Suborbits in Infinite Primitive Permutation Groups

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Abstract: For every infinite cardinal κ, we construct a primitive permutation group which has a finite suborbit paired with a suborbit of size κ. This answers a question of Peter M. Neumann.

1 Introduction

In this paper we prove the following (see Corollary 2.10):

Theorem *For every infinite cardinal κ, there is a primitive permutation group which has a finite suborbit paired with a suborbit of size κ.*

The existence of such groups answers a question raised by Peter M. Neumann ([8]), and completes a rough taxonomy of infinite, primitive permutation groups which we shall outline below. It may also retrieve some 'lost' examples of Ehud Hrushovski (see [8]).

As is easy to see, the theorem is equivalent to constructing a directed graph (we shall write *digraph*) with finite out-valency and in-valency κ, whose automorphism group is primitive on vertices and transitive on directed edges. The construction we give of such graphs is very similar to constructions used throughout model theory: we construct a digraph by amalgamating certain smaller digraphs in a systematic way, so that partial automorphisms of the large digraph can be extended to automorphisms. The reader unfamiliar with these types of constructions could look at [2] or [4] for the prototype (due to R. Fraïssé [5]), and also [7] for more recent variations due to Hrushovski. Our exposition is, however, reasonably self-contained.

In the remainder of the Introduction we give some background material about infinite permutation groups (although this is not necessary for the constructions in Section 2). The unreferenced material tends to be considered as folk-lore, but it might more properly be attributed to unpublished lectures of Peter M. Neumann from the 1980's.

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1.1 Suborbits and Orbitals

We shall assume a basic familiarity with permutation groups, all of which can be found in [2], for example. When we say that (G, Ω) is a permutation group, we mean that G is a subgroup of $\mathrm{Sym}(\Omega)$, the full symmetric group on Ω. Assume henceforth that G is transitive on Ω, and let α be an element of Ω. Let $G_\alpha = \{g \in G : g\alpha = \alpha\}$. Then the G_α-orbits on Ω are called the *suborbits*. The singleton $\{\alpha\}$ is the *trivial* suborbit. The G-orbits on Ω^2 are called the *orbitals* and $\{(\omega, \omega) : \omega \in \Omega\}$ is the *trivial* orbital. If $\Delta \subseteq \Omega^2$ is a non-trivial orbital, then we may consider the corresponding *orbital digraph*. This is a directed graph with vertex set Ω and edge set Δ. It is clear that G acts as a group of automorphisms of this, and is transitive on vertices and on directed edges. If we ignore the orientation on the edges we obtain the *orbital graph* with vertex set Ω and edge set $\{\{\beta, \gamma\} : (\beta, \gamma) \in \Delta\}$.

There is a natural correspondence between the suborbits and the orbitals. Suppose Γ is a suborbit and $\beta \in \Gamma$. Then the corresponding orbital is the G-orbit on Ω^2 which contains (α, β). It is easy to check that this is well defined, and moreover, if Δ is any orbital then the corresponding suborbit is the one containing γ, where γ is any element of Ω such that $(\alpha, \gamma) \in \Delta$.

If Δ is an orbital then so is $\Delta^* = \{(\gamma, \beta) : (\beta, \gamma) \in \Delta\}$. This is called the *paired* orbital. This pairing of the orbitals induces a pairing on the suborbits. If Γ is a suborbit and $\beta \in \Gamma$, let $g \in G$ be such that $g\beta = \alpha$, and let $\gamma = g\alpha$. Then Γ^*, the paired suborbit, is the suborbit which contains γ.

The orbital digraphs provide a rather more enlightening way of saying this. Suppose Δ is an orbital with corresponding suborbit Γ. Then Γ is the set of out-vertices for α in the orbital digraph with edge set Δ. Moreover, Γ^* is the set of in-vertices for α in this digraph. Hence we have the connection between the theorem and the construction of digraphs, as stated in the opening paragraphs.

We conclude this subsection with a very useful criterion for primitivity, due to D. G. Higman ([6]).

Lemma 1.1 *The transitive permutation group (G, Ω) is primitive if and only if all its non-trivial orbital graphs are connected.* $\Box$

1.2 A Taxonomy

Throughout this subsection Ω is an infinite set and G is a primitive permutation group on Ω. We claim that such groups divide into three cases:

Case A: There is no (non-trivial) finite suborbit;

Case B: There is a (non-trivial) finite suborbit Γ and Γ^* is finite;

Case C: There is a finite suborbit Γ with Γ^* infinite.

