$\aleph_0$-categorical structures with a predimension

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1 Predimensions

Our aim in this paper is to give an axiomatic framework for the non-modular simple $\aleph_0$-categorical structures discovered by Ehud Hrushovski ([4] and [7]). This allows us to verify some of their properties (weak elimination of imaginaries, simplicity, the nature of forking, CM-triviality and stable forking) in a uniform way, and to show that these properties are preserved by 'iterations' of the construction (Section 3). In particular, all the simple $\aleph_0$-categorical structures produced have the following property, which we refer to as *CM-triviality* (in the stable context, it is easily seen to be equivalent to the definition in ([6], Proposition 10), cf. Remarks 2.22):

> Suppose E_1, E_2 are algebraically closed and independent over $E_1 \cap E_2$. Let $D \subseteq E_1 \cup E_2$ be such that $\mathrm{acl}(D) \cap (E_1 \cup E_2) = D$ and let $D_i = D \cap E_i$. Then D_1 and D_2 are independent over $D_1 \cap D_2$.

Here, algebraic closure includes imaginaries, and independence is in the sense of non-forking. The axiomatization we give postulates properties of independence with respect to a *predimension*: a real-vauled submodular function on the type of a finite subset. One of the axioms essentially states that CM-triviality holds with respect to this notion of independence. In the actual constructions, this axiom holds for the following reason. The structure is built up as a sequence of 'free amalgamations' of finite structures. The predimension detects when something is a free amalgam, and the amalgamation procedure is such that a subset of a free amalgam is again a free amalgam.

It would of course be of considerable interest to know whether there are (super)simple $\aleph_0$-categorical structures which are not CM-trivial, and presumably this is related to the question of whether there are non-CM-trivial strongly minimal sets which do not interpret an infinite field (see [2], [10]).

Axiomatizations of these constructions in various contexts and levels of generality have been given before (cf. [1], [12] for example), but the approach we give here is different in several respects.

There is also overlap between this work and the Doctoral Thesis of Massoud Pourmahdian [11], and of course much of what we do here is at least implicit, and much is explicit, in Hrushovski's work. In contrast to [7] and [12] the axiomatization makes no explicit mention of generic structures or free amalgamation. These notions are replaced here by axioms guaranteeing d-free extensions of types, and a version of the independence theorem over finite algebraically closed sets. The axioms (P1-P5) can all be checked on finite subsets of the countable model.

In the remainder of this introduction we summarise the axioms and main results. The second section contains the proof of the main theorems, and in the third section we give a slightly more general version of Hrushovski's construction (showing how it can be 'iterated'). We then show that the stable forking conjecture holds for these structures. The final section contains a few further observations about the structures produced.

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1.1 Summary of the axioms and results

For convenience, we collect together the axioms.

Throughout, M is a countably infinite $\aleph_0$-categorical structure and N a large, highly saturated, highly homogeneous elementary extension of M. The letters A, B will usually be used to denote finite subsets of M or small (that is of cardinality less than the degree of saturation of N) subsets of N; c will be a finite tuple. We denote by $\mathrm{acl}(A)$ the algebraic closure of A in N. If A is a finite subset of M, then by the Ryll-Nardzewski Theorem this is the union of the finite $\mathrm{Aut}(M/A)$-orbits on M, and is finite. We assume that there is an $\mathrm{Aut}(M)$-invariant $\mathbb{R}^{\geq 0}$-valued function d_0 defined on the finite subsets of M. By $\aleph_0$-categoricity we may assume that d_0 is also defined on finite subsets of N.

Axioms (P1), (P2) say what we mean by a *predimension*. The function d which is determined by d_0 is called the corresponding *dimension* function. Axiom (P4) postulates existence of d-non-forking extensions over finite sets, and (P5) is the d-independence theorem over finite algebraically closed sets. Axiom (P3) looks contrived, but is what gives CM-triviality. It is used only in the proof of Lemma 2.11, where it can be replaced by (P3$'$) and (P7) (see Remark 2.12). The point of using (P3) in preference to these, is that it is checkable on finite subsets of the countable model. Axioms (P6) and (P7) follow from (P1-P5). Note also that (P7) holds automatically if the subgroup of $\mathbb{R}$ generated by $\{d(A) : A \subseteq M$ finite $\}$ is discrete.

(P1) For finite subsets A and B of M

$$d_0(A \cup B) \leq d_0(A) + d_0(B) - d_0(A \cap B).$$

In other words, d_0 is *submodular*. If equality holds here, we say that A, B is a *modular pair*. We shall always assume that (P1) holds.

If A, B are finite subsets of M we write $A \leq B$ to mean that $d_0(A) < d_0(B')$ whenever $A \subset B' \subseteq B$. We write $A \leq M$ if $A \leq B$ for all finite subsets B of M which contain A.

(P2$^-$) For all finite $A \subseteq M$ there exists a finite $B \leq M$ containing A.

Note that as M is $\aleph_0$-categorical this is a first-order property (the size of B can be bounded in terms of the size of A). It follows from Lemma 2.1 that for any finite A there is a least $B \leq M$ containing it. Denote this by $\mathrm{cl}(A)$. Let $d(A) = d_0(\mathrm{cl}(A))$ and note that this is equal to $\min(d_0(B) : A \subseteq B)$. It is clear that $\mathrm{cl}(A) \subseteq \mathrm{acl}(A)$. The next axiom is a strengthening of (P2$^-$).

(P2) For all finite A we have $\mathrm{cl}(A) = \mathrm{acl}(A)$.

If c is a (finite) tuple of elements of M let $d(c/A) = d(cA) - d(A)$, and similarly define $d_0(B/A) = d_0(B \cup A) - d_0(A)$. Note that A, B is a modular pair if and only if $d_0(A/B) = d_0(A/A \cap B)$.

(P3) For any finite $A_2 \leq M$ and c there exists $\epsilon > 0$ such that if B is finite with $A_1, A_2 \leq B \leq M$, and $\mathrm{acl}(cA_i) \cap B = A_i$, and

$$d_0(\mathrm{acl}(cA_1)) + d_0(B) - d_0(A_1) - d_0(\mathrm{acl}(cA_1) \cup B) < \epsilon$$

then $\mathrm{acl}(cA_1) \cap \mathrm{acl}(cA_2)$, A_2 is a modular pair.

(P3$'$) Suppose $X, Y \leq M$ is a modular pair of finite sets. If $A \leq X \cup Y$ then $A \cap X$, $A \cap Y$ is a modular pair.

(P4) If $A \leq B \leq M$ are finite and c is a finite tuple of elements of M with $\mathrm{acl}(cA) \cap B = A$, then there exists c_1 with

(i) $\mathrm{tp}(c_1/A) = \mathrm{tp}(c/A)$;

(ii) $d(c_1/B) = d(c_1/A)$;

(iii) if $C \leq B$ is such that $\mathrm{acl}(cC) \cap A = C \cap A$, then $\mathrm{acl}(c_1 C) \cap A = A \cap C$.

(P5) Let $A \leq M$ and $B_1, B_2 \leq M$ be finite and such that $B_1 \cap B_2 = A$ and $d(B_1/B_2) = d(B_1/A)$. Suppose c_1, c_2 are tuples such that $\mathrm{tp}(c_1/A) = \mathrm{tp}(c_2/A)$ and $d(c_i/B_i) = d(c_i/A)$. Then there exists c with $\mathrm{tp}(c/B_i) = \mathrm{tp}(c_i/B_i)$ and $d(c/A) = d(c/B_1 \cup B_2)$.

For (possibly infinite) $S \subseteq N$ we let

$$d(c/S) = \inf(d(c/A) : A \text{ is a finite subset of } S).$$

(P6) Let $A \subseteq B$ be small algebraically closed subsets of N and let c be a finite tuple. Then there exists c_1 such that $\mathrm{tp}(c/A) = \mathrm{tp}(c_1/A)$, $d(c_1/A) = d(c_1/B)$, and $\mathrm{acl}(c_1 A) \cap B = A$.

(P7) If a, b are finite tuples and A is a small subset of N such that $d(a/Ab) = d(a/A)$ and $\mathrm{acl}(aA) \cap \mathrm{acl}(bA) = A$, then for every finite $A_0 \subseteq A$ there exists a finite subset $A_0 \subseteq A_1 \subseteq A$ with $d(a/A_1 b) = d(a/A_1)$.

The main results can be stated as follows. if (P1, P2, P5, P6, P7) hold then $Th(M)$ is simple, and d describes forking (Corollary 2.20). If (P1-P5) hold then (P6) and (P7) hold (Corollary 2.21), and moreover $Th(M)$ is CM-trivial (Corollary 2.15). Weaker versions of (P3) and (P4) suffice to prove weak elimination of imaginaries (Lemma 2.9). The discrete case, leading to supersimple structures, is more straightforward: see Section 2.8.

2 Consequences of the axioms

2.1 Consequences of the axioms for the predimension

Lemma 2.1 *Assume (P1) and (P2$^-$) hold.*
(i) If $A \leq B$ and $X \subseteq B$ then $A \cap X \leq X$.
(ii) If $A \leq B$ and $B \leq C$ then $A \leq C$.
(iii) If $A, B \leq M$ then $d(A \cup B) \leq d(A) + d(B) - d(A \cap B)$.
(iv) For any finite A, B, C, $d_0(A \cup B/C) \leq d_0(A/C) + d_0(B/C) - d_0(A \cap B/C)$.
(v) If $A \leq B \leq M$ then $d(c/A) \geq d(c/B)$.

Proof. These are well-known and, apart from (iii) and (v), are all consequences of (P1).

(i) Let $A \cap X \subset Z \subseteq X$. So $A \subset A \cup Z$ and $A \cap Z = A \cap X$. Then by (P1) we have $d_0(A) + d_0(Z) - d_0(A \cap Z) \geq d_0(A \cup Z)$. As $A \leq B$ we therefore get $d_0(Z) > d_0(A \cap X)$, as required.

(ii) Let $A \subset X \subseteq C$. If $X \subseteq B$ then we already have $d_0(X) > d_0(A)$. On the other hand if $X \not\subseteq B$, then

$$d_0(X) + d_0(B) - d_0(X \cap B) \geq d_0(X \cup B) > d_0(B)$$

so $d_0(X) > d_0(X \cap B) \geq d_0(A)$, as required.

(iii) By (i) and (ii), $A \cap B \leq M$. The inequality now follows as $d(A \cup B) \leq d_0(A \cup B)$.

(iv) $d_0(A \cup B/C) = d_0(A \cup B \cup C) - d_0(C)$. By (P1) this is at most $d_0(A \cup C) + d_0(B \cup C) - d_0((A \cap B) \cup C) - d_0(C)$, and this is $d_0(A/C) + d_0(B/C) - d_0(A \cap B/C)$.

(v) By (iii), and the fact that $\mathrm{cl}(cA) \cap B \geq A$ we have:

$$d(cA) + d(B) \geq d(cB) + d(\mathrm{cl}(cA) \cap B) \geq d(cB) + d(A),$$

as required. □

For $A \subseteq B$, write $A \leq^* B$ if $d_0(A') \geq d_0(A)$ for all $A \subseteq A' \subseteq B$. Replacing the strict inequalities in the above proofs by weak inequalities gives:

Lemma 2.2 *Assume (P1) holds.*
(i) If $A \leq^* B$ and $X \subseteq B$ then $A \cap X \leq^* X$.
(ii) If $A \leq^* B$ and $B \leq^* C$ then $A \leq^* C$. □

We say that $X, Y \leq M$ form a *modular pair* if $d_0(X \cup Y) = d_0(X) + d_0(Y) - d_0(X \cap Y)$.

Lemma 2.3 *Assume (P1) and (P2$^-$) hold. Let c be a tuple of elements of M and $A \leq B \leq M$ be finite sets. Then $d(c/A) = d(c/B)$ if and only if the following three conditions hold:*

(I1) $\mathrm{cl}(cA) \cap B = A$;

(I2) $\mathrm{cl}(cA)$, B *is a modular pair in* M;

(I3) $d(cB) = d_0(\mathrm{cl}(cA) \cup B)$.

In particular, in this case we have $\mathrm{cl}(cA) \cup B \leq^* \mathrm{cl}(cB)$.

Proof. $d(c/A) = d(c/B)$ if and only if $d_0(\mathrm{cl}(cA)) - d_0(A) = d_0(\mathrm{cl}(cB)) - d_0(B)$. But

$$d_0(\mathrm{cl}(cB)) - d_0(B) \leq d_0(\mathrm{cl}(cA) \cup B) - d_0(B)$$

$$\leq d_0(\mathrm{cl}(cA)) - d_0(\mathrm{cl}(cA) \cap B),$$

by submodularity. Moreover as $A \leq \mathrm{cl}(cA) \cap B$ we have $d_0(\mathrm{cl}(cA) \cap B) \geq d_0(A)$, with equality if and only if $A = \mathrm{cl}(cA) \cap B$. Thus $d(c/B) = d(c/A)$ if and only if there is equality at all three places in the above, hence the result. □

2.2 Weak canonical bases

In this section we consider a predimension satisfying (P1), (P2$^-$) and the extra axiom:

(P3$^-$) Suppose $A_1, A_2 \leq B \leq M$ and c is a tuple in M such that $\mathrm{cl}(cA_i) \cap B = A_i$ (for $i = 1, 2$). If $\mathrm{cl}(cA_1)$, B is a modular pair, then so is $\mathrm{cl}(cA_1) \cap \mathrm{cl}(cA_2)$, A_2.

This may seem slightly contrived. Note however that it is implied by the following (take $X = \mathrm{cl}(cA_1)$, $Y = B$ and $A = \mathrm{cl}(cA_2) \cap (X \cup Y) = (\mathrm{cl}(cA_1) \cap \mathrm{cl}(cA_2)) \cup A_2$ in (P3$'$) to get (P3$^-$)).

