of Kim [8], ω-categoricity implies that the independence theorem holds over algebraically closed subsets, and canonical bases exist in $\mathbb{C}^{eq}$. If $a \in \mathbb{C}^{eq}$ and A is a small subset, we denote by $\mathbf{cb}(a/A)$ the algebraic closure of the canonical base of $\mathrm{tp}(a/\mathrm{acl}(A))$.

Definition 3. The simple ω-categorical theory T is *CM-trivial* if whenever $\mathrm{acl}(cA) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$, then $\mathbf{cb}(c/A) \subseteq \mathbf{cb}(c/B)$.

This notion was introduced for stable theories by Hrushovski in [6], but transfers immediately to our context. (If T is not ω-categorical, we have to replace the *algebraic* by the *bounded* closure, see [4].) It appears that a significant variation on the constructions in [7] would be required in order to produce ω-categorical simple theories which are not CM-trivial. The notion makes sense localised to a type p (over $\emptyset$). We say that p is CM-trivial if the above holds where everything is restricted to lie in p^{eq}.

Theorem 8. *If p is a CM-trivial regular type of T defined over $\emptyset$, then p is non-orthogonal to a type of SU-rank 1.*

Although the details require some care, the idea of the proof is very straightforward. CM-triviality forbids the interpretation of a 'pseudoplanar space': a set of points, lines and planes essentially behaving like a sufficiently rich collection of points, curves and surfaces (compare [6] and [11], for example). But forking gives a pregeometry on the set of realisations of p, and by ω-categoricity the dependence relations of the corresponding geometry are interpretable. If p is orthogonal to all types of SU-rank 1, then the geometry is not locally finite, and its points, lines and planes (over some finite base) contradict CM-triviality.

From now on, let p be a regular type defined over $\emptyset$. Let C be the set of realisations of p in $\mathbb{C}^{eq}$. For any $x \in C$ and $X \subseteq C$ write $x \in \mathrm{cl}(X)$ precisely when $x \not\mathrel{\unicode{x2AEB}} X$. As p is regular, this is a closure operation, and by ω-categoricity, if X is finite, it is a definable relation. Let C_1 be the set of closures of elements of C and cl_1 the closure operation induced on this by cl. So (C_1, cl_1) is a geometry. Note that C_1 is the locus of a complete (imaginary) type p_1 over $\emptyset$, and $p \not\perp p_1$. Furthermore, since realizations of p_1 are definable over realizations of p, we have $p_1^{eq} \subseteq p^{eq}$. Hence CM-triviality of p implies CM-triviality of p_1.

Lemma 9. *For $x \in C_1$ and $X \subseteq C_1$ we have $x \in \mathrm{cl}_1(X)$ if and only if $x \not\mathrel{\unicode{x2AEB}} X$.* □

Lemma 10. *If the closure operation cl_1 over every finite parameter set is locally finite, then $SU(p_1) \leq 1$.*

Proof: Suppose the geometry is locally finite over all finite sets, $x \models p_1$ and A is a set of parameters with $x \not\mathrel{\unicode{x2AEB}} A$. We claim that $x \in \mathrm{acl}(A)$. This will clearly prove the assertion.

So suppose $x \notin \mathrm{acl}(A)$, and let $(x_i : i < \alpha)$ be a Morley sequence in $\mathrm{tp}(x/A)$. By local finiteness and Lemma 9, we may replace the sequence by an infinite subsequence of the same cardinality and may assume that $(x_i : i < \alpha)$ is independent over $\emptyset$. But $x_i \not\mathrel{\unicode{x2AEB}} A$, which is impossible for α large enough. □

It follows that if p is orthogonal to all types of SU-rank 1, then over some finite set the associated geometry is not locally finite. Localizing further, we may assume that the closure of some pair of points in (C_1, cl_1) contains infinitely many points. We assume (for ease of notation) that $C = C_1$ and work entirely in p^{eq}.

We use the following geometric terminology. The closure of a pair of points is definable over the points (by ω-categoricity), and so is interdefinable with an element of p^{eq}. We refer to this as a line of the geometry. Similarly a plane is (a canonical parameter for) the closure of three independent points of C_1. Incidence between points, lines and planes means containment of the closed subsets of the geometry. Thus we have just reduced to the case where some line is incident with infinitely many points. The theorem is proved in a series of claims.

Claim: *There exist $a, b, c \in p^{eq}$ representing an incident point, line, plane such that*

$$b \notin \mathrm{acl}(ac),\ a \notin \mathrm{acl}(bc) \text{ and } c \mathop{\unicode{x2AEB}}_{b} a.$$

Proof of Claim: First, by assumption, there exists an infinite line b. Let a^0 be a point on it with $a^0 \notin \mathrm{acl}(b)$. Let k be the maximum number of points in the algebraic closure of any two lines, and let $a^0, \dots, a^k$ be part of a Morley sequence of $\mathrm{tp}(a^0/\operatorname{acl}(b))$. Let b' realise a non-forking extension of $\mathrm{tp}(b/\operatorname{acl}(a^0))$ to $\operatorname{acl}(a^0, \dots, a^k)$. Note that $b \notin \operatorname{acl}(a^0)$ so b' is a line through a^0 distinct from b, and we may assume that $a^1 \notin \operatorname{acl}(bb')$ by the choice of k. Let c be the unique plane containing b', a^1. As $c \in \operatorname{acl}(bb')$ we therefore have $a^1 \notin \operatorname{acl}(bc)$.

