Amalgamation Constructions in Permutation Group Theory and Model Theory

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Ambleside, August 2007.

Amalgamation class method

Input: Amalgamation class

Class $\mathcal{C}$ of (finite) structures and a 'distinguished' notion of substructure $A \leq B$ ('A is a self-sufficient substructure of B')

Output: Fraïssé limit

Countable structure M whose automorphism group is $\leq$-homogeneous: any isomorphism between finite $A_1, A_2 \leq M$ extends to an automorphism of M.

Structure: graphs, digraphs, orderings, groups, ...
Substructure: full induced substructure

Overview

- Describe general method
- Focus on two basic examples: 2-out digraphs and an example of a Hrushovski construction
- Mention how variations on these basic examples give some interesting infinite permutation groups and combinatorial structures
- Connection between the 2-out digraphs and the Hrushovski construction
- Connection via matroids

Amalgamation classes

$(\mathcal{C}, \leq)$ is an amalgamation class if

- $\mathcal{C}$ has countably many isomorphism types
- if $A \leq B \leq C$ then $A \leq C$
- $\emptyset \leq A$ and $A \leq A$ for all $A \in \mathcal{C}$
- Amalgamation Property: if $f_1 : A \xrightarrow{\leq} B_1$ and $f_2 : A \xrightarrow{\leq} B_2$ are in $(\mathcal{C}, \leq)$ there exist $C \in \mathcal{C}$ and $\leq$-embeddings $g_1 : B_1 \xrightarrow{\leq} C$ and $g_2 : B_2 \xrightarrow{\leq} C$ with $g_1 \circ f_1 = g_2 \circ f_2$.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f_1\ } & B_1 \\
{\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\
B_2 & \xrightarrow{\ g_2\ } & C
\end{array}
$$

The Fraïssé limit

Theorem (Fraïssé, Jónsson, Shelah, Hrushovski ...)

Suppose $(\mathcal{C}, \leq)$ is an amalgamation class. Then there is a countable structure M such that:

1. *M is a union of a chain of finite substructures each in $\mathcal{C}$:*

$$M_1 \leq M_2 \leq M_3 \leq \ldots$$

2. *whenever $A \leq M_i$ and $A \leq B \in \mathcal{C}$ there is $j > i$ and a $\leq$-embedding $f : B \to M_j$ which is the identity on A*

3. *any element of $\mathcal{C}$ is a $\leq$-substructure of some M_i.*

Moreover
M is determined up to isomorphism by these conditions and any isomorphism between finite $\leq$-substructures of M extends to an automorphism of M.

M is called the Fraïssé limit of $(\mathcal{C}, \leq)$.

The simplest example

In the original Fraïssé construction $\mathcal{C}$ is a class of relational structures and is closed under substructures; $\leq$ is just $\subseteq$.

Example: let $\mathcal{C}$ be the class of all finite graphs.

AP: take C to be the disjoint union of B_1 and B_2 over A with edges just those in B_1 or B_2 . (The free amalgam.)

The Fraïssé limit of this amalgamation class is the random graph: it is the graph on vertex set $\mathbb{N}$ which you get with probability 1 by choosing independently with fixed probability p ($\neq 0, 1$) whether each pair $\{i, j\}$ is an edge.

2-out digraphs

We work with the class $\mathcal{D}$ of finite, simple, loopless directed graphs (digraphs) in which all vertices have at most two successors.
If B is one of these and $X \subseteq B$ then we write $\mathrm{cl}'_B(X)$ for the closure of X in B under taking successors and write $X \sqsubseteq B$ if $X = \mathrm{cl}'_B(X)$. Note that this closure is *disintegrated*:

$$\mathrm{cl}'_B(X) = \bigcup_{x \in X} \mathrm{cl}'_B(\{x\})$$

.

Properties of $(\mathcal{D}, \sqsubseteq)$

Let $\mathcal{D}$ be the class of 2-out digraphs. The following is just a matter of checking the definitions:

LEMMA: For $D, E \in \mathcal{D}$ we have:

(i) If $C \sqsubseteq D$ and $X \subseteq D$ then $C \cap X \sqsubseteq X$.

(ii) If $C \sqsubseteq D \sqsubseteq E$ then $C \sqsubseteq E$.

(iii) (Full Amalgamation) Suppose $D, E \in \mathcal{D}$ and C is a sub-digraph of both D and E and $C \sqsubseteq E$. Let F be the disjoint union of D and E over C (with no other directed edges except those in D and E). Then $F \in \mathcal{D}$ and $D \sqsubseteq F$.