(P3$'$) Suppose $X, Y \leq M$ is a modular pair, and $A \leq X \cup Y$. Then $A \cap X$, $A \cap Y$ is a modular pair.

If we read 'are independent over their intersection' in place of 'is a modular pair,' then we see that this corresponds to CM-triviality of the predimension. Moreover note that in the above situation $B \cap \mathrm{cl}(cA_1) \cap \mathrm{cl}(cA_2) = A_1 \cap A_2 = A_0$. So $\mathrm{cl}(cA_0)$, A_2 is a modular pair, by the following lemma.

Lemma 2.4 *Suppose (P1) holds and $A, B \leq M$ is a modular pair. If $A \cap B \leq A' \subseteq A$, then A', B is a modular pair.*

Proof. We have

$$d_0(A) + d_0(B) - d_0(A \cap B) = d_0(A \cup B) = d_0((A' \cup B) \cup A) \leq$$

$$d_0(A' \cup B) + d_0(A) - d_0(A').$$

Thus $d_0(A' \cup B) \geq d_0(A') + d_0(B) - d_0(A \cap B) = d_0(A') + d_0(B) - d_0(A' \cap B)$. So by submodularity, we have equality here. $\square$

Lemma 2.5 *Assume (P1, P2$^-$, P3$^-$). Let $A_1, A_2 \leq B \leq M$, and suppose c is such that $d(c/A_1) = d(c/B)$ and $\mathrm{cl}(cA_2) \cap B = A_2$. Let $A_0 = A_1 \cap A_2$. Then $d(c/A_0) = d(c/A_2)$.*

Proof. We show that (I1-3) of Lemma 2.3 hold for (c, A_0, A_2).

For (I1) note that

$$A_1 \cap A_2 \subseteq \mathrm{cl}(c(A_1 \cap A_2)) \cap A_2 \subseteq \mathrm{cl}(cA_1) \cap A_2 \subseteq A_1 \cap A_2,$$

using (I1) for (c, A_1, B).

For (I2) we can use (P3$^-$) and the remark following it (after observing that $\mathrm{cl}(cA_1) \cap B = A_1$, by Lemma 2.3).

For (I3), first note that as (I3) holds in (c, A_1, B), we have $\mathrm{cl}(cA_1) \cup B \leq^* \mathrm{cl}(cB)$. Intersecting this with $\mathrm{cl}(cA_2)$ and applying Lemma 2.2 gives:

$$(\mathrm{cl}(cA_1) \cap \mathrm{cl}(cA_2)) \cup A_2 \leq^* \mathrm{cl}(cA_2).$$

Thus $d_0(A_2 \cup (\mathrm{cl}(cA_1) \cap \mathrm{cl}(cA_2)) = d(cA_2)$. So it remains to show that $\mathrm{cl}(cA_1) \cap \mathrm{cl}(cA_2) = \mathrm{cl}(cA_0)$.

By (P3$^-$), as above we have that A_2 and $\mathrm{cl}(cA_0)$ form a modular pair, and their intersection is A_0. Thus we have

$$d_0(A_2) + d_0(\mathrm{cl}(cA_0)) - d_0(A_0) = d_0(A_2 \cup \mathrm{cl}(cA_0)) \geq d(A_2c)$$

$$= d_0(A_2 \cup (\mathrm{cl}(A_1c) \cap \mathrm{cl}(A_2c))).$$

By (P3$^-$) again, this equals

$$d_0(A_2) + d_0(\mathrm{cl}(cA_1) \cap \mathrm{cl}(cA_2)) - d_0(A_0).$$

So we get $d_0(\mathrm{cl}(cA_0)) \geq d_0(\mathrm{cl}(cA_1) \cap \mathrm{cl}(cA_2))$. But clearly $\mathrm{cl}(cA_0) \leq \mathrm{cl}(cA_1) \cap \mathrm{cl}(cA_2)$ so in fact we have equality (by definition of $\leq$). □

Definition 2.6 Let $B \leq M$ and c be a finite tuple from M. Suppose there exists $A_0 \leq B$ such that for $A \leq B$, $d(c/A) = d(c/B)$ if and only if $A_0 \subseteq A$. We refer to this A_0 as the *weak canonical base* of c over B, and denote it by $\mathbf{cb}(c/B)$. If $\mathbf{cb}(c/B)$ exists for all such c, B, we say that (M, d_0) has weak canonical bases.

Corollary 2.7 *Suppose that (P1, P2$^-$, P3$^-$) hold. Then (M, d_0) has weak canonical bases.* □

2.3 CM-triviality over finite sets

Corollary 2.8 *Suppose M satisfies axioms (P1, P2$^-$, P3$^-$). Let $A \leq B \leq M$ and suppose c is a tuple of elements of M such that* $\mathrm{cl}(cA) \cap B = A$. *Then* $\mathbf{cb}(c/A) \leq \mathbf{cb}(c/B)$.

Proof. Let $B_0 = \mathbf{cb}(c/B)$. Then $d(c/A \cap B_0) = d(c/A)$ by Lemma 2.5. So $B_0 \geq \mathbf{cb}(c/A)$, as required. □

2.4 Weak elimination of imaginaries

We impose the following extra axioms to obtain weak elimination of imaginaries. Strengthening (P4$^-$) will give that M is simple and d describes non-forking.

(P4$^-$) Let $A \leq B \leq M$ and c a tuple of elements of M. Then there exists a tuple c_1 from M such that $\mathrm{tp}(c/A) = \mathrm{tp}(c_1/A)$ and $d(c/A) = d(c_1/B)$.

(P5) Let $A \leq M$ and $B_1, B_2 \leq M$ be such that $B_1 \cap B_2 = A$ and $d(B_1/A \cup B_2) = d(B_1/A)$. Suppose c_1, c_2 are tuples such that $\mathrm{tp}(c_1/A) = \mathrm{tp}(c_2/A)$ and $d(c_i/B_i) = d(c_i/A)$. Then there exists c with $\mathrm{tp}(c/B_i) = \mathrm{tp}(c_i/B_i)$ and $d(c/A) = d(c/B_1 \cup B_2)$.

Obviously (P4$^-$) corresponds to existence of 'd-non-forking extensions' and (P5) to the the Independence Theorem over finite algebraically closed sets. We use them to prove weak elimination of imaginaries. It is also easy to see that (P4) and (P2$^-$) imply (P2).

Lemma 2.9 *Suppose (M, d_0) has weak canonical bases and satisfies axioms (P1, P2, P4$^-$, P5). Let $e \in M^{eq}$. Then $e \in \mathrm{dcl}^{eq}(\mathrm{acl}^{eq}(e) \cap M)$. In particular, this holds if M satisfies (P1, P2, P3$^-$, p4$^-$, P5).*

Proof. The proof follows Hrushovski's proof of weak elimination of imaginaries for PAC structures (over a submodel) ([5], Proposition 3.1).

There exists a $\emptyset$-definable equivalence relation E on a set of tuples from M such that $e = b/E$, for some tuple b. Let $C = \mathrm{acl}^{eq}(e) \cap M$, and let Q be the $\mathrm{Aut}(M/C)$-orbit containing b. It is enough to show that $E(b, b')$ holds for all $b' \in Q$.

Step 1. There exist $b_1, b_2 \in Q$ such that $E(b_1, b_2)$ and $d(b_1/b_2C) = d(b_1/C)$.

First, it follows from Neumann's Separation Lemma that there exist $\mathrm{Aut}(M/Ce)$-translates b_1 and b_2 of b such that

$$\mathrm{acl}(b_1C) \cap \mathrm{acl}(b_2C) = C.$$

Choose these so that $d(b_2/b_1C)$ is as large as possible (this can be done, as there are only finitely many $\mathrm{Aut}(M/Ce)$-orbits on Q^2, by $\aleph_0$-categoricity). Note that $E(b_1, b_2)$. Now let b_3 have the same type as b_2 over Cb_1, and be such that $d(b_3/b_1b_2C) = d(b_3/b_1C)$ (by (P4$^-$)). Then $E(b_3, b_1)$ and therefore $E(b_3, b_2)$. Moreover, by Lemma 2.3 and (P2), we have

$$\mathrm{acl}(b_3b_1C) \cap \mathrm{acl}(b_1b_2C) = \mathrm{acl}(b_1C).$$

Thus

$$\mathrm{acl}(Cb_3) \cap \mathrm{acl}(Cb_2) \subseteq \mathrm{acl}(Cb_3) \cap \mathrm{acl}(Cb_1) = C.$$

It follows (by choice of b_1, b_2) that $d(b_3/b_2C) \leq d(b_2/b_1C) = d(b_3/b_1b_2C)$. So by Lemma 2.1 (v), we have $d(b_3/b_1b_2C) = d(b_3/b_2C)$. Also, $d(b_3/b_1b_2C) = d(b_3/b_1C)$, by construction. So by existence of weak canonical bases,

$$\mathbf{cb}(b_3/\,\mathrm{acl}(b_1b_2C)) \subseteq \mathrm{acl}(b_1C) \cap \mathrm{acl}(b_2C) = C.$$

So $d(b_3/b_1C) = d(b_3/C)$. This completes Step 1.

Step 2. $E(b', b'')$ holds for all $b', b'' \in Q$ with $d(b'/b''C) = d(b'/C)$.

By Step 1, there exist $b_1, b_2 \in Q$ such that $E(b_1, b')$, $E(b_2, b'')$ and $d(b_1/b'C) = d(b_1/C)$, $d(b_2/b''C) = d(b_2/C)$. Thus by (P5) there exists $b_3 \in Q$ of the same type as b_1 over Cb' and the same type as b_2 over Cb''. In particular, $E(b_3, b')$ and $E(b_3, b'')$. So $E(b', b'')$.

Step 3. $E(b_1, b_2)$ holds for all $b_1, b_2 \in Q$.

By (P4$^-$), there exists $b_3 \in Q$ such that $d(b_3/b_1C) = d(b_3/b_2C) = d(b_3/C)$. So by step 2, $E(b_3, b_1)$ and $E(b_3, b_2)$. Thus $E(b_1, b_2)$. $\square$

Remarks 2.10 The proof works under a significantly weaker assumption than (P5). Let C be a finite algebraically closed subset of M and Q the set of realisations of a complete n-type over C (so an $\mathrm{Aut}(M/C)$-orbit on M^n). We say that a non-empty $R \subseteq Q \times Q$ is a *d-independent relation over* C, if it is $\mathrm{Aut}(M/C)$-invariant, and has the property that for all $(b, b') \in R$, $d(b/Cb') = d(b/C)$. It is *connected* if the (undirected) graph on Q with the pairs in R as edges is connected. If this is the case for all C, Q, R, then the above proof works (assuming existence of weak canonical bases, and (P1, P2, P4$^-$)).

2.5 Dimension over infinite sets

We work in a large model N of $Th(M)$. Note that as $d_0(A)$ depends only on $\mathrm{tp}(A/\emptyset)$ (for a finite subset A of M), we can consider d_0 and d as defined on finite subsets of N. For (possibly infinite) $S \subseteq N$ we let

$$d(c/S) = \inf(d(c/A) : A \subseteq S \text{ finite }).$$

We shall be assuming (P1, P2) throughout. We would like to have analogous results to Corollaries 2.7 and 2.8 without assuming B there to be finite. This appears to require a strengthening of Axiom (P3$^-$).

(P3) For any finite $A_2 \leq M$ and c there exists $\epsilon > 0$ such that if B is finite with $A_1, A_2 \leq B \leq M$, and $\mathrm{acl}(cA_i) \cap B = A_i$, and

$$d_0(\mathrm{acl}(cA_1)) + d_0(B) - d_0(A_1) - d_0(\mathrm{acl}(cA_1) \cup B) < \epsilon$$

then $\mathrm{acl}(cA_1) \cap \mathrm{acl}(cA_2)$, A_2 is a modular pair.

Note that this implies (P3$^-$), and that it implies the same property for finite subsets of N.

Analogously with Lemma 2.5 we have:

Lemma 2.11 *Assume axioms (P1, P2, P3) hold. Let $A_2 \leq N$ be finite, let c be a finite tuple in N, and suppose $X, A_2 \leq B \leq N$. If $d(c/X) = d(c/B)$, $\mathrm{acl}(cX) \cap B = X$, and $\mathrm{acl}(cA_2) \cap B = A_2$, then $d(c/A_2) = d(c/A_2 \cap X)$.*

Proof. We show that there exists a finite set A_1 with $A_2 \cap X \subseteq A_1 \leq X$ such that $\mathrm{acl}(cA_1) \cap B = A_1$, and

$$d(cA_2) = d_0(\mathrm{acl}(cA_1) \cap \mathrm{acl}(cA_2)) + d_0(A_2) - d_0(A_1 \cap A_2).$$

Then, as $A_1 \cap A_2 = A_2 \cap X$ and $\mathrm{acl}(cA_1) \cap \mathrm{acl}(cA_2) \geq \mathrm{acl}(c(A_1 \cap A_2))$ we have

$$d(c/A_2) = d(\mathrm{acl}(cA_1) \cap \mathrm{acl}(cA_2)/A_1 \cap A_2) \geq d(c/A_2 \cap X).$$

But, (by Lemma 2.1) $d(c/A_2) \leq d(c/A_2 \cap X)$, so we have equality, as required.

Let $\delta > 0$. There exists a finite $B_1 \leq B$ containing A_2 with $d(c/B_1) - d(c/B) < \delta$. As $d(c/X) = d(c/B)$ there exists a finite $X_1 \leq X$ with $d(c/X_1) \leq d(c/B_1)$. Without loss, we may assume that $A_2 \cap X \subseteq X_1$ and (as $\mathrm{acl}(cX) \cap B = X$) that $\mathrm{acl}(cX_1) \cap B = X_1$. Let $B_2 = \mathrm{acl}(X_1 B_1)$. We may assume that $\mathrm{acl}(cB_2) \cap B = B_2$.