To see that Cases B and C do not overlap, we have:

Lemma 1.2 *In Case B all suborbits are finite, and Ω is countable.*

Proof. Let Δ be the orbital corresponding to Γ and consider the orbital graph from this (so this has edge set $\Delta \cup \Delta^*$). This is locally finite and connected. So Ω is countable, and any element of Ω lies in a finite G_α-orbit (as there are only finitely many points at any given distance from α in the graph). $\Box$

It is reasonable to view Case A as the 'typical' case. Corollary 2.10 says that Case C is non-empty. In Case B the orbital digraphs are locally finite, and their automorphism groups are (in a natural way) locally compact groups. There are some highly non-trivial results concerning such permutation groups. For example, the 'Tarski monsters' of Ol'shanskii provide examples of infinite groups satisfying the hypotheses of the following result.

Theorem 1.3 (G. Schlichting [10], G. Bergman and H. Lenstra [1]) *If there is a (finite) bound on the size of the suborbits, then G_α is finite.*

The following is due to C. Praeger ([9]). See also the work of V. Trofimov [11], and [3] (Praeger's paper is cited incorrectly in the latter).

Theorem 1.4 *In Case B, we have $|\Gamma| = |\Gamma^*|$, for all suborbits Γ.*

It should be clear that Lemma 1.2 generalises in various ways. For example, if G (still primitive) has a suborbit of size $\kappa \geq \aleph_0$ paired with a suborbit of size $\lambda \geq \kappa$, then $|\Omega| = \lambda$. Examples with $\kappa < \lambda$ are described in [8].

2 The Construction

Throughout, our digraphs are anti-symmetric, loopless, and have no multiple edges. Suppose we have a digraph with vertex set A and edge set $E \subseteq A^2$. If $B \subseteq A$ then the *subdigraph* on B has vertex set B and edge set $E \cap B^2$. Henceforth, we shall use the same notation for a digraph and its vertex set (thus, the above digraphs would be called simply A and B). If A_1, A_2 are digraphs a function $\alpha : A_1 \to A_2$ is a *digraph embedding* if it is an isomorphism between A_1 and the subdigraph of A_2 on $\alpha(A_1)$.

If $a \in A$ then (as in [3]) $a^{\Rightarrow}$ is the set of vertices which are reachable from a by an outward-directed path (this includes a). We say that A is *finitely generated* (f.g.) if A is the union of a finite number of subsets of this form. If $X \subseteq A$ then $X^{\Rightarrow} = \bigcup\{a^{\Rightarrow} : a \in X\}$. The (outward) directed binary tree is denoted by T. We work with digraphs A satisfying:

C1 for all $a \in A$ we have $a^{\Rightarrow} \cong T$.

Notation 2.1 Suppose $X \subseteq A$. Write $X \leq A$ to mean that if $a \in A$ and $a^{\Rightarrow} \setminus X^{\Rightarrow}$ is finite then $a \in X$. Write $X \leq^+ A$ if additionally $a^{\Rightarrow} \cap X$ is f.g. for all $a \in A$.

Lemma 2.2 *(i) If $X \leq A$ then $X = X^{\Rightarrow}$.*
(ii) If $X \leq A$ and $Y \subseteq A$ is such that $Y^{\Rightarrow} = Y$, then $X \cap Y \leq Y$.
(iii) If $X \leq Y \leq A$, then $X \leq A$.
(iv) If $X \leq^+ Y \leq^+ A$, then $X \leq^+ A$.

Proof. (i)-(iii) are left to the reader. For (iv), let $a \in A$. Then $a^{\Rightarrow} \cap Y = \bigcup_{i \leq n} y_i^{\Rightarrow}$ for some $y_1, \ldots, y_n \in Y$. Then $a^{\Rightarrow} \cap X = \bigcup_{i \leq n}(y_i^{\Rightarrow} \cap X)$, which is finitely generated. Then (iv) follows, by (iii). □

Remarks 2.3 In any digraph A satisfying **C1**, the following two relations on vertices are equivalence relations invariant under all automorphisms of A:

$$a \sim b \Leftrightarrow a^{\Rightarrow} \triangle b^{\Rightarrow} \text{ is finite}$$

$$a \simeq b \Leftrightarrow a^{\Rightarrow} \cap b^{\Rightarrow} \text{ is dense in } a^{\Rightarrow} \text{ and } b^{\Rightarrow}.$$

(Here, $X \subseteq T$ is dense if $(\forall t \in T) t^{\Rightarrow} \cap X \neq \emptyset$.) The conditions below will ensure that these relations are trivial in the digraphs we work with.