We refer to F in the above as the *free amalgam* of D and E over C.

The Fraïssé Limit

PROPOSITION: There exists a countably infinite digraph N satisfying the following properties:

(D1): N is the union of a chain of finite sub-digraphs $C_1 \sqsubseteq C_2 \sqsubseteq C_3 \sqsubseteq \cdots$ all in $\mathcal{D}$.

(D2): If $C \sqsubseteq N$ is finite and $C \sqsubseteq D \in \mathcal{D}$ is finite, then there is an embedding $f : D \to N$ which is the identity on C and has $f(D) \sqsubseteq N$. Moreover, N is uniquely determined up to isomorphism by these two properties and is $\sqsubseteq$-homogeneous (i.e. any isomorphism between finite closed subdigraphs extends to an automorphism of N).

We refer to N given by the above as the Fraïssé limit of the amalgamation class $(\mathcal{D}, \sqsubseteq)$.

A Hrushovski predimension

Work with undirected, loopless graphs. If A is a finite graph we let $e(A)$ be the number of edges in A and define the predimension

$$\delta(A) = 2|A| - e(A).$$

Let $\mathcal{G}$ be the class of finite graphs B in which $\delta(A) \geq 0$ for all $B \subseteq A$. If $A \subseteq B \in \mathcal{C}$ we write $A \leq B$ and say that A is self-sufficient in B if $\delta(A) \leq \delta(B')$ whenever $A \subseteq B' \subseteq B$.
Note that we can express the condition that $A \in \mathcal{G}$ by saying $\emptyset \leq A$.

Properties of $\leq$

Submodularity: If B, C are finite subgraphs of a graph D then

$$\delta(B \cup C) \leq \delta(B) + \delta(C) - \delta(B \cap C).$$

Moreover there is equality here iff B, C are freely amalgamated over $B \cap C$ (i.e. there no adjacencies between $B \setminus C$ and $C \setminus B$).

LEMMA: We have:

(i) If $A \leq B$ and $X \subseteq B$ then $A \cap X \leq X$.

(ii) If $A \leq B \leq C$ then $A \leq C$.

(iii) (Full amalgamation) Suppose $A, B \in \mathcal{G}$ and C is a subgraph of A and B and $C \leq B$. Let D be the disjoint union (i.e. free amalgam) of A and B over C. Then $D \in \mathcal{G}$ and $A \leq D$.

Self-sufficient closure

(i) and (ii) here imply that if $A, B \le C$ then $A \cap B \le C$. Thus for every $X \subseteq B$ there is a smallest self-sufficient subset of B which contains X. Denote this by $\mathrm{cl}_B(X)$: the self-sufficient closure of X in B.

Note that cl is *not* disintegrated.

The Fraïssé limit

THEOREM: There is a countably infinite graph M satisfying the following properties:

(G1): M is the union of a chain of finite subgraphs
$B_1 \leq B_2 \leq B_3 \leq \cdots$ all in $\mathcal{G}$.

(G2): If $B \leq M$ is finite and $B \leq C \in \mathcal{G}$ is finite, then there is an embedding $f : C \to M$ which is the identity on B and has $f(C) \leq M$.

Moreover, M is uniquely determined up to isomorphism by these two properties and is $\leq$-homogeneous (i.e. any isomorphism between finite self-sufficient subgraphs extends to an automorphism of M).

Applications to permutation groups

Many nice applications of the original Fraïssé method (particularly by Peter Cameron) in the 1980's. The following require the more general method (with an extra twist).

Unbalanced primitive groups

Theorem (DE, 2001)

There is a countable digraph having infinite in-valency and finite out-valency whose automorphism group is primitive on vertices and transitive on directed edges. (It can be taken to be highly arc-transitive.)

- $\mathcal{C}$ is a collection of finitely generated 2-out digraphs with descendant set a 2-ary tree; $\leq$ is descendent closure + ...
- Daniela Amato (D Phil Thesis, Oxford 2006): Construct other examples where the descendant set is not a tree.
- Josephine de la Rue (UEA, 2006): Construct $2^{\aleph_0}$ examples where the descendant set is the 2-ary tree.

Exotic combinatorial structures

... constructed using variations on the Hrushovski construction include:

- (John Baldwin, 1994) New projective planes
- (Katrin Tent, 2000) For all $n \geq 3$, thick generalised n-gons with automorphism group transitive on $(n+1)$-gons.
- (DE, 2004) An $2 - (\aleph_0, 4, 1)$ design with a group of automorphisms which is transitive on blocks and has 2 orbits on points.