Note that $d(c/X_1) - d(c/B_2) < \delta$, that is

$$d_0(\mathrm{acl}(cX_1)) - d_0(X_1) - d_0(\mathrm{acl}(cB_2)) + d_0(B_2) < \delta.$$

But

$$d_0(\mathrm{acl}(cB_2)) \leq d_0(\mathrm{acl}(cX_1) \cup B_2) \leq d_0(\mathrm{acl}(cX_1)) + d_0(B_2) - d_0(X_1).$$

By the above, this is less than $d_0(\mathrm{acl}(cB_2)) + \delta$. So $\mathrm{acl}(cX_1)$ and B_2 are within δ of being a modular pair, and $d_0(\mathrm{acl}(cX_1) \cup B_2) - d_0(\mathrm{acl}(cB_2)) < \delta$. Let $Z_1 = \mathrm{acl}(cX_1) \cup B_2$. Note that $\mathrm{acl}(Z_1) = \mathrm{acl}(cB_2)$. Then $d_0(Z_1 \cup \mathrm{acl}(cA_2)) \leq d_0(Z_1) + d_0(\mathrm{acl}(cA_2)) - d_0(Z_1 \cap \mathrm{acl}(cA_2))$, and as $Z_1 \subseteq Z_1 \cup \mathrm{acl}(cA_2) \subseteq \mathrm{acl}(Z_1)$, we have $d_0(Z_1 \cup \mathrm{acl}(cA_2)) \geq d_0(\mathrm{acl}(cB_2)) > d_0(Z_1) - \delta$. Thus $d_0(Z_1 \cap \mathrm{acl}(cA_2)) - d_0(\mathrm{acl}(cA_2)) < \delta$. Now, $Z_1 \cap \mathrm{acl}(cA_2) = (\mathrm{acl}(cX_1) \cap \mathrm{acl}(cX_2)) \cup A_2$. Thus

$$0 \leq d_0((\mathrm{acl}(cX_1) \cap \mathrm{acl}(cA_2)) \cup A_2) - d_0(\mathrm{acl}(cA_2)) < \delta.$$

But this inequality relates to the fixed finite set $\mathrm{acl}(cA_2)$. So if we choose δ small enough here, we must have $d_0((\mathrm{acl}(cX_1) \cap \mathrm{acl}(cA_2)) \cup A_2) = d_0(\mathrm{acl}(cA_2))$. Recall that $\mathrm{acl}(cX_1)$ and B_2 are within δ of being a modular pair. Moreover, $\mathrm{acl}(cX_1) \cap B_2 = X_1$ and $\mathrm{acl}(cA_2) \cap B_2 = A_2$. Let ϵ be as given by (P3) for A_2 and c, and choose $\delta < \epsilon$. Then $\mathrm{acl}(cX_1) \cap \mathrm{acl}(cA_2), A_2$ is a modular pair. This gives what we need (let $A_1 = X_1$). $\square$

Remark 2.12 As stated, axiom (P3) looks rather technical. The only point where it is used (as opposed to (P3$^-$), which itself follows from the more readily intelligible (P3$'$)) is in the proof of the above lemma. Note however that the lemma holds if we assume (P3$^-$) and (P7). Indeed, with the notation as in the statement of the lemma, by (P7) there exists a finite $A_1 \supseteq X \cap A_2$ contained in X with $d(c/A_1A_2) = d(c/A_1)$. Let $B_1 = \mathrm{acl}(A_1A_2)$. Then, as $A_1 \cap A_2 = A_2 \cap X$, the required result follows from Lemma 2.5.

Definition 2.13 Let B be an algebraically closed subset of N. Consider the subset

$$\bigcup\{\mathbf{cb}(c/A) : A \leq B \text{ finite, } \mathrm{acl}(cA) \cap B = A\}.$$

Note that if B is finite then Corollary 2.8 states that this is equal to $\mathbf{cb}(c/B)$. Thus, we denote this set by $\mathbf{cb}(c/B)$ for arbitrary B, and refer to it as the weak canonical base of c over B.

Corollary 2.14 *Suppose (P1, P2, P3) hold. Suppose $B_1 \leq B \leq N$ satisfies $\mathrm{acl}(cB_1) \cap B = B_1$. Then $d(c/B_1) = d(c/B)$ if and only if $\mathbf{cb}(c/B) \subseteq B_1$.*

Proof. First suppose $\mathbf{cb}(c/B) \subseteq B_1$. Given a finite $C \leq B$, there exists a finite A with $C \leq A \leq B$ and $\mathrm{acl}(cA) \cap B = A$. Then $d(c/C) \geq d(c/A) = d(c/A_0)$ where $A_0 = \mathbf{cb}(c/A) \subseteq B_1$. Thus $d(c/C) \geq d(c/A_0) \geq d(c/B_1)$. So $d(c/B_1) \geq d(c/B) = \inf(d(c/C) : C \subseteq B \text{ finite}) \geq d(c/B_1)$, and we therefore have equality.

Conversely, suppose $d(c/B_1) = d(c/B)$, $A \leq B$ is finite and $\mathrm{acl}(cA) \cap B = A$. By Lemma 2.11 $d(c/A) = d(c/B_1 \cap A)$, so $\mathbf{cb}(c/A) \subseteq B_1 \cap A \subseteq B_1$, by Corollary 2.7. $\square$

Corollary 2.15 *Suppose (P1-P3) hold. Suppose* $A \leq B \leq N$ *and* $\mathrm{acl}(cA) \cap B = A$. *Then* $\mathbf{cb}(c/A) \subseteq \mathbf{cb}(c/B)$.

Proof. By definition, it is enough to prove this when A is finite. But this is then Lemma 2.11. $\square$

2.6 Existence of d-non-forking extensions

We now wish to impose conditions guaranteeing that $Th(M)$ is simple and that the dimension function d detects forking. In particular we want to pass from the local condition (P4$^-$) (over finite sets) to a global one (over arbitrary sets) via some sort of compactness argument. Making this work in general appears to require a strengthening of (P4$^-$), and curiously CM-triviality is involved in the proof.

(P4) If $A \leq B \leq M$ are finite and c is a finite tuple of elements of M with $\mathrm{acl}(cA) \cap B = A$, then there exists c_1 with

(i) $\mathrm{tp}(c_1/A) = \mathrm{tp}(c/A)$;

(ii) $d(c_1/B) = d(c_1/A)$;

(iii) if $C \leq B$ is such that $\mathrm{acl}(cC) \cap A = C \cap A$, then $\mathrm{acl}(c_1C) \cap A = A \cap C$.

Once again, we work in a large highly saturated model N of $Th(M)$. We shall show that the following holds.

(P6) Let $A \subseteq B$ be small algebraically closed subsets of N and let c be a finite tuple. Then there exists c_1 such that $\mathrm{tp}(c/A) = \mathrm{tp}(c_1/A)$, $d(c_1/A) = d(c_1/B)$, and $\mathrm{acl}(c_1A) \cap B = A$.

Lemma 2.16 *Suppose (P1, P2, P3*$^-$*, P4) hold. Then (P6) holds.*

Proof. Let $A \leq B \leq N$ be small and c a finite tuple from N. By Neumann's lemma, we may assume that $\mathrm{acl}(cA) \cap B = A$. We must show that there is some c_1 with $\mathrm{tp}(c/A) = \mathrm{tp}(c_1/A)$ and $d(c_1/B) = d(c_1/A)$. We first do this in the case where $B = A \cup b$ for some finite tuple b.

Note first that for every finite $X \subseteq A$ there exists a finite A_1 with $X \subseteq A_1 \subseteq A$ such that

$$\mathrm{acl}(cA_1) \cap A = A_1 \text{ and } \mathrm{acl}(bA_1) \cap A = A_1.$$

Indeed, let $A_1 = \operatorname{acl}(bcX) \cap A$. Then $\operatorname{acl}(cA_1) \cap A \subseteq \operatorname{acl}(bcX) \cap A = A_1$, and likewise $\operatorname{acl}(bA_1) \cap A = A_1$.

Now let $\mathcal{A} = \{A_i : i \in I\}$ be the directed set of finite A_1 with the above properties (X varying). Consider the set of formulas (in the language for N with constant symbols added for elements of A) expressing that

$$\operatorname{tp}(c_1/A) = \operatorname{tp}(c/A)$$

and, for $i \in I$,

$$d(c_1/A_i) = d(c_1/A_i \cup b).$$

We show that this is consistent. Let $A_1, \ldots, A_r \in \mathcal{A}$. There exists $A' \in \mathcal{A}$ with $A_i \leq A'$ for $i = 1, \ldots, r$.

By (P4), there exists c_1 with with the same type over A' as c, and such that $d(c_1/\operatorname{acl}(bA')) = d(c_1/A')$, and moreover $\operatorname{acl}(c_1C) \cap A' = C \cap A'$ whenever $C \leq \operatorname{acl}(bA')$ is such that $\operatorname{acl}(cC) \cap A' = C \cap A'$. Consider (for $i = 1, \ldots, r$) $C = \operatorname{acl}(bA_i)$. Then (by definition of $\mathcal{A}$) $C \cap A' = A_i$ and $\operatorname{acl}(cC) \cap A' = \operatorname{acl}(cbA_i) \cap A' = A_i = C \cap A'$. So $\operatorname{acl}(c_1C) \cap A' = C \cap A' = A_i$. We want to show $\mathbf{cb}(c_1/C) \subseteq A_i$. Let $Y = \operatorname{acl}(c_1C) \cap \operatorname{acl}(bA')$. Then $\operatorname{acl}(c_1Y) \cap \operatorname{acl}(bA') = Y$, so by Corollary 2.8 $\mathbf{cb}(c_1/Y) \subseteq \mathbf{cb}(c_1/A'b) \subseteq A'$. Thus $\mathbf{cb}(c_1/Y) \subseteq \operatorname{acl}(c_1C) \cap \operatorname{acl}(bA') \cap A' \subseteq A_i$. So $d(c_1/Y) = d(c_1/A_i)$ and as $A_i \subseteq C \subseteq Y$ we have $d(c_1/C) = d(c_1/A_i)$, as required.

This establishes the consistency of the above set of formulas. Let c_1 realise them. Then of course $\operatorname{tp}(c_1/A) = \operatorname{tp}(c/A)$ and

$$d(c_1/A \cup b) = \inf(d(c_1/A_i \cup b) : i \in I)$$

$$= \inf(d(c_1/A_i) : i \in I) = d(c_1/A).$$

Moreover, by Lemma 2.3 $\operatorname{acl}(c_1A_i) \cap \operatorname{acl}(c_1A_ib) = A_i$. Thus $\operatorname{acl}(c_1A) \cap \operatorname{acl}(c_1Ab) = A$.

Finally, the statement for arbitrary $B \supseteq A$ follows by the above and compactness, by allowing b to range over all finite tuples from B. □

2.7 Simplicity

The terminology in the following lemma is that of Kim and Pillay ([8], Definition 4.1). It is explained in the proof of the lemma.

Lemma 2.17 *Let Γ' be the collection of triples (c, B, A) where c is a (finite) tuple from N, and $A \subseteq B$ are small subsets of N such that $d(c/B) = d(c/A)$. Denote by Γ the subset of this where additionally* $\mathrm{acl}(cA) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$.

(a) If d_0 satisfies axioms (P1) and (P2), then Γ, Γ' have the properties of invariance under automorphisms, local character, finite character, symmetry and transitivity.

(b) If additionally d_0 satisfies (P6), then Γ, Γ' are notions of independence.

Proof. (a) We check the axioms of Kim and Pillay for Γ'. The modifications required for Γ are easy and omitted.

(i) Invariance under automorphisms: this is clear.

(ii) Local character: this follows from separability of $\mathbb{R}^{\geq 0}$.

(iii) Finite character: Suppose $(c, B, A) \in \Gamma'$. Then $d(c/A') = d(c/A)$ for all $A \subseteq A' \subseteq B$. On the other hand, if $(c, B, A) \notin \Gamma'$, then $d(c/B) < d(c/A)$, so $\inf(d(c/S) : S \subseteq B \text{ finite }) < d(c/A)$. Thus for some finite $S \subseteq B$, we have $d(c/S) < d(c/A)$. So $d(c/S \cup A) < d(c/A)$, that is, $(c, S \cup A, A) \notin \Gamma'$.

(v) Symmetry: Suppose $(c, A \cup B, A) \in \Gamma'$. We must show that $(b, A \cup c, A) \in \Gamma'$ for all finite tuples b from B. It is enough to show that for any finite $A_0 \subseteq A$ and $\epsilon > 0$ there exists a finite $A_2 \supseteq A_0$ contained in A such that $d(b/A_2) - d(b/A_2c) < \epsilon$. But there exists A_2 with $A_0 \subseteq A_2 \subseteq A$ and $d(c/A_2) - d(c/A) < \epsilon$. So as $d(c/Ab) = d(c/A) \leq d(c/A_2b) \leq d(c/A_2)$ we get $d(c/A_2) - d(c/A_2b) < \epsilon$. Thus

$$d(cA_2) - d(A_2) - d(cbA_2) + d(A_2b) < \epsilon$$

whence $d(b/A_2) - d(b/A_2c) < \epsilon$, as required.

(vi) Transitivity: Suppose $A \subseteq B \subseteq C$. Then $d(c/C) \leq d(c/B) \leq d(c/A)$, so $(c, C, A) \in \Gamma'$ if and only if $(c, C, B) \in \Gamma'$ and $(c, B, A) \in \Gamma'$.