Definition 2.4 Let $\mathcal{C}$ be the following category. The objects are digraphs A satisfying **C1** and:

C2 we have $a \leq^+ A$ for all $a \in A$;

C3 for all finite $X \subseteq A$ there exists a finitely generated $Y \leq A$ with $X \subseteq Y$.

The morphisms of $\mathcal{C}$ are digraph embeddings $\phi : A \to B$ with $\phi(A) \leq^+ B$ (we shall refer to these simply as *embeddings*).

Remarks 2.5 (i) By Lemma 2.2 the composition of two morphisms is a morphism. Also, by **C2**, it is easy to see that if $Y \leq A$ is f.g. then $Y \leq^+ A$.
(ii) If $A \leq^+ B$ and $B \in \mathcal{C}$ then $A \in \mathcal{C}$. Indeed, as $A \leq B$ we have $A^{\Rightarrow} = A$, so **C1** holds. For **C2** we have $a^{\Rightarrow} \leq A$ for $a \in A$ (by Lemma 2.2) and $a^{\Rightarrow} \cap b^{\Rightarrow}$ is f.g. for all $b \in B$. Finally, suppose $X \subseteq A$ is finite. Let $Y \leq^+ B$ be f.g. and contain X. Then $Y \cap A \leq A$ and $Y \cap A$ is f.g. (as $A \leq^+ B$). Thus **C3** holds.
(iii) Suppose $A \in \mathcal{C}$. Then for all finite $X \subseteq A$ there is a smallest $Y \leq A$ with $X \subseteq Y$, and moreover this is finitely generated. To see this, note that by **C3** in A, we may assume that A itself is finitely generated. It is then clear that $\{a \in A : a^{\Rightarrow} \setminus X^{\Rightarrow} \text{ is finite }\} \leq A$ and this is finitely generated. Thus, this is equal to Y. We denote this by $\mathrm{cl}_A(X)$. Note that $X^{\Rightarrow}$ is cofinite in this.

Lemma 2.6 (Amalgamation) *Suppose $A, B_1, B_2 \in \mathcal{C}$ and $A \leq^+ B_i$. Let C be the disjoint union of B_1 and B_2 over A. Then $C \in \mathcal{C}$ and $B_i \leq^+ C$.*

Proof. Note that for any $c \in C$, if $c \in B_i$ then $c^{\Rightarrow} \subseteq B_i$. Thus C satisfies **C1**. We show $B_i \leq^+ C$, and then **C2** follows. Note that if $c \in B_2$ then $c^{\Rightarrow} \cap B_1 = c^{\Rightarrow} \cap A$. So $c^{\Rightarrow} \cap B_1$ is f.g., and if it is cofinite in $c^{\Rightarrow}$, then $c \in A$, as $A \leq^+ B_2$. It follows that $B_1 \leq^+ C$, as required.

For **C3** let X be a finite subset of C and $X_i = X \cap B_i$. As $A \leq^+ B_i$ we have that $A_i = X_i^{\Rightarrow} \cap A$ is finitely generated. In the notation of Remark 2.5(iii), let $A^0 = \mathrm{cl}_A(A_1 \cup A_2)$. Let $X^i = \mathrm{cl}_{B_i}(A^0 \cup X_i)$. Note that these are finitely generated.
Claim 1: $X^i \cap A = A^0$.
By Remark 2.5(iii) it is enough to show that if $a \in A$ and $a^{\Rightarrow} \setminus (A^0 \cup X_i^{\Rightarrow})$ is finite, then $a \in A^0$. But this is immediate, as $X_i^{\Rightarrow} \cap A \subseteq A^0$ and $A^0 \leq A$.
Claim 2: $X^1 \cup X^2 \leq C$.
Let $c \in C$ be such that $c^{\Rightarrow} \setminus (X^1 \cup X^2)$ is finite. Without loss of generality, $c \in B_1$. We show $c \in X^1$. Now,

$$c^{\Rightarrow} \cap (X^1 \cup X^2) = (c^{\Rightarrow} \cap X^1) \cup (c^{\Rightarrow} \cap X^2).$$