Next, $b' \mathop{\smile\!\!\!\!\!\mid}_{a^0} a^1$ and $a^1 \mathop{\smile\!\!\!\!\!\mid} a^0$, so $b'a^0 \mathop{\smile\!\!\!\!\!\mid} a^1$, whence $a^0 \mathop{\smile\!\!\!\!\!\mid}_{b'} a^1$. As $a^0 \notin \operatorname{acl}(b')$, we get $a^0 \notin \operatorname{acl}(a^1 b')$; so $a^0 \in \operatorname{acl}(bb')$ implies $b \notin \operatorname{acl}(a^1 b')$, whence $b \notin \operatorname{acl}(a^1 c)$.

Finally let a_2 be a point of b' not equal to a^0. Then by definition of the closure, $a_2 \mathop{\smile\!\!\!\!\!\mid} a^1 a^0$, so $a_2 \mathop{\smile\!\!\!\!\!\mid} a^1 b$. Thus $a_2 \mathop{\smile\!\!\!\!\!\mid}_b a^1$, whence, as $c \in \operatorname{acl}(a_2 b)$, we get $c \mathop{\smile\!\!\!\!\!\mid}_b a^1$. Taking $a = a^1$ gives what we want. $\square$

Claim: *If $X = \mathbf{cb}(a/\operatorname{acl}(b))$, then $b \in X$.*

Proof of Claim: This is standard. Suppose not. Then there exists $b' \neq b$ with $\mathrm{tp}(b/X) = \mathrm{tp}(b'/X)$ and $b \mathop{\smile\!\!\!\!\!\mid}_X b'$. Let $r = \mathrm{tp}(a/\operatorname{acl}(b))$ and r' a translate of this (over X) to $\operatorname{acl}(b')$. These have the same restrictions to X and neither forks over X. So by the independence theorem over X, there exists a' realising $r \cup r'$ whose type over $\operatorname{acl}(bb')$ does not fork over X. But a' is a point on both lines b and b', and $a' \in \operatorname{acl}(bb')$. Thus $a' \in X$, so $a \in X \subseteq \operatorname{acl}(b)$, a contradiction. $\square$

Claim: *Let d be a canonical parameter for $\operatorname{acl}(ac) \cap \operatorname{acl}(bc)$ (intersected with p^{eq}). Then c is the unique plane in $\operatorname{acl}(d)$; $\operatorname{acl}(ad) \cap \operatorname{acl}(bd) = \operatorname{acl}(d)$; and $\mathbf{cb}(a/\operatorname{acl}(bd)) = \operatorname{acl}(b)$.*

Proof of Claim: It is easy to see that c is the unique plane in $\operatorname{acl}(bc)$, so the first statement follows. The second is trivial, and the third follows from the fact that $a \mathop{\smile\!\!\!\!\!\mid}_b c$ and $\operatorname{acl}(bd) = \operatorname{acl}(bc)$. $\square$

Claim: *There is some element of p^{eq} which lies in $\mathbf{cb}(a/d)$, but not in $\operatorname{acl}(b)$.*

Proof of Claim: Let $Y = \mathbf{cb}(a/d)$. If $c \in Y$, we are done as $c \notin \operatorname{acl}(b)$. Hence we may assume $c \notin Y$, so $d \notin Y$. Let $d_1 = d, d_2, d_3$ be the start of a Morley sequence of $\mathrm{tp}(d/Y)$. By the independence theorem over Y twice, there exists a point a' with $\mathrm{tp}(a'/\operatorname{acl}(d_1 d_2 d_3 Y))$ non-forking over Y and $\mathrm{tp}(a' d_i) = \mathrm{tp}(ad)$. Let c_i be the unique plane in $\operatorname{acl}(d_i)$. Then (as a is on the plane $c = c_1$), a' lies in all three planes c_1, c_2 and c_3. Moreover, a' is not algebraic over $c_1 c_2 c_3$: if it were, then $a \in Y$, so $a \in \operatorname{acl}(d) \subseteq \operatorname{acl}(bc)$. Thus the three planes have infinitely many points in common, so intersect in a line b_1. As d_1, d_2, d_3 are independent over Y and b_1 lies in the algebraic closure of any pair of them, we get that $b_1 \in Y$. As $b \notin \operatorname{acl}(d)$ (because $b \notin \operatorname{acl}(ac)$), we have $b_1 \neq b$. Thus $Y \not\subseteq \operatorname{acl}(b)$, as otherwise we have that $\operatorname{acl}(b)$ contains two distinct lines of the plane c, and so contains c, a contradiction. $\square$

So now we have $\operatorname{acl}(ad) \cap \operatorname{acl}(bd) = \operatorname{acl}(d)$ and $\mathbf{cb}(a/d) \not\subseteq \mathbf{cb}(a/bd)$. All of this is restricted to p^{eq} and so p is not CM-trivial. $\square$

Corollary 11. *Let T be an ω-categorical supersimple CM-trivial theory. Then T has finite SU-rank.*

Proof: If there were a type of infinite SU-rank, we could find a finitely based type p of SU-rank ω. Then p is regular, CM-trivial, and orthogonal to all types of SU-rank 1; adding the parameters of its domain to the language, we contradict Theorem 8. □

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