(b) All that is missing is the existence of Γ and Γ' -non-forking extensions. This is exactly what (P6) gives for Γ. For Γ', use Neumann's separation lemma first to get the extra hypothesis for (P6). $\square$

It is convenient to isolate the following property.

(P7) If a, b are finite tuples and A is a small algebraically closed subset of N such that $d(a/Ab) = d(a/A)$ and $\mathrm{acl}(aA) \cap \mathrm{acl}(bA) = A$, then for every finite $A_0 \subseteq A$ there exists a finite subset $A_0 \subseteq A_1 \subseteq A$ with $d(a/A_1b) = d(a/A_1)$.

Lemma 2.18 *If (P1, P2, P3) hold, then (P7) holds.*

Proof. Let $B = \mathrm{acl}(Ab)$ and $B_1 = \mathrm{acl}(abA_0) \cap B$. Then $\mathrm{acl}(aB_1) \cap B = B_1$, so by CM-triviality (Corollary 2.15) and hypothesis we have $\mathbf{cb}(a/B_1) \subseteq \mathbf{cb}(a/B)$. By Corollary 2.14 this is contained in A. Thus if we let $A_1 = B_1 \cap A$ we have $\mathbf{cb}(a/B_1) \subseteq A_1$ and in particular $d(a/A_1b) = d(a/A_1)$. $\square$

We are now in a position to prove simplicity. It is tempting to try to do this by showing that the Independence Theorem holds over a model (for the notion of independence Γ), and then invoke Theorem 4.2 of [8]. But there appear to be difficulties passing from (P5) to IT over a model via compactness. Instead, we proceed directly.

Lemma 2.19 *Let $A \subseteq B$ be small subsets of N and c a tuple of elements from N.*

(a) Assume (P1, P2, P6, P7) hold. If $d(c/B) < d(c/A)$ then $\mathrm{tp}(c/B)$ divides over A. If $\mathrm{acl}(cA) \cap \mathrm{acl}(B) \neq \mathrm{acl}(A)$ then $\mathrm{tp}(c/B)$ divides over A.

(b) Assume (P1, P2, P5, P6, P7) hold. If $d(c/B) = d(c/A)$ and $\mathrm{acl}(cA) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$, then $\mathrm{tp}(c/B)$ does not divide over A.

(c) Assume that (P1, P2, P5, P6, P7) hold. If $\mathrm{tp}(c/B)$ does not divide over A, it does not fork over A.

Proof. (a) The first part follows exactly as in the proof of Claim I of Theorem 4.2 of [8] using Γ'. For the second part, suppose A is algebraically closed and the inequality is witnessed by $\phi(c, b, a)$, $b \in \mathrm{acl}(B) \setminus A$. By Neumann's separation lemma, there are b_i $(i < \omega)$ with $\mathrm{tp}(b_i/A) = \mathrm{tp}(b/A)$ and $\mathrm{acl}(Ab_i) \cap \mathrm{acl}(Ab_j) = \mathrm{acl}(A)$ if $i \neq j$. Thus $\{\phi(x, b_i, a) : i < \omega\}$ is 2-inconsistent.

(b) We modify the proof of Lemma 4.8 of [7]. Without loss, assume $\mathrm{acl}(A) = A$. Suppose, for a contradiction, that there is $\phi(x, b) \in \mathrm{tp}(c/B)$ and an infinite A-indiscernible sequence $(b_i : i \in I)$ with $b_0 = b$ and $\{\phi(x, b_i) : i \in I\}$ being k-inconsistent (for some natural number k). Then $d(c/Ab) = d(c/A)$ and $\mathrm{acl}(cA) \cap \mathrm{acl}(Ab) = A$, by assumption, so by (P7) we may assume that b is an extension of a tuple a of elements of A, such that $d(c/b) = d(c/a)$.

We may assume that the indexing set I is $\mathbb{Z}$. Let $B_1 = \mathrm{acl}(a, b_i : i \leq 0)$. Then $(b_i : i > 0)$ is indiscernible over B_1, and as there is an automorphism of N fixing a, b_{i+1} and taking $\ldots, b_0, \ldots, b_i$ to $\ldots, b_0$, we have

$$d(b_{i+1}/B_1 b_1 \ldots b_i) = d(b_{i+1}/B_1)$$

for $i \geq 0$.

By extending the B_1-indiscernible sequence $(b_i : i \geq 1)$ and applying the Erdös-Rado Theorem, we may assume that $X = \mathrm{acl}(B_1 b_{i_k}) \cap \mathrm{acl}(B_1 b_{i_1} \dots b_{i_{k-1}})$ is constant for all $1 \leq i_1 < \dots < i_k$, and $(b_i : i \geq 1)$ is X-indiscernible. Then $d(b_k/Xb_1 \dots b_{k-1}) = d(b_k/X)$ and $\mathrm{acl}(Xb_k) \cap \mathrm{acl}(Xb_1 \dots b_{k-1}) = X$. So by (P7) again, there is a finite $B_0 \subseteq X$ which contains a and is such that $d(b_k/B_0 b_1 \dots b_{k-1}) = d(b_k/B_0)$. By indiscernibility (over X), we therefore have $d(b_{i+1}/B_0 b_1 \dots b_i) = d(b_{i+1}/B_0)$ for $1 \leq i < k$.

Let c' be such that $\mathrm{tp}(c'b_1/A) = \mathrm{tp}(cb/A)$. Thus $\phi(x, b_1) \in \mathrm{tp}(c'/b_1)$ and $d(c'/b_1) = d(c'/a)$. Let c'' be such that $\mathrm{tp}(c'/b_1) = \mathrm{tp}(c''/b_1)$ and $d(c''/Xb_1) = d(c''/b_1)$ (by (P6)). So $d(c''/B_0 b_1) = d(c''/b_1) = d(c''/a) = d(c''/B_0)$ and $\phi(x, b_1) \in \mathrm{tp}(c''/b_1)$. Thus there exist $c_1, \dots, c_k$ such that $d(c_i/B_0 b_i) = d(c_i/B_0)$, $\mathrm{tp}(c_i/B_0) = \mathrm{tp}(c''/B_0)$ and $\phi(x, b_i) \in \mathrm{tp}(c_i/b_i)$ (simply take $\mathrm{tp}(c_i b_i/B_0) = \mathrm{tp}(c'' b_1/B_0)$). So by (P5) (applied $k-1$ times) there exists c_0 with $\mathrm{tp}(c_0/B_0 b_i) = \mathrm{tp}(c_i/B_0 b_i)$ for $i = 1, \dots, k$. But this contradicts the inconsistency of $\{\phi(x, b_i) : i = 1, \dots, k\}$.

(c) Suppose $\phi \in \mathrm{tp}(c/B)$ and $\phi \to \phi_1 \vee \dots \vee \phi_r$, where each ϕ_i divides over A. Let $B_1 \supseteq B$ be algebraically closed and include the extra parameters needed for these ϕ_i. By (P6) we may assume that $d(c/B_1) = d(c/B)$ and $\mathrm{acl}(cB) \cap B_1 = B$. By (a), $d(c/B) = d(c/A)$ and $\mathrm{acl}(cA) \cap B = \mathrm{acl}(A)$. Thus $d(c/B_1) = d(c/A)$ and $\mathrm{acl}(cA) \cap B_1 = \mathrm{acl}(A)$. As $\phi \in \mathrm{tp}(c/B_1)$ we must have $\phi_i \in \mathrm{tp}(c/B_1)$ for some i. So $\mathrm{tp}(c/B_1)$ divides over A, which contradicts (b). □

Corollary 2.20 *Suppose (P1, P2, P5, P6, P7) hold. Then $Th(M)$ is simple and, for small algebraically closed A, B and any tuple c,*

$$\mathrm{tp}(c/B) \textit{ does not fork over } A \Leftrightarrow d(c/A) = d(c/B) \textit{ and } \mathrm{acl}(cA) \cap B = A.$$

Proof. The last statement follows at once from Lemma 2.19. Simplicity then follows from separability of $\mathbb{R}$. □

Corollary 2.21 *Suppose (P1-P5) hold. Then $Th(M)$ is simple and, for small algebraically closed A, B and any tuple c,*

$$\mathrm{tp}(c/B) \textit{ does not fork over } A \Leftrightarrow d(c/A) = d(c/B) \textit{ and } \mathrm{acl}(cA) \cap B = A.$$

Proof. This follows from the above and Lemmas 2.16 and 2.18. □

Remarks 2.22 If (P1-P5) hold, then $Th(M)$ is CM-trivial, as defined in the introduction. Indeed, suppose E_1, E_2 are algebraically closed and independent over $E_1 \cap E_2$. Let $D \subseteq E_1 \cup E_2$ be such that $\mathrm{acl}(D) \cap (E_1 \cup E_2) = D$ and let $D_i = D \cap E_i$. We must show that D_1 and D_2 are independent over $D_1 \cap D_2$. So let c be a tuple of elements in D_1. We know (by Corollary 2.21) that $d(c/E_2) = d(c/E_1 \cap E_2)$ and $\mathrm{acl}(c(E_1 \cap E_2)) \cap E_2 = E_1 \cap E_2$. Thus by Corollary 2.14, $\mathbf{cb}(c/E_2) \subseteq E_1 \cap E_2$. Now, $\mathrm{acl}(cD_2) \cap E_2 = D_2$, so by Corollary 2.15, $\mathbf{cb}(c/D_2) \subseteq \mathbf{cb}(c/E_2)$ and it follows easily (using Corollary 2.21 again) that c and D_2 are independent over $D_1 \cap D_2$.

The statement of Corollary 2.15 is closer to the usual definition of CM-triviality in a simple theory (with elimination of hyperimaginaries), expressed as a property of canonical bases: if $A \subseteq B$ are algebraically closed and $\mathrm{acl}(cA) \cap B = A$, then $\mathrm{Cb}(c/A) \subseteq \mathrm{acl}(\mathrm{Cb}(c/B))$.

2.8 Supersimplicity

In this subsection, we suppose that the set of values of $d(c/A)$ (where c is a finite tuple from M and A is a finite subset of M) is well-ordered. In this case we say that d is *discrete*. For example, this happens if the subgroup of $\mathbb{R}$ generated by $\{d_0(A) : A \subseteq M \text{ finite }\}$ is discrete. It is clear that if d is discrete then for all c and small $B \leq N$ there is a finite $A \leq B$ with $d(c/B) = d(c/A)$. Moreover, Lemma 2.3 shows that $\mathrm{acl}(cA) \cap B = A$. So by Corollary 2.21, if (P1-P5) also hold then $Th(M)$ is supersimple and of finite SU-rank. However, more is true.

Lemma 2.23 *Suppose (M, d_0) satisfies (P1), (P2) and d is discrete.*
(i) If (P4$^-$) holds, then (P6) holds.
(ii) (P7) holds.

Proof. (i) A compactness argument similar to (but easier than) Lemma 2.16.
(ii) Trivial. □

Corollary 2.24 *Suppose (M, d_0) satisfies (P1, P2, P4$^-$, P5) and d is discrete. Then $Th(M)$ is supersimple of finite SU-rank. Moreover, for small algebraically closed A, B and a tuple c,*

$$\mathrm{tp}(c/B) \textit{ does not fork over } A \ \Leftrightarrow d(c/A) = d(c/B) \textit{ and } \mathrm{acl}(cA) \cap B = A. \square$$

Remarks 2.25 There is clearly a converse to this. Suppose M is supersimple, $\aleph_0$-categorical and of finite SU-rank. For finite $A \subseteq M$ define $d_0(A) = SU(\mathrm{tp}(A/\emptyset))$. This satisfies (P1, P2, P4$^-$), and if M has weak elimination of imaginaries, then (P5) also holds (strong type and Lascar strong type over finite sets coincide in $\aleph_0$-categorical theories, so the independence theorem holds over finite algebraically closed sets, by ([8], Theorem 5.8)).

Solutions to the following might shed some light on the problem of finding supersimple $\aleph_0$-categorical structures which are not CM-trivial.

Question 2.26 Find $\aleph_0$-categorical structures (with weak elimination of imaginaries) without the strict order property which have a predimension satisfying (P1, P2, P4$^-$), but not (P3). If this can be done, impose the extra condition of having weak canonical bases for the predimension.

3 Examples

In this section we give a slight modification of Hrushovski's construction from [4] and [7]. Throughout, M_0 is a countable $\aleph_0$-categorical L-structure having quantifier elimination in a relational language, and y_0 is a predimension on M_0 (with corresponding dimension y) satisfying (P1-P5). We assume that $y(X) \to \infty$ as $|X| \to \infty$ (for $X \subseteq M_0$), and, for convenience, we assume that $y(\mathrm{acl}(\emptyset)) = 0$. For example, M_0 could be a pure set, and $y_0(X)$ the cardinality of X; or M_0 could be a vector space over a finite field (Morleyized, to give quantifier elimination in a relational language), and $y_0(X)$ the dimension of the subspace spanned by X. It could also be a structure produced by a previous application of this construction (with its language enriched to give the quantifier elimination).

Now let L^+ be a first-order language extending L by relation symbols, with finitely many of each arity. Let $\mathcal{C}'_0$ be the collection of finite L-structures A such that: the reduct $A \mid L$ to L is isomorphic to an algebraically closed substructure of M_0; none of the new relations involve elements of $\mathrm{acl}_{M_0}(\emptyset)$; and for all n-ary relation symbols R in $L^+ \setminus L$,

$$A \models R(x_1, \ldots, x_n) \to \bigwedge_{i \neq j} x_i \neq x_j.$$

(We could also insist that some of the relations be invariant under permutation of coordinates.) Let $\mathcal{C}''_0$ be the set of substructures of elements of $\mathcal{C}'_0$.