By Claim 1,

$$c^{\Rightarrow} \cap X^2 = c^{\Rightarrow} \cap A \cap X^2 = c^{\Rightarrow} \cap A^0.$$

Thus $c^{\Rightarrow} \cap (X^1 \cup X^2) = c^{\Rightarrow} \cap X^1$. This is supposed to be cofinite in $c^{\Rightarrow}$, so as $X^1 \leq B_1$ we have $c \in X^1$, as required. This completes the verification of **C3**. □

Corollary 2.7 *The category $\mathcal{C}$ has direct limits and the amalgamation property.*

Proof. This is now very easy. □

Definition 2.8 Let $M \in \mathcal{C}$. A *partial automorphism* of M is an isomorphism $\phi : A \to B$ where $A, B \leq^+ M$. If A (and B) is f.g. then (for want of better terminology) refer to this as a *finite* partial automorphism.

Theorem 2.9 *(a) For every cardinal μ there exists a digraph $M \in \mathcal{C}$ with in-valency at least μ and satisfying the following:*

(1) For every f.g. $A \leq^+ M$ and embedding $\phi : A \to B$ in $\mathcal{C}$ with B f.g., there exists an embedding $\psi : B \to M$ with $\psi \circ \phi = id_A$.

(2) Any finite partial automorphism of M can be extended to an automorphism of M.

(b) With M as in (a), $\mathrm{Aut}(M)$ *is highly arc transitive and primitive on M.*

Corollary 2.10 *For every infinite cardinal κ there exists a primitive permutation group (G, D) which has a finite suborbit paired with a suborbit of size κ.*

Proof of the Theorem. (a) This is fairly routine, but we give a few details. The main point is to prove the following:

Lemma 2.11 *Let $L \in \mathcal{C}$ and suppose $\Phi = (\varphi_i : i \in I)$ is a family of partial automorphisms of L. Then there exists $N \in \mathcal{C}$ with $L \leq^+ N$ and $\bar{\varphi}_i \in \mathrm{Aut}(N)$, for all $i \in I$, with $\varphi_i \subseteq \bar{\varphi}_i$.*

Once we have this, take $M_1 \in \mathcal{C}$ of in-valency at least μ and such that all finitely generated $A \in \mathcal{C}$ embed into M_1. Now construct $M_1 \leq^+ M_2 \leq^+ \cdots$, where M_{i+1} is obtained from M_i by applying the lemma with $L = M_i$ and Φ all partial automorphisms of M_i (it is possible to be more economical). Let M be the direct limit $\bigcup_{i<\omega} M_i$. Then (2) holds, and (1) follows.

To prove the lemma, it is enough to deal with a single partial automorphism φ (for the general case, we build N as a direct limit $N = \bigcup_{j<\beta} N_j$; at stage j we replace a single partial automorphism φ_i by an automorphism of N_{j+1}, and for each $i \in I$ the set of stages j at which φ_i is replaced is unbounded in β). We construct $L = L_0 \leq^+ L_1 \leq^+ L_2 \leq^+ \cdots$ in $\mathcal{C}$ and partial automorphisms θ_i of L_i such that $\theta_0 = \varphi$, $\theta_i \subseteq \theta_{i+1}$, $L_i \subseteq \mathrm{im}(\theta_{i+1}) \cap \mathrm{dom}(\theta_{i+1})$. Letting $N = \bigcup_{i<\omega} L_i$ and $\bar{\varphi} = \bigcup_{i<\omega} \theta_i$ gives what we require.

Suppose L_i and θ_i have been constructed. Let $A = \mathrm{dom}(\theta_i)$ and $B = \mathrm{im}(\theta_i)$. There is $L' \in \mathcal{C}$ with $B \leq^+ L'$ and an isomorphism $\psi : L_i \to L'$ such that $\theta_i(a) = \psi(a)$ for all $a \in A$. Amalgamate L_i and L' over B, using Lemma 2.6, and let L_{i+1} be the resulting structure. Then $\theta_{i+1} = \psi$ is a partial automorphism of L_{i+1} which has L_i in its domain and extends θ_i. We can similarly arrange that L_i is in the image of θ_{i+1} (repeat the process with θ_{i+1}^{-1} and adjust the notation).

