Geometric properties of forking in stable theories

David Evans

School of Mathematics, UEA, Norwich.

Stability and independence

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L countable language;
T complete first-order L-theory;
$\mathbb{M}$ monster model of T;
$a, b, c \ldots$ elements or tuples from $\mathbb{M}$ (or $\mathbb{M}^{eq}$);
$A, B, C, \ldots$ small subsets of $\mathbb{M}$ (or $\mathbb{M}^{eq}$).

Assume T is stable: there exists $\lambda \geq \aleph_0$ such that $|S_1(A)| \leq \lambda$ when $|A| \leq \lambda$.

Write $c \mathop{\smile\!\!\!\!\!\!|}_A B$ to mean:
Suppose $\phi(x, y) \in L(A)$ and $\phi(x, b) \in \mathrm{tp}(c/A \cup B)$. Suppose $(b_i : i < \omega)$ is an infinite A-indiscernible sequence of $\mathrm{tp}(b/A)$. Then $\bigwedge_i \phi(x, b_i)$ is consistent.

Say that $\mathrm{tp}(c/A \cup B)$ does not fork over A, or c is independent from B over A.
REMARK: This is really non-dividing... .

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Examples:

(1) Let $T = ACF_p$. Then $c \mathop{\smile\hspace{-0.9em}\mid}_A B \Leftrightarrow \text{tr.deg}(c/A \cup B) = \text{tr.deg}(c/A)$.

(2) Let $T_{V(K)}$ be the theory of (infinite) vector spaces over a field K. This is stable and for subspaces C, B of $\mathbb{M}$ we have $C \mathop{\smile\hspace{-0.9em}\mid}_{C \cap B} B$.

(3) L: 2-ary relation symbol R
T_D: directed graphs; each vertex has one directed edge going out, infinitely many coming in; no (undirected) cycles.
T_D is complete and stable.
Write $A