Let $\alpha > 0$ and $\alpha_R \geq 0$ (for each such R) be real numbers. For $A \in \mathcal{C}'_0$ and $B \subseteq A$ let

$$d_0(B) = \alpha y(B) - \sum_R \alpha_R \mid R[B] \mid$$

(the sum ranges over the new relation symbols, and $\mid R[B] \mid$ is the number of realisations of R in B). Note that this depends only on the L^+-isomorphism type of B (and not on A) by quantifier elimination in the restricted language. For $B \subseteq C \subseteq A \in \mathcal{C}'_0$ write

$$B \leq_d C \Leftrightarrow d_0(B) < d_0(D) \text{ for all } B \subset D \subseteq C.$$

It is easy to see that d_0 satisfies $(P1)$ and if $B \leq_d C$ then $B \mid L \leq_y C \mid L$. So (by Lemma 2.1) for any $B \subseteq A \in \mathcal{C}'_0$ one can define cl_d^A (respectively cl_y^A) to be the smallest subset C of A containing B such that $C \leq_d A$ (respectively $C \leq_y A$). Clearly $\mathrm{cl}_y^A(B) \subseteq \mathrm{cl}_d^A(B)$. Let $\mathcal{C}_0 = \{A \in \mathcal{C}'_0 : \mathrm{cl}_y^{A'}(\emptyset) \leq_d A' \text{ for all } A' \subseteq A\}$.

Definition 3.1 *For $x \geq 0$ let $\kappa(x) = min(1, d_0(D/A) : D \in \mathcal{C}_0, A \leq_d D, A \neq D, y(D) \leq x)$.*

For example, if y, α and α_R are integer valued, then $\kappa(x) = 1$.

Definition 3.2 *Let $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be a continuous function with $f(x) \to \infty$ as $x \to \infty$ whose right derivative f' exists at all points, and is ≤ 1. Suppose further that $0 < f'(x) \leq \kappa(x)/x$ and $f'(x)$ is monotone decreasing for all x. Let*

$$\mathcal{C}_f = \{X \in \mathcal{C}_0 : d_0(B) \geq f(y(B)) \textit{ for all } B \subseteq X\}.$$

Lemma 3.3 (Amalgamation Lemma) *Let $A, B_1, B_2 \in \mathcal{C}_f$ and $\beta_i : A \to B_i$ be embeddings with $\beta_i A \leq_d B_i$. Then there exists $C \in \mathcal{C}_f$ and embeddings $\gamma_i : B_i \to C$ with:*
(i) $\gamma_1\beta_1 = \gamma_2\beta_2$;
(ii) $\gamma_i B_i \leq_d C$;
(iii) $\gamma_1 B_1 \cap \gamma_2 B_2 = \gamma_1\beta_1 A$ and $d_0(C) = d_0(B_1) + d_0(B_2) - d_0(A)$.

Proof: To ease notation, assume that the β_i are inclusions and $B_1 \cap B_2 = A$. By (P4) for y_0, we can assume that there exists and L-structure C^- isomorphic to an algebraically closed subset of M_0, such that $B_i \mid L \leq_y C^-$ and

$$y(B_1 \cup B_2) = y(C^-) = y(B_1) + y(B_2) - y(A).$$

Now let C be the L^+ structure obtained by expanding C^- by the relations on B_1 and B_2 (and no others). We call this a *free amalgam* of B_1 and B_2 over A. It is clear that $B_i \leq_y C$ and part (iii) of the Lemma holds.

To show that $B_1 \leq_d C$, let $D = \mathrm{cl}_d^C(B_1)$. By y-modularity (and Lemma 2.4 for y), $y(D) \geq y(D \cap (B_1 \cup B_2)) = y(B_1) + y(D \cap B_2) - y(A)$. As any instance of a new relation in D lies in $D \cap (B_1 \cup B_2)$ this gives $d_0(D) \geq d_0(B_1) + d_0(D \cap B_2) - d_0(A)$. If $A \subset D \cap B_2$ then $d_0(A) < d_0(D \cap B_2)$, and so $d_0(D) > d_0(B_1)$, contradicting the definition on d-closure. Thus $D \cap B_2 = A$, so $y(D) = y(B_1)$, whence $D = B_1$.

Next, we show that if $D \subseteq C$ then $d_0(D) \geq f(y(D))$. First, suppose that $D \leq_y B_1 \cup B_2$. We may assume $D \not\subseteq B_i$. Then by CM-triviality in M_0 we have $y(D) = y(D \cap B_1) + y(D \cap B_2) - y(D \cap A)$. Thus $d_0(D) = d_0(D \cap B_1) + d_0(D \cap B_2) - d_0(D \cap A)$. A familiar argument, which we repeat for convenience, then shows that $d_0(D) \geq f(y(D))$. For $X \in \mathcal{C}_0''$ let $\langle X \rangle = (y(X), d_0(X))$. Let $D_0 = D \cap A$ and $D_i = D \cap B_i$. Then

$$\langle D \rangle = \langle D_0 \rangle + (\langle D_1 \rangle - \langle D_0 \rangle) + (\langle D_1 \rangle - \langle D_0 \rangle).$$

So the line segment in the (y, d_0)-plane joining $\langle D_0 \rangle$ and $\langle D \rangle$ lies above the line segment joining $\langle D_0 \rangle$ and $\langle D_1 \rangle$, say. As $D_0 \leq_d D_1$ the latter has slope at least $\kappa(y(D_1))/y(D_1) \geq f'(y(D_1))$. As f' is non-increasing, the half line joining $\langle D_0 \rangle$ and $\langle D_1 \rangle$ lies above the graph of f, so the same is true of $\langle D_0 \rangle$ and $\langle D \rangle$. In particular $d_0(D) \geq f(y(D))$.

Now let D be any subset of C. We may assume that $D \leq_y C$. Let $D_0 = D \cap (B_1 \cup B_2)$. Then $d_0(D) = y(D) - y(D_0) + d_0(D_0)$. By the above $d_0(D_0) \geq f(y(D_0))$ and so it remains to show that $y(D) - y(D_0) \geq f(y(D)) - f(y(D_0))$. But this follows as $f'(x) \leq 1$. □

Lemma 3.4 *Let $E \in \mathcal{C}_0'$ and $X, Y \subseteq E$ be a y-modular pair. Suppose that any instance of a relation of $L^+ \setminus L$ on $X \cup Y$ is in X or in Y, and that $X \cap Y \leq_d^* X$. Then $Y \leq_d^* X \cup Y$.*

Proof. The hypotheses imply that if $X \cap Y \subseteq T \subseteq X$ then $d_0(Y \cup T) = d_0(Y) + d_0(T) - d_0(X \cap Y) \geq d_0(Y)$. □

Theorem 3.5 *There exists a countable L^+-structure M_f with a predimension d_0 satisfying (P1) and (P2) and having the following properties:*
(i) The set of isomorphism types of finite $\leq_d$-closed substructures of M_f is $\mathcal{C}_f$ (and d_0 agrees with the d_0 defined above).
(ii) There exist finite $X_1 \leq_d X_2 \leq_d \cdots$ with $M_f = \bigcup_{i<\omega} X_i$.
(iii) If $\beta : X \to Y$ is as embedding of structures in $\mathcal{C}_f$ with $\beta X \leq_d Y$, then for all embeddings $\delta : X \to M_f$ with $\delta X \leq_d M_f$ there exists an embedding $\gamma : Y \to M_f$ with $\gamma\beta = \delta$ and $\gamma Y \leq_d M_f$.

Moreover, M_f is determined by the above properties. It is $\aleph_0$-categorical and any isomorphism between finite algebraically closed substructures of M_f extends to an automorphism of M_f.

Proof. Existence and uniqueness of a countable structure M_f with properties (i), (ii), (iii) is standard, using Lemma 3.3, and we omit this. Moreover, using these properties we obtain $\leq_d$-homogeneity: any isomorphism between finite structures $A, B \leq_d M_f$ extends to an automorphism of M_f.

For $\aleph_0$-categoricity, note that for finite subsetes $X \subseteq Y$ of M_f, if $d_0(Y) \leq d_0(X)$, then $y(Y) \leq f^{-1}(d_0(X)) \leq f^{-1}(\alpha y(X))$. Thus, there exists $Z \leq_d M_f$ containing X with $y(Z) \leq f^{-1}(\alpha y(X))$. So, because $y(X) \to \infty$ as $|X| \to \infty$, we have that $|\operatorname{cl}_d^{M_f}(X)|$ is uniformly bounded by a function of $|X|$. This, together with $\leq_d$-homogeneity, gives $\aleph_0$-categoricity.

For (P2) we need to know that if $A \leq_d M_f$ and $b \notin A$ then $b \notin \operatorname{acl}(A)$. Let $B = \operatorname{cl}_d(Ab)$. Then $A \leq_d B \leq_d M_f$ and $A, B \in \mathcal{C}_f$. By Lemma 3.3 there exist $B_i \leq_d M_f$ (for $i \in \omega$) with pairwise intersection A and $\operatorname{tp}(B_i/A) = \operatorname{tp}(B/A)$ for all i. Thus $\operatorname{tp}(b/A)$ is non-algebraic. □

It should be noted that the reduct of M_f to L is isomorphic to M_0 (this is an easy back-and-forth argument using quantifier elimination). Also, for $X \subseteq M_f$ we have $d(X) \to \infty$ as $|X| \to \infty$.

Theorem 3.6 *(i) The predimension d_0 on M_f satisfies (P1-P4).*

(ii) If additionally $f(3t) \leq f(t) + \kappa(t)$ for all $t \geq 0$, then d_0 has property (P5), and M_f is simple.

Remarks 3.7 Suppose there exists $c_0 > 0$ such that $\kappa(x) \geq c_0$ for sufficiently large x. Then the condition on f in (ii) simplifies to $f(3t) \leq f(t) + c_0$ for sufficiently large t. Thus, if M_0 is supersimple and we take y to be given by SU-rank, there is no problem in finding such a function f. So the construction can be iterated to produce supersimple structures (all of which will be CM-trivial). It is possible, however, that there is no such c_0, and then the existence of a suitable function f is not guaranteed. In fact, an obvious additional condition is that the series $\sum_{k<\omega} \kappa(t_0 3^k)$ should diverge (for sufficiently large t_0).

Proof of 3.6. (i) Note that A, B is a d_0-modular pair if and only if it is a y-modular pair, and the only instances of relations R of $L^+ \setminus L$ with $\alpha_R > 0$ lie entirely in A or entirely in B. Now we show (P3) holds. For fixed $A_2 \leq_d M_f$ and c, we must show that there exists $\epsilon > 0$ such that if $A_1, A_2 \leq_d B \leq_d M_f$, $\mathrm{acl}(cA_i) \cap B = A_i$ and A_1 and B are within ϵ of being a d_0-modular pair, then $C_0 = \mathrm{acl}(cA_1) \cap \mathrm{acl}(cA_2)$ and A_2 is a d_0-modular pair. Note that this fails to be modular only if it is not y-modular, or there is a relation $R(a)$(with $\alpha_R > 0$) holding on $C_0 \cup A_2$ with a not contained entirely within one of the parts. But the latter possibility can be excluded by taking $\epsilon < \alpha_R$ for all $R \in L^+ \setminus L$ of arity at most $\mid \mathrm{acl}(cA_2) \mid$. To exclude the possibility that the pair is not y-modular, note that $\mathrm{acl}(cB)$ is L-isomorphic to an algebraically closed subset C_1 of M_0. So it will suffice to take ϵ no greater that the ϵ guaranteed by (P3) in M_0 for (the finitely many possibilities of) the pair C_0, A_2.

For (P4) first note that by (P4) in M_0 and the properties (i) and (iii) of M_f in Theorem 3.5 we may assume that $y(c/B) = y(c/A)$. Let $D = \mathrm{acl}(cB)$. Then using Lemma 3.3 and (ii) of Theorem 3.5 there exists $D_1 \leq_d M_f$ and an L-isomorphism $\phi : D \to D_1$ such that $\phi \mid \mathrm{acl}(cA)$ is an L^+-isomorphism, $\phi \mid B$ is the identity, and there are no instances of relations of $L^+ \setminus L$ on D_1 except those lying in $\phi(\mathrm{acl}(cA))$ or B. Let $c_1 = \phi(c)$. Then (i), (ii) of (P4) hold, and if $C \subset B$ then $\mathrm{acl}(c_1 C) = \phi(\mathrm{acl}(cC))$, which gives (iii) of (P4).

(ii) The proof is an elaboration of the proof of Lemma 4.14 in [7].

Note that by Theorem 3.5, the type in M_f of a tuple c over a finite set X is determined by the quantifier-free type of $\mathrm{acl}(cX)$ over X. We use the same notation as in the statement of (P5). Let $\psi : \mathrm{acl}(c_1 A) \to \mathrm{acl}(c_2 A)$ be an elementary map which is the identity on A and $\psi(c_1) = c_2$. By (P4) and (P5) in M_0, there exists $E \in \mathcal{C}_0'$ and L^+-embeddings

$$\phi_{12} : \mathrm{acl}(B_1 B_2) \to E$$

$$\phi_{j3} : \mathrm{acl}(c_j B_j) \to E \text{ for } j = 1, 2$$

such that:

- $\phi_{12} \mid B_j = \phi_{j3} \mid B_j$ (for $j = 1, 2$);
- $\phi_{13}(x) = \phi_{23}(\psi(x))$ for all $x \in \mathrm{acl}(c_1 A) = B_3$;
- $E_{ij} := \mathrm{im}(\phi_{ij}) \leq_y E$;
- $E_{ij} \cap E_{kj} = E_j = \phi_{ij}(B_j)$ if $j = 1, 2$, and $= \phi_{13}(B_3)$ if $j = 3$;
- E_1, E_2, E_3 are y-independent over their intersection E_0 and if $C = \bigcup \mathrm{cl}_y(E_i E_j)$, then the E_{ij} are y-independent over C;

- any instance of an $L^+ \setminus L$-relation on E is entirely in some E_{ij}.