Forgetting the direction

We have a countable directed graph *N* and a countable graph *M* obtained as Fraïssé limits of the amalgamation classes $(\mathcal{D}, \sqsubseteq)$ and $(\mathcal{G}, \leq)$.

THEOREM:

If we forget the direction on the edges in *N*, the resulting graph is isomorphic to *M*.

Thus *M* is a reduct of *N*.

Orientation

Suppose A is a graph.

A $\mathcal{D}$-orientation of A is a directed graph $A^+ \in \mathcal{D}$ with the same vertex set as A and such that if we forget the direction on the edges, we obtain A.

We say that $A_1, A_2 \in \mathcal{D}$ are equivalent if they have the same vertex set and the same graph-reduct (i.e. they are $\mathcal{D}$-orientations of the same graph).

Two lemmas

The theorem follows from two lemmas.

LEMMA A:

(1) Suppose B is a finite graph. Then $B \in \mathcal{G}$ iff B has a $\mathcal{D}$-orientation.
(2) If $B \in \mathcal{G}$ and $A \subseteq B$, then $A \leq B$ iff there is a $\mathcal{D}$-orientation of B in which A is closed.

LEMMA B:

(1) If $C \sqsubseteq D \in \mathcal{D}$ and we replace the digraph structure on C by an equivalent structure $C' \in \mathcal{D}$, then the resulting digraph D' is still in $\mathcal{D}$.
(2) If $A \leq B \in \mathcal{C}$ then any $\mathcal{D}$-orientation of A extends to a $\mathcal{D}$-orientation of B.

Ternary structures

Work with finite 3-uniform hypergraphs *A*. Define:

- $e(A)$: the number of hyperedges in *A*
- Predimension: $\delta(A) = |A| - e(A)$.
- $A \leq B$: $\delta(A') \geq \delta(A)$ for all $A \subseteq A' \subseteq B$
- $\mathcal{T}$: the class of *A* which satisfy $\emptyset \leq A$.

Then $(\mathcal{T}, \leq)$ is an amalgamation class; call the Fraïssé limit *H*.

If $X \subseteq A \in \mathcal{T}$ define its dimension to be:

$$d_A(X) = \delta(\mathrm{cl}_A(X)).$$

This gives the rank function of a matroid on *A*.

Note: If $X \subseteq A \leq B$ then $d_B(X) = d_A(X)$.

Matroids

... aka 'Pregeometries.'

Definition

A matroid $\mathbb{M} = (E, \mathcal{I})$ consists of a finite set E and a non-empty collection $\mathcal{I}$ of subsets of E which is closed under subsets and satisfies:

If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ there is $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

The sets in $\mathcal{I}$ are called the independent subsets of $\mathbb{M}$.

Example

Take E a finite set of vectors in some vector space and $\mathcal{I}$ the linearly independent subsets of E.

More definitions

If $A \subseteq E$, a basis of A is a maximal independent subset of A.

The rank of A is the size of a basis of A

Transversal matroids

- *E* finite set
- $\mathcal{A} = (A_i : i \in S)$ family of non-empty subsets of *E*
- Transversal of $\mathcal{A}$: image of an injection $\psi : S \to E$ with $\psi(i) \in A_i$
- Partial transversal: transversal of a subfamily of $\mathcal{A}$.

Theorem (Edmonds and Fulkerson, 1965)

Let $\mathcal{I}$ be the set of partial transversals of $\mathcal{A}$. Then $(E, \mathcal{I})$ is a matroid. (The transversal matroid associated to the family $\mathcal{A}$.)

Matroid dual

If $\mathbb{M} = (E, \mathcal{I})$ is a matroid, let:

$$\mathcal{J} = \{C \subseteq E \setminus B : \ B \text{ a basis of } \mathbb{M}\}.$$

Theorem (Whitney, 1935)

$\mathbb{M}^* = (E, \mathcal{J})$ *is a matroid.*
(The dual matroid of $\mathbb{M}$.)

– The bases of $\mathbb{M}^*$ are complements of bases of $\mathbb{M}$.

OBSERVATION: Let $A \in \mathcal{T}$ and $\mathbb{M}$ the transversal matroid coming from the family of hyperedges in A. Then the dimension function in $\mathbb{M}^*$ is the Hrushovski dimension function d_A.

– So the matroids coming from the Hrushovski predimension are cotransversal matroids.