(b) Note that the high arc transitivity follows from property (2) in (a) and **C1**,**C2** in M. To show that $\mathrm{Aut}(M)$ is primitive on M, we show that all orbital graphs are connected. This is similar to the proof of ([3], Lemma 2.7), with some simplifications suggested by Peter M. Neumann. We already know that $\mathrm{Aut}(M)$ is transitive on M. To show that it is primitive, we prove that if $a \neq b$ then the orbital graph Σ with vertices M and edges $\{\{ga, gb\} : g \in \mathrm{Aut}(M)\}$ is connected. Note that as M itself is connected, it is enough to show that if $x, y \in M$ are such that (x, y) is an edge of M, then x, y lie in the same connected component of Σ. By vertex transitivity, we may assume that $x = a$. Let $C_1 = \mathrm{cl}_M(a, b)$. Thus $C_1 \leq^+ M$ and C_1 is finitely generated.

Case 1: $a^{\Rightarrow} \cap b^{\Rightarrow} = \emptyset$.

Amalgamate C_1 with a copy C_2 of C_1, identifying $b^{\Rightarrow}$ with $y^{\Rightarrow}$. Let a' be the copy of a in C_2. So we have a digraph with $a'^{\Rightarrow} \cap b^{\Rightarrow} = \emptyset$. Adjoining new vertices we obtain a digraph in which the closure C_3 of a', b is isomorphic to C_1. Let E be the resulting graph. So E is the union of C_1, C_2, C_3, and $C_1 \cap C_3 = b^{\Rightarrow}$, $C_3 \cap C_2 = a'^{\Rightarrow}$ and $C_2 \cap C_1 = y^{\Rightarrow}$. Moreover any edge is contained entirely in some C_i. We claim that $E \in \mathcal{C}$. Once we have this it follows (from properties (1) and (2) of M) that x, y are at distance at most 3 in the orbital graph Σ.

Clearly **C1** holds for E as $e^{\Rightarrow}$ is contained in some C_i, for every $e \in E$. For **C2**, let $e, f \in E$. Then $e^{\Rightarrow} \cup f^{\Rightarrow} \subseteq C_i \cup C_j$ for some i, j, and by the amalgamation lemma this is in $\mathcal{C}$. So $e^{\Rightarrow} \cap f^{\Rightarrow}$ is finitely generated, and if $f^{\Rightarrow} \setminus e^{\Rightarrow}$ is finite, then $f \in e^{\Rightarrow}$. Thus **C2** holds. As E is f.g., there is nothing to prove for **C3**.

Case 2: $b \in a^{\Rightarrow}$.

Let b_0 be the predecessor of b in $a^{\Rightarrow}$ (that is, (b_0, b) is an edge in M), and let $b_1 \in a^{\Rightarrow}$ be its other successor. There is an automorphism of M which fixes a and interchanges b and b_1. Thus b_1 and b are connected in the orbital graph Σ. As $b^{\Rightarrow} \cap b_1^{\Rightarrow} = \emptyset$, Case 1 implies that the orbital graph having $\{b, b_1\}$ as an edge is connected. Thus Σ is connected also.

Case 3: $b^{\Rightarrow} \setminus a^{\Rightarrow}$ and $a^{\Rightarrow} \setminus b^{\Rightarrow}$ are infinite.

Let $x_1, \ldots, x_r$ be a minimal generating set for $a^{\Rightarrow} \cap b^{\Rightarrow}$ (so $x_i^{\Rightarrow} \cap x_j^{\Rightarrow} = \emptyset$ for $i \neq j$), and let z_i be the predecessor of x_i in $a^{\Rightarrow}$ and y_i its other successor in $a^{\Rightarrow}$ (for $i = 1, \ldots, r$). Note that $\{x_1, \ldots, x_r\}$ and $\{y_1, \ldots, y_r\}$ are disjoint. By properties (1) and (2) of M, there is an automorphism g of M fixing a and interchanging each x_i and y_i, and such that $gb^{\Rightarrow} \cap b^{\Rightarrow} = \emptyset$. Now b and gb are connected in Σ, and by Case 1, the orbital graph having $\{b, gb\}$ as an edge is connected. So Σ is connected. $\square$

Proof of the Corollary. Take $\mu \geq \kappa$ and M as in Theorem 2.9. Let $(\Omega_i : i \in I)$ be the (non-trivial) orbital digraphs for $\mathrm{Aut}(M)$ on M. Let D_0 be a subdigraph of M of cardinality κ and in-valency κ, with $D_0^{\Rightarrow} = D_0$. Let $G_0 = \{1\}$. We build D as a union of a chain of subgraphs $D = \bigcup_{i<\omega} D_i$ and construct a chain of subgroups G_i of $\mathrm{Aut}(M)$.