By Theorem 3.5 it is enough to prove:

(i) $E_{ij} \leq_d E$;

(ii) $E \in \mathcal{C}_f$;

(iii) $d(E_3/E_1E_2) = d(E_3/E_0)$.

Let $F = \bigcup E_{ij}$. Clearly $F \leq_d^* E$. Let i, j, k denote the indices $1, 2, 3$ in some order (and of course, $E_{ij} = E_{ji}$, by definition).

Claim 1. $y(F) = y(E_{ik}) + y(E_{ij} \cup E_{kj}) - y(E_i \cup E_k)$.

To see this, note that $y(F) = y(F/C) + y(C)$ and use the above y-independence properties to get $y(F) = \sum_{s<t} y(E_{st}) - \sum_{s=1}^{3} y(E_s) + y(E_0)$. But also E_{ij}, E_{kj} is a y-modular pair, as is E_i, E_j. The claim follows.

Claim 2. $E_{ij} \cup E_{kj} \leq_d^* F$.

Note that the d-independence conditions on $B_1, B_2, \mathrm{acl}(Ac_l)$ imply $E_i \cup E_k \leq_d^* E_{ik}$ (by Lemma 2.3). So the claim follows from Claim 1 and Lemma 3.4.

Claim 3. $E_{ij} \leq_d E_{ij} \cup E_{kj}$.

As $E_j \leq_d E_{kj}$ this follows from the amalgamation lemma.

Claim 4. $E_{ij} \leq_d F$.

Let $E_{ij} \subset X \leq F$. Without loss of generality, $X \cap (E_{kj} \setminus E_{ij}) \neq \emptyset$. Then by Claim 3, $d_0(E_{ij}) < d_0(X \cap (E_{ij} \cup E_{kj}))$, and by Claim 2 and Lemma 2.2 $d_0(X \cap (E_{ij} \cup E_{kj})) \leq d_0(X)$.

We can now prove (i). Let $E_{ij} \subset X \subseteq E$. As $F \leq_d^* E$ we have $d_0(X \cap F) \leq d_0(X)$. First suppose $E_{ij} \subset X \cap F$. Then by Claim 4, $d_0(E_{ij}) < d_0(X \cap F)$. On the other hand, suppose $X \cap F = E_{ij}$. As all instances in E of relations of $L^+ \setminus L$ are in F, we have $d_0(X) - d_0(E_{ij}) = y(X) - y(E_{ij})$. But $E_{ij} \leq_y E$ so we again get $d_0(X) > d_0(E_{ij})$.

(ii) Let $g = f^{-1}$. Making the substitution $t = g(x)$ the extra condition on f gives $x + \kappa(g(x)) \geq f(3g(x))$, thus $g(x + \kappa(g(x))) \geq 3g(x)$.

Now let $D \subseteq E$. We must show that $d_0(D) \geq f(y(D))$, that is, $y(D) \leq g(d_0(D))$. As in the proof of the amalgamation lemma, we may assume that $D \leq_d F$.

Let $D_{ij} = D \cap E_{ij}$ and $d_{ij} = d_0(D_{ij})$. Suppose, without loss of generality, that d_{12} is the largest of these. By the amalgamation lemma $E_{ij} \cup E_{kj}$ is a substructure of an element of $\mathcal{C}_f$ so we may assume that $D \neq D_{12} \cup D_{23}$. Thus $D \cap E_1 \subset D_{13}$. Let $D^1 = D_{12} \cup D_{13}$. By Claim 2 and Lemma 2.2 we have $D^1 \leq_d^* D$. Moreover, D^1 is a free amalgam over $D \cap E_1$ therefore:

$$d_0(D^1) = d_{12} + d_0(D_{13}/D \cap E_1) \geq d_{12} + \kappa(y(D_{13})).$$

Also,

$$y(D_{13}) \leq g(d_{13}) \leq g(d_{12}),$$

so

$$d_{12} \leq d_0(D^1) - \kappa(g(d_{12})).$$

The assumption on g implies $g(d_{12} + \kappa(g(d_{12}))) \geq 3g(d_{12})$. Thus we have:

$$y(D) \leq \sum y(D_{ij}) \leq 3g(d_{12}) \leq g(d_0(D^1)) \leq g(d_0(D)),$$

the last inequality following as $D^1 \leq_d^* D$. This completes the proof of (ii).

(iii) We leave the reader to show this. □

Remark 3.8 It is useful to have a version of this construction where the conditions on the function f are only assumed to hold from some point onwards. It is then necessary to check by hand 'small' cases of the amalgamations in Lemma 3.3 and Theorem 3.6. For example, suppose in Lemma 3.3 that we only have $f'(x) \leq \kappa(x)/x$ for $x \geq n_0$. Then for the Lemma, we also have to check that if $A \leq_d B_1, B_2 \in \mathcal{C}_f$ and $y(B_i) \leq n_0$ then $d_0(B_1) + d_0(B_2) - d_0(A) \geq f(y(B_1) + y(B_2) - y(A))$. Similarly, if the additional condition in Theorem 3.6 (ii) is only known to hold for $t \geq n_1$ then the amalgamations in the proof have to be checked to lie in $\mathcal{C}_f$ when $d_0(D) \leq n_1$.

Lemma 3.9 *Suppose M_0 is stable and for all $A \subseteq B \in \mathcal{C}_f$, if $y(A) < y(B)$ then $d_0(A) \neq d_0(B)$. Then M_f is stable.*

Proof: Note that (P1-P4) hold even without the extra condition in Theorem 3.6. We show uniqueness of d-non-forking extensions of types over algebraically closed sets. Thus (P5) holds in a very strong sense, and Corollary 2.21 shows that d-non-forking is actually non-forking. So we have stability. Moreover, note that by property (P7) (Lemma 2.18) it is enough to check the uniqueness over finite algebraically closed subsets of M_f.

So let $A \leq_d B \leq_d M_f$ be finite and c_1, c_2 have the same type over A, and be such that $d(c_i/B) = d(c_i/A)$. Let $C_i = \mathrm{acl}(c_i A)$. Then (by Lemma 2.3) $d_0(\mathrm{acl}(c_i B)) = d_0(C_i \cup B)$ and C_i, B is a d_0-modular pair. So by assumption $y(\mathrm{acl}(c_i B)) = y(C_i \cup B)$, and this equals $y(C_i) + y(B) - y(A)$. Furthermore, there are no $L^+ \setminus L$-relations on $\mathrm{acl}(c_i B)$ outside C_i and B (note that the assumption implies all α_R are non-zero). It follows that $d_0(\mathrm{cl}_y(c_i B)) = d_0(C_i) + d_0(B) - d_0(A) = d(C_i B)$. Thus, by our assumption again, $\mathrm{cl}_y(C_i B) = \mathrm{acl}(c_i B)$.

As M_0 is stable with weak elimination of imaginaries (Lemma 2.9) and quantifier elimination, it follows that the L-types of C_1 and C_2 over B are the same (they are non-forking extensions of $\mathrm{tp}^{M_0}(C_i/A)$ and A is algebraically closed in M_0). So the L-types of $\mathrm{cl}_y(C_1 B)$ and $\mathrm{cl}_y(C_2 B)$ are the same. By the construction of M_f and the above, it then follows that c_1 and c_2 have the same L^+-type over B. $\square$

Remarks 3.10 It is clear that without the extra condition in the above lemma, we do not have uniqueness of non-forking extensions of types over algebraically closed sets. As the structures have weak elimination of imaginaries, we do not have stability.

As is well-known, the extra conditions in Lemma 3.9 can be guaranteed as follows. Without loss, assume $\alpha = 1$ and let K be the subfield of $\mathbb{R}$ generated by $\{y(A) : A \subseteq M_0 \text{ finite}\}$. Let $\mathcal{R}$ be the set of new relations in $L^+ \setminus L$ and suppose that $\{1\} \cup \{\alpha_R : R \in \mathcal{R}\}$ is K-linearly independent. It follows that if $Y \subseteq X \in \mathcal{C}_0'$ and $d_0(X) = d_0(Y)$, then $R[X] = R[Y]$ for all $R \in \mathcal{R}$ and $y(X) = y(Y)$.

4 Stable forking

Recall that a simple L-theory T has *stable forking* if forking is witnessed by stable formulas. More precisely, whenever $N \models T$, $\bar{c}$ is a tuple from N, and $A \subseteq B \subseteq N$ are such that $\mathrm{tp}(\bar{c}/B)$ forks over A, there exists a *stable* $L(A)$-formula $\phi(\bar{x}, \bar{y})$ such that some $\phi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{c}/B)$ forks over A. We show that the simple L^+-theories $Th(M_f)$ constructed in Section 3 have stable forking. We first give the formulas which will witness forking; then we prove stability of these formulas (relative to $Th(M_f)$, of course). The formulas we use are just the ones which express d-independence in Lemma 2.3: we shall refer to these as *dimension formulas* for d. Using the notation of Section 3, the point is that if the dimension formulas for y are stable, then the dimension formulas for d are stable. So iterations of the construction in Section 3 starting off with a stable structure (or even a modular simple structure) result in structures with stable forking. Thus it seems that all $\aleph_0$-categorical simple structures which are currently known have stable forking.

It should be noted that Pourmahdian [11] has an alternative and far more elegant approach to proving stable forking. At present, this approach works (in a wider context) when M_0 is a pure set (i.e. y is simply cardinality). It would be very interesting to generalise these methods.

Henceforth, the notation will be as in Section 3, but we rehearse it slightly before proceeding. We have an L-structure M_0 with a predimension y (which satisfies P1-P5). Introducing new relation symbols we obtain L^+, and using a suitable auxiliary function f we define a predimension d_0, and a $\leq_d$-amalgamation class with Fraïssé limit M_f. Conditions on f ensure that d_0 on M_f satisfies (P1-P5). In particular M_f is simple and d detects forking (Theorem 3.7 and Corollary 2.21). We let $M = M_f$ and work in a large model N of $Th(M)$, as before. Note that $M_f|L \cong M_0$, so $N|L \models Th(M_0)$. Also, if a pair of finite subsets A, B is d_0-modular, then it is y-modular.

By $\aleph_0$-categoricity, algebraic closure is uniformly definable in N. Explicitly, for each $n < \omega$ there is an L^+-formula $\beta(x_1, \ldots, x_n, y)$ such that for all $a_1, \ldots, a_n, b \in N$,

$$N \models \beta(a_1, \ldots, a_n, b) \Leftrightarrow b \in \operatorname{acl}(a_1, \ldots, a_n).$$

Thus, for a finite subset X of N, and all $m, n < \omega$ there exist $L^+(X)$-formulas $\phi_1(\bar{x}, \bar{y})$, $\phi_2(\bar{x}, \bar{y})$, $\phi_3(\bar{x}, \bar{y})$ (with $\bar{x}$ and $\bar{y}$ being tuples of variables of lengths m and n respectively) such that for all m-tuples $\bar{c}$ and n-tuples $\bar{b}$ from N:

$$N \models \phi_1(\bar{c}, \bar{b}) \Leftrightarrow \operatorname{acl}(\bar{c}X) \cap \operatorname{acl}(\bar{b}X) = \operatorname{acl}(X)$$

$$N \models \phi_2(\bar{c}, \bar{b}) \Leftrightarrow \operatorname{acl}(\bar{c}X), \operatorname{acl}(\bar{b}X) \text{ is a modular pair in } N \text{ with intersection } \operatorname{acl}(X)$$

$$N \models \phi_3(\bar{c}, \bar{b}) \Leftrightarrow d_0(\operatorname{acl}(\bar{c}\bar{b}X)) = d_0(\operatorname{acl}(\bar{c}X)) + d_0(\operatorname{acl}(\bar{b}X)) - d_0(\operatorname{acl}(X)).$$

We refer to these formulas as *dimension formulas* for d (of course, we have such formulas for each parameter set X). We also use $L^+(X)$ formulas $\alpha(\bar{x}, \bar{y}, \bar{z})$ such that

$$\alpha(\bar{x}, \bar{y}, \bar{z}) \Leftrightarrow \operatorname{acl}(X\bar{x}\bar{y}) \cap \operatorname{acl}(X\bar{y}\bar{z}) = \operatorname{acl}(X\bar{y}).$$

Henceforth, we do not put bars over tuples.

The following definition is rather technical. The point of it is to isolate a property which appears in an argument in [6] (cf. Proposition 16 there).

Definition 4.1 Say that (M, d_0) is *quasi-flat* if the following holds. Suppose $(D_{ij} : j < i < \omega)$ are finite and algebraically closed and

- $D_{ij} \cap D_{ij'} = \tilde{C}_i$ is constant for distinct $j, j' < i$
- $D_{ij} \cap D_{i'j} = \tilde{B}_j$ is constant for distinct $i, i' > j$
- $D_{ij} \cap D_{i'j'} = \tilde{X}$ is constant if $i \neq i'$ and $j \neq j'$.

Moreover, assume that $d_0(\tilde{C}_i) = \tilde{c}$, $d_0(\tilde{B}_j) = \tilde{b}$, $d_0(D_{ij}) = \tilde{d}$ for $i > j$ are independent of i, j, and the same holds for the y-values of these sets. Let $\tilde{a} = d_0(\tilde{X})$. Then $\tilde{d} \geq \tilde{c} + \tilde{b} - \tilde{a}$.

Remark 4.2 A pure set and a vector space are quasi-flat.

Lemma 4.3 *If (M_0, y) is quasi-flat, then (M, d_0) is quasi-flat.*

Proof. Write $I = \{(i, j) : 1 \leq j < i \leq r\}$. If $\emptyset \neq S \subseteq I$ let $D_S = \bigcap_{(i,j)\in S} D_{ij}$. By definition of d_0:

$$\begin{aligned} d_0(\bigcup_{(i,j)\in I} D_{ij}/\tilde{X}) &= y(\bigcup D_{ij}/\tilde{X}) - \sum_R \alpha_R(|R[\bigcup D_{ij}] \setminus R[\tilde{X}]|) \\ &\leq y(\bigcup D_{ij}/\tilde{X}) - \sum_{\emptyset\neq S\subseteq I} (-1)^{|S|+1} y(D_S/\tilde{X}) + \sum_{\emptyset\neq S\subseteq I} (-1)^{|S|+1} d_0(D_S/\tilde{X}). \end{aligned} \tag{1}$$

Note that if $| S | \geq 2$ then $D_S = \tilde{C}_i$, $\tilde{B}_j$ or $\tilde{X}$. Moreover, the first happens iff all elements of S have first coordinate i; the second happens iff they all have second coordinate j. For a particular $i \geq 3$, the contribution of the first case to the sums on the right hand side of the above inequality is

$$\left(\begin{pmatrix} i-1 \\ 2 \end{pmatrix} - \begin{pmatrix} i-1 \\ 3 \end{pmatrix} + \cdots \right) = (i-2).$$

Thus the total contribution is

$$(1 + \cdots + (r-2)) = \frac{(r-1)(r-2)}{2}.$$

Likewise for the $\tilde{B}_j$.

Let $y(D_{ij}) = \delta$, $y(\tilde{B}_j) = \beta$, $y(\tilde{C}_i) = \gamma$, $y(\tilde{X}) = \alpha$. Then

$$\sum_{S\neq\emptyset} (-1)^{|S|+1} y(D_S/\tilde{X}) = \frac{r(r-1)}{2}(\delta - \alpha) - \frac{(r-1)(r-2)}{2}(\beta + \gamma - 2\alpha)$$

and

$$\sum_{S\neq\emptyset}(-1)^{|S|+1}d_0(D_S/\tilde{X}) = \frac{r(r-1)}{2}(\tilde{d}-\tilde{a}) - \frac{(r-1)(r-2)}{2}(\tilde{b}+\tilde{c}-2\tilde{a}).$$

Moreover, by submodularity

$$y(\bigcup D_{ij}/\tilde{X}) \leq r(\delta-\alpha).$$

Thus the inequality (1) gives

$$d_0(\bigcup_{(i,j)\in I} D_{ij}/\tilde{X}) \leq r(\delta-\alpha) + \frac{r(r-1)}{2}(\tilde{d}-\tilde{a}-\delta+\alpha) - \frac{(r-1)(r-2)}{2}(\tilde{c}+\tilde{b}-2\tilde{a}-\gamma-\beta+2\alpha).$$

This is quadratic in r and non-negative, thus the coefficient of r^2 on the right-hand side here is non-negative. So

$$\tilde{d}-\tilde{c}-\tilde{b}+\tilde{a} \geq \delta-\beta-\gamma+\alpha.$$

But $\delta = y(D_{ij}) \geq \beta+\gamma-\alpha$, by quasi-flatness of y. So

$$\tilde{d}-\tilde{c}-\tilde{b}+\tilde{a} \geq 0.$$

□

Lemma 4.4 *(1) If the dimension formulas for (M_0, y) are stable, then the dimension formulas for (M, d) are stable.*
(2) If also (M_0, y) is quasi-flat, then the formulas $\alpha(x, y, a)$ are stable.

Proof. (1) The proof is in 3 parts, dealing with each of the three types of formula in turn. In each part, we suppress the subscript on the formula. We will assume that X is finite, algebraically closed, and that $(c_ib_i : i \in \mathbb{Z})$ is an X-indiscernible sequence such that $\models \phi(c_i, b_j)$ if and only if $i \leq j$. Write $C_i = \mathrm{acl}(c_iX)$ and $B_i = \mathrm{acl}(b_iX)$. Indiscernibility (and finiteness of B_i, C_i) implies that we may assume that $B_i \cap B_j = B'$ and $C_i \cap C_j = C'$ are independent of i, j (if $i \neq j$), and also that $B_i \cap C_j$ is constant and is contained in $B' \cap C'$. In fact, we may assume that $C_i \cap B_j = X$ for all i, j.

Case 1: Assume $\phi = \phi_1$. Thus $C_i \cap B_j = X$ iff $i \leq j$. This is a contradiction to the above remarks.

Case 2: Assume $\phi = \phi_2$. As $C_i \cap B_i = X$, we have $C' \cap B' = X$ and so $C_i \cap B_j = X$ for all i, j. Thus C_i, B_j is a modular pair if and only if $i \leq j$. Using that C', B_j is a modular pair we get that

C_i, $C' \cup B_j$ is d_0-modular iff $i \leq j$. As the corresponding dimension formula for y is stable, C_i, $C' \cup B_j$ is y-modular for all i, j. By definition of d_0, it follows that there is some $L^+ \setminus L$-relation R with $\alpha_R > 0$, and for every $j < 1$ a realisation of R in $C_1 \cup B_j$ that is not contained in C_1 or $C' \cup B_j$. It follows that for large enough r, $C' \cup B_0 \cup \cdots \cup B_{-r} \not\leq_d C_1 \cup B_0 \cup \cdots \cup B_{-r}$. So $\mathrm{acl}(C' \cup B_0 \cup \cdots \cup B_{-r}) \cap C_1$ properly contains C'. Using indiscernibility (and finiteness of C_1) this contradicts the definition of C'.

Case 3: We are assuming (for a contradiction) that $d_0(\mathrm{acl}(C_i \cup B_j)) = d_0(C_i) + d_0(B_j) - d_0(X)$ if and only if $i \leq j$. In particular if $i \leq j$ we have that C_i, B_j is a modular pair with intersection X. Indiscernibility and the previous case then imply that if $i > j$, then C_i, B_j is also a modular pair, and $d_0(\mathrm{acl}(C_i \cup B_j)) < d_0(C_i) + d_0(B_j) - d_0(A)$.

Let $D_{ij} = \mathrm{acl}(C_i B_j)$ (for $i > j$). Indiscernibility implies that we may assume

- $D_{ij} \cap D_{ij'} = \tilde{C}_i$ is constant for distinct $j, j' < i$
- $D_{ij} \cap D_{i'j} = \tilde{B}_j$ is constant for distinct $i, i' > j$
- $D_{ij} \cap D_{i'j'} = \tilde{X}$ is constant if $i \neq i'$ and $j \neq j'$.

Moreover, we may assume that $d_0(\tilde{C}_i) = \tilde{c}$, $d_0(\tilde{B}_j) = \tilde{b}$, $d_0(D_{ij}) = \tilde{d}$ for $i > j$ are independent of i, j. Stability of the dimension formulas (for y) implies that

$$y(D_{ij}) \geq y(\mathrm{cl}_y(\tilde{C}_i \tilde{B}_j)) = y(\tilde{C}_i) + \tilde{B}_j - y(\tilde{X}).$$

So the proof of Lemma 4.3 gives that

$$d_0(\mathrm{acl}(\tilde{C}_i \tilde{B}_j)) = d_0(\tilde{C}_i) + d_0(\tilde{B}_j) - d_0(\tilde{X}) = d_0(\tilde{C}_i \cup \tilde{B}_j)$$

if $i > j$.

Now, we may assume that $(\tilde{C}_i \tilde{B}_i : i < \omega)$ is $\tilde{X}$-indiscernible. So by cases (1) and (2) $\tilde{C}_i$, $\tilde{B}_j$ is a modular pair for all i, j. Recall that we are assuming that if $i > j$ then

$$d(C_i B_j) < d_0(C_i) + d_0(B_j) - d_0(X).$$

Note also that $\mathrm{acl}(\tilde{C}_i \tilde{B}_j) = \mathrm{acl}(C_i B_j)$. Thus, if $i > j$,

$$d_0(\tilde{C}_i \tilde{B}_i) = d_0(\tilde{C}_i) + d_0(\tilde{B}_i) - d_0(\tilde{X}) = d_0(\tilde{C}_i) + d_0(\tilde{B}_j) - d_0(\tilde{X}) = d_0(\mathrm{acl}(\tilde{C}_i \tilde{B}_j))$$

$$= d_0(\mathrm{acl}(C_iB_j)) < d_0(C_i) + d_0(B_j) - d_0(X) = d_0(C_i) + d_0(B_i) - d_0(X) = d_0(C_iB_i).$$

So $d_0(\tilde{C}_i\tilde{B}_i) < d_0(C_iB_i)$. But $d(C_iB_i) = d(\tilde{C}_i\tilde{B}_i) \leq d_0(\tilde{C}_i\tilde{B}_i)$. So we have a contradiction.

(2) Suppose $((c_ib_i) : i < \omega)$ is Xa-indiscernible and such that

$$\mathrm{acl}(c_ib_jX) \cap \mathrm{acl}(b_jaX) = \mathrm{acl}(b_jX) \Leftrightarrow i \leq j.$$

Let $E_{ij} = \mathrm{acl}(c_ib_jX)$ for $i \leq j$. We may assume that $E_{ij} \cap E_{i'j} = B_j$ (for distinct i, i'), $E_{ij} \cap E_{ij'} = C_i$ (for distinct j, j') and $E_{ij} \cap E_{i'j'} = X$ (for $i \neq i'$ and $j \neq j'$) as before. By quasi-flatness (and indiscernibility) we get that

$$d(C_iB_j) = d_0(C_i) + d_0(B_j) - d_0(X) \tag{2}$$

for all $i \leq j$. (Quasi-flatness gives $d(E_{ij}) \geq d_0(C_i) + d_0(B_j) - d_0(X)$ for $i \leq j$. As $E_{ij} = \mathrm{acl}(C_iB_j)$ we have equality here.) Thus, by part (1), this holds for all i, j.

Claim: We may replace c_i, b_i by C_i, B_i (containing X) satisfying equation 2.
First, note that, with the above notation, if $i \leq j$ then $\mathrm{acl}(C_iB_j) = \mathrm{acl}(c_ib_jX) = E_{ij}$ and $B_j = E_{ij} \cap \mathrm{acl}(B_ja)$. Next, note that if $i > j$ then $\mathrm{acl}(C_iB_j) \cap \mathrm{acl}(B_ja) \neq B_j$. In fact, let $x \in (\mathrm{acl}(c_ib_jX) \cap \mathrm{acl}(b_jaX)) \setminus \mathrm{acl}(b_jX)$. Then $x \in \mathrm{acl}(C_iB_j) \cap \mathrm{acl}(B_ja)$. Take $i' < j$. As $x \notin E_{i'j}$ we have $x \notin B_j$, as required. This gives the claim.

So we may assume that $c_i = C_i$, and $b_i = B_i$, and we have equation 2 holding for c_i, b_j.

Now let $D_{ij} = \mathrm{acl}(c_ib_j)$ for $i > j$. As before we may assume that $D_{ij} \cap D_{i'j} = \tilde{B}_j$, $D_{ij} \cap D_{ij'} = \tilde{C}_i$ and $D_{ij} \cap D_{i'j'} = \tilde{X}$. By assumption, $B_j \subset D_{ij} \cap \mathrm{acl}(B_ja)$ and by indiscernibility $D_{ij} \cap \mathrm{acl}(B_ja) = D_{i'j} \cap \mathrm{acl}(B_ja)$. Thus $B_j \subset \tilde{B}_j$.

By quasi-flatness, we get that

$$d(\tilde{C}_i\tilde{B}_j) = d_0(\tilde{C}_i) + d_0(\tilde{B}_j) - d_0(\tilde{X}) \tag{3}$$

for all $i > j$. So by part (1) again, this holds for all i, j.

Thus, if $k \geq i > j$:

$$d(\tilde{C}_i\tilde{B}_j) = d(\tilde{C}_i) + d(\tilde{B}_j) - d(\tilde{X}) = d(\tilde{C}_i) + d(\tilde{B}_k) - d(\tilde{X}) = d(\tilde{C}_i\tilde{B}_k)$$

$$\geq d(c_ib_k) = d(c_i) + d(b_k) - d(X) = d(c_i) + d(b_j) - d(X).$$

So by submodularity we have equality everywhere here, and in particular $\mathrm{acl}(\tilde{C}_i\tilde{B}_k) = \mathrm{acl}(c_ib_k)$ for all i, k. So $\tilde{B}_j \subseteq E_{ij}$ for $i \leq j$, whence $\tilde{B}_j \subseteq B_j$, which is a contradiction. □

Lemma 4.5 *Let A, B be small algebraically closed subsets of N and c a tuple from N. Suppose $\mathrm{tp}(c/B)$ forks over A and $\mathrm{tp}^L(c/B)$ does not fork over A. Then there exists a dimension formula $\phi_j(x,b)$ and some $\alpha(x,b,a)$ (where the parameters X, a are from A, and b is from B), such that $\alpha(x,b,a) \wedge \neg\phi_j(x,b) \in \mathrm{tp}(c/B)$ forks over A.*

Proof. We have that $\mathbf{cb}(c/B) \not\subseteq A$, so there is a finite $Y \subseteq B$ with $\mathrm{acl}(cY) \cap B = Y$ such that $\mathbf{cb}(c/Y) \not\subseteq A$. Let $X = A \cap Y$, and let b enumerate Y. Then $\mathrm{tp}(c/Xb)$ forks over X.

It follows that $\alpha(x,b,a) \in \mathrm{tp}(c/B)$ for all tuples a from A, and $\neg\phi_j(x,b) \in \mathrm{tp}(c/B)$ for some $j = 1,2,3$.