To construct D_{i+1} and G_{i+1} from D_i and G_i, do the following:

(1) Increase D_i to D_i' so that if $x, y \in D_i$ and $(x, y) \in \Omega_j$ then $\Omega_j | D_i'$ (the restriction of Ω_j to D_i')

is connected.
(2) If $(x, y), (x', y') \in (D'_i)^2$ lie in the same $\mathrm{Aut}(M)$-orbit then include $g \in G_{i+1}$ such that $g(x, y) = (x', y')$.
(3) Increase D'_i to D_{i+1} so that $gD_{i+1} = D_{i+1}$ for all $g \in G_{i+1}$.
Note that we can do all this and keep $|D_i|, |G_i| \leq \kappa$ for all $i < \omega$. Let $G^0 = \bigcup_{i<\omega} G_i$. Then $D = \bigcup_{i<\omega} D_i$ is invariant under G^0. Let G be the restriction of G^0 to D. By construction, all orbital graphs of G on D are restrictions of the Ω_j to D, and are connected. Thus G is primitive on D. But also G is transitive on directed edges of D, so D is an orbital digraph. As it has out-valency 2 and in-valency κ, it satisfies our requirements. □

Remark 2.12 The construction can be done more economically in the countable case, and in fact, a similar simplification can be made in the general case if we assume the existence of a strongly inaccessible cardinal at least as large as κ.

Theorem 2.13 *There is a countable $M \in \mathcal{C}$ such that for all f.g. $A, B \in \mathcal{C}$ and embeddings $\alpha : A \to M$, $\phi : A \to B$, there exists an embedding $\psi : B \to M$ with $\psi \circ \phi = \alpha$. Such an M is unique and any finite partial isomorphism of M extends to an automorphism of M.*

It is easy to see that M in the above has infinite in-valency, and by the previous results $\mathrm{Aut}(M)$ is primitive and highly arc transitive. The theorem follows from the amalgamation lemma (but we only need this for f.g. elements of $\mathcal{C}$, and it is rather easier in this case as clause **C3** becomes redundant), and the following, which shows that we only have to amalgamate a countable number of embeddings to achieve the required property of M:

Lemma 2.14 *(1) There are countably many isomorphism types of f.g. digraph in $\mathcal{C}$.*

(2) Given f.g. $A, B \in \mathcal{C}$ there are (up to equivalence) only countably many embeddings ϕ : $A \to B$.

Proof. (1) If $C \in \mathcal{C}$ is f.g. then we can write $C = c_1^{\Rightarrow} \cup \ldots \cup c_n^{\Rightarrow}$ for some natural number n. We show by induction on n that the number of isomorphism types of such n-generator digraphs in $\mathcal{C}$ is countable. Let $C_1 = c_1^{\Rightarrow} \cup \ldots \cup c_{n-1}^{\Rightarrow}$. Then, in the notation of ([3], 2.2 and 2.3), for some finite tuples $\bar{x}, \bar{y}$ of (independent) elements in C_1 and T we have that $C \cong (C_1, \bar{x}) * (T, \bar{y})$. As there are only countably many possibilities for $\bar{x}, \bar{y}$ we have the inductive step.

(2) We call embeddings $\phi : A \to B$ and $\chi : A \to B'$ *equivalent* if there exists an isomorphism $\theta : B \to B'$ with $\theta \circ \phi = \chi$. We prove by induction on n, the number of generators of A, that up to equivalence there are only countably many embeddings $\phi : A \to B$, for fixed B. Write $A = A_1 \cup a_n^{\Rightarrow}$, where A_1 can be generated by $n-1$ elements. Regard $\phi | A_1$ and $\phi(a_n) = b$ as fixed. Then for some natural number k, any automorphism of $b^{\Rightarrow}$ which fixes all vertices at out-distance at most k from b can be extended to an automorphism of B which fixes every element of $\phi(A_1)$ (as B is f.g.). Thus, there are now only finitely many possibilities for ϕ, up to equivalence. □

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