Note that as $\mathrm{acl}(cb) \cap B = Y$ we have $\mathrm{acl}(cb) \cap A = X$. Thus (working temporarily in the reduct to L) $\mathrm{acl}^y(cY) \cap B = Y$, whence $\mathbf{cb}^y(c/Y) \subseteq A \cap Y = X$, so $\mathrm{tp}^L(c/Ab)$ does not fork over A.

Let $(b_i : i < \omega)$ be an A-indiscernible Morley sequence of $\mathrm{tp}(b/A)$ with $b = b_0$.

Claim: For $j = 1,2,3$

$$\bigwedge_{i<\omega} \left(\bigwedge_a \alpha(x,b_i,a) \wedge \neg\phi_j(x,b_i) \right)$$

is inconsistent.

Suppose not, and c' is a realisation. Let $C = \mathrm{acl}(c'X)$, $Y_i = \mathrm{acl}(b_i)$. Thus $\mathrm{acl}(CY_i) \cap \mathrm{acl}(AY_i) = Y_i$ for all i. We consider the cases $j = 1,2,3$ in turn.

Case $j = 1$ Thus $C \cap Y_i \neq X$. By independence of the b_i over A, $\mathrm{acl}(AY_i) \cap \mathrm{acl}(AY_j) = A$ if $i \neq j$, so $Y_i \cap Y_j \subseteq A \cap Y_i = X$. But as C is finite, we may assume that $C \cap Y_i = C \cap Y_j$ (for all i, j). Thus $C \cap Y_i = X$, a contradiction.

Case $j = 2$ By the above, $C \cap Y_i = X$, but we are assuming $d_0(C/Y_i) < d_0(C/X)$. Now, $y(C/Y_0) = y(C/X)$. As in the proof of Case (2) of the previous lemma, we get that $\mathrm{acl}(B_{i_1} \dots B_{i_r}) \cap C \neq X$ if r is large enough, and as C is finite, we may assume this intersection is constant. By independence over A, $\mathrm{acl}(AB_1 \dots B_r) \cap \mathrm{acl}(AB_{r+1} \dots B_{2r}) = A$, thus

$$\mathrm{acl}(B_1 \dots B_r) \cap C \subseteq C \cap A = X.$$

This is a contradiction.

Case $j = 3$ By the above, C, Y_i is a modular pair with intersection X. By assumption, there is $\gamma > 0$ such that $d(C/Y_i) = d(C/X) - \gamma$. Fix $r \in \omega$. Let $Y' = \bigcup_{i<r} Y_i$, $D_i = \mathrm{acl}(CY_i)$ and

$D' = \bigcup_{i<r} D_i$. Note that $D_0 \cap Y_i \subseteq A$ so $D_0 \cap Y_i = X$ if $i > 0$. It follows that $d_0(D_0/Y'C) \leq d_0(D_0/Y_0C)$, by submodularity.

Then:

$$d_0(D'/Y') = d_0(D'/Y'C) + d_0(C/Y') \leq rd_0(D_0/Y'C) + d_0(C/Y') \leq rd_0(D_0/Y_0C) + d_0(C/Y').$$

As Y_0, C is modular, we get $d_0(D'/Y') \leq d_0(C/Y') - r\gamma$.

So if r is large enough, $\mathrm{acl}(Y') \cap D_0 \neq Y_0$. We may assume (by finiteness of D_0) that $\mathrm{acl}(b_0 \ldots b_{r-1}) \cap D_0 = \mathrm{acl}(b_0 b_r \ldots b_{2r-2}) \cap D_0$, and so this is contained in $\mathrm{acl}(Ab_0) \cap D_0$. We obtain $\mathrm{acl}(AY) \cap \mathrm{acl}(Yc) \neq Y$, and this is a contradiction, as $\mathrm{acl}(Yc) \cap B = Y$.

This proves the claim. So by compactness, there is some $j = 1, 2, 3$ and some a (containing X) such that $\alpha(x, b, a) \wedge \neg\phi_j(x, b) \in \mathrm{tp}(c/B)$ and which forks over A. $\square$

Putting these together, we have:

Theorem 4.6 *Suppose (M_0, y) is quasi-flat, has stable dimension formulas and has stable forking. Then the same is true of (N, d).* $\square$

5 Further observations

5.1 A rank 1 set without weak elimination of imaginaries

We use the construction of Section 3 to produce an $\aleph_0$-categorical graph M which has SU-rank 2 (this is essentially the unstable pseudoplane from the end of [4]). The point is that this has a definable subset C which, when given the full induced substructure, has SU-rank 1 and does not have weak elimination of imaginaries. In fact, weak elimination of imaginaries requires the inclusion of elements of C^{eq} of SU-rank 2. This is a phenomenon which does not arise in the ω-stable case (or the smoothly approximated case).

In the notation of Section 3, let M_0 be a pure set with $y(X)$ the cardinality of X (for $X \subseteq M_0$). The language L^+ has one additional binary relation symbol R and $\mathcal{C}_0'$ is the class of finite (loopless, symmetric) graphs. The predimension d_0 is given by $d_0(B) = 2y(B) - \mid R[B] \mid /2$ (i.e. twice the number of vertices, minus the number of edges). Let $f(0) = 0$, $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 4.5$ and for $x \geq 5$, let $f(x) = \log_3(x) + 5 - \log_3(5)$. (We do not have $f'(x) \leq 1$ here, but this condition is only used in the proof if algebraic closure in M_0 is non-trivial.) Then using Theorem

3.6 and Remarks 3.8 one can check that $M = M_f$ has properties (P1-P5). It then follows that M is supersimple of SU-rank 2. Moreover, because of the choice of initial vaues of f, the smallest cycle in M is a pentagon, and points and adjacent pairs of points are algebraically closed in M. So by homogeneity of M for algebraically closed sets, $\mathrm{Aut}(M)$ is vertex and edge-transitive.

Claim 1: M is of diameter 3.

If a, b are vertices at distance greater than 2, then $\{a, b\} \leq M$. Also if Y is a path with 4 vertices having endpoints a and b, then $\{a, b\} \leq Y \in \mathcal{C}_f$. So by Theorem 3.5 (iii), there is a path of length 3 from a to b in M.

Claim 2: $\mathrm{Aut}(M)$ has 2 orbits on pairs of non-adjacent vertices.

Homogeneity gives that $\mathrm{Aut}(M)$ is transitive on pairs of vertices at distance 3. Suppose a, b are at distance 2. Then there is a unique vertex c adjacent to both a and b, as M contains no 4-cycle. Easily, $\{a, b, c\} = \mathrm{acl}(ab)$, and the claim then follows from the homogeneity.

Now let a be any vertex and C the set of vertices of M adjacent to a. Note that $SU(c/a) = 1$ for all $c \in C$. Consider the restriction map $\mu : \mathrm{Aut}(M/a) \to \mathrm{Sym}(C)$.

Claim 3: μ is one-to-one and has closed image.

Let D and E be the sets of vertices at distance 2 and 3 from a. As in Claim 2, $\mathrm{Aut}(M/a)$ is transitive on each of these, and on C. Suppose $\gamma \in \mathrm{Aut}(M/a)$ fixes every element of $C \cup D$. If $e \in E$, there exist distinct $d, d' \in D$ which are adjacent to e. But then e is the unique vertex adjacent to both d and d', so γ fixes e.

To show that μ is one-to-one and the image of μ is closed it is therefore enough to show that $D \cap \mathrm{dcl}(a, c_1, \ldots, c_n) \neq \emptyset$ for some $c_1, \ldots, c_n \in C$. There is an algebraically closed subgraph A of M with vertex set

$$\{a\} \cup \{c_1, \ldots, c_{10}\} \cup \{d_1, \ldots, d_{10}\}$$

such that $c_i \in C$ and $d_i \in D$ and the other adjacencies are given by

$$R(c_i, d_i), R(d_i, d_{i+1 \pmod{10}}), R(d_1, d_5), R(d_4, d_8).$$

One computes that $d_0(ac_1 \ldots c_{10}) = 12$, $d_0(A) = 10$. Thus the algebraic closure of $a, c_1, \ldots, c_{10}$ contains a point of $d_1, \ldots, d_{10}$, which clearly lies in the definable closure.

Thus the canonical structure on C induced by $\mathrm{Aut}(M/a)$ is biinterpretable with (M, a), and is clearly supersimple of SU-rank 1. On the other hand, C does not have weak elimination of imaginaries: the elements of (M, a) can be regarded as living in C^{eq} and $\mathrm{acl}(ad_1) \cap C = c_1$, whereas $d \notin \mathrm{acl}(ac_1)$.

5.2 Growth rates of algebraic closure

It is shown in [3] that Hrushovski's construction yields (multiply transitive) $\aleph_0$-categorical structures with arbitrarily fast growth rates of algeraic closure. Whilst it seems clear (using Theorem 3.6) that that structures in ([3], Theorems 1 and 2) can be taken to be supersimple, there appear to be significant difficulties in using the construction to exhibit stable $\aleph_0$-categorical structures with arbitrarily fast growth rates of algebraic closure (see [9] for some attempts to do this). Although it may transpire that these are only techincal difficulties, the following easy observations show that (as far as these constructions are concerned) stability does impose some restrictions.

Definition 5.1 Let k be a natural number, M a structure and X a subset of M. The *k-ary algebraic closure of X* (in M), denoted by $\mathrm{acl}^k(X)$, is the smallest subset of M containing X which is closed under taking algebraic closure of k-element subsets. Clearly this is a well-defined closure operation, and $\mathrm{acl}^k(X) \subseteq \mathrm{acl}(X)$.

The construction in ([3], Section 2.1) shows that k-ary algebraic closure can grow arbitrarily quickly in supersimple $\aleph_0$-categorical structures (this does not appear to follow from the simpler argument of ([3], Section 3). With the notation of Section 3, the following shows that growth of k-ary algebraic closure in the structure M_f of Lemma 3.9 is constrained by growth of algebraic closure in M_0.

Lemma 5.2 *Assume the notation and hypotheses of Lemma 3.9. Suppose the function ϕ is such that the algebraic closure of any n points in M_0 has size at most $\phi(n)$. Then there exists a constant c such that for all finite subsets A of M_f we have $\mid \mathrm{acl}^k(A) \mid \leq \phi(c \mid A \mid)$.*

Proof. Let ϵ_k be the minimum value for M_f of $d_0(X') - d_0(\mathrm{acl}(X))$ where $\mid X \mid \leq k$, $X \subseteq X' \subseteq \mathrm{acl}(X)$ and $y(X') < y(\mathrm{acl}(X))$. Note that $\epsilon_k > 0$.

Write $\mathrm{acl}^k(A) = \bigcup_{i=0}^{r} A_i$ where $A_0 = A$ and $A_{i+1} = \mathrm{cl}_y(A_i \cup \mathrm{acl}(X_i))$ for some $X_i \subseteq A_i$ of size k, and assume that $\mathrm{acl}(X_i) \not\subseteq A_i$ for $i < r$. Then by definition of d_0,

$$d_0(A_{i+1}) \leq d_0(A_i \cup \mathrm{acl}(X_i)) \leq d_0(A_i) + d_0(\mathrm{acl}(X_i)) - d_0(A_i \cap \mathrm{acl}(X_i)).$$

Moreover $A_i \cap \operatorname{acl}(X_i)$ and $\operatorname{acl}(X_i)$ are y-closed, so $y(A_i \cap \operatorname{acl}(X_i)) < y(\operatorname{acl}(X_i))$. Thus $d_0(A_i \cap \operatorname{acl}(X_i)) \geq d_0(\operatorname{acl}(X_i)) + \epsilon_k$, whence $d_0(A_{i+1}) \leq d_0(A_i) - \epsilon_k$. In particular, as $d_0(A_r) \geq 0$ we have $r \leq d_0(A_0)/\epsilon_k$.

Note that $A_{i+1} = \operatorname{cl}_y(A_0 \cup \operatorname{acl}(X_0) \cup \cdots \cup \operatorname{acl}(X_i))$, so $\mid \operatorname{acl}^k(A) \mid \leq \phi(\mid A \mid + rt)$, where t bounds the size of the algebraic closure of k points in M_f. As $d_0(A)$ is bounded by a constant multiple of $\mid A \mid$ (by submodularity) we get the required result. □

References

[1] John T. Baldwin and Niandong Shi, 'Stable generic structures', Annals of Pure and Applied Logic 79 (1996), 1–35.

[2] Andreas Baudisch and Anand Pillay, 'A free pseudospace', J. Symbolic Logic 65 (2000), 443–463.

[3] David M. Evans and M. E. Pantano, '$\aleph_0$-categorical structures with arbitrarily fast growth of algebraic closure', Preprint, Norwich, 1996.

[4] Ehud Hrushovski, 'A stable $\aleph_0$-categorical pseudoplane', Unpublished notes, 1988.

[5] Ehud Hrushovski, 'Pseudofinite fields and related structures', Unpublished notes, 1991.

[6] Ehud Hrushovski, 'A new strongly minimal set', Annals of Pure and Applied Logic 62 (1993), 147–166.

[7] Ehud Hrushovski, 'Simplicity and the Lascar group', Preprint, December 1997.

[8] Byunghan Kim and Anand Pillay, 'Simple theories', Annals of Pure and Applied Logic, 88 (1997), 149–164.

[9] M. E. Pantano, Algebraic Closure in $\aleph_0$-Categorical Structures, Ph. D. Thesis, UEA Norwich, 1995.

[10] Anand Pillay, 'A note on CM-triviality and the geometry of forking', J. Symbolic Logic 65 (2000), 474–480.

[11] Massoud Pourmahdian, D. Phil. Thesis, Oxford, 2000.

[12] Frank O. Wagner, 'Relational structures and dimensions', in Automorphisms of First-Order Structures, R. Kaye and D. Macpherson (eds.), Oxford University Press, 1994, pp. 153–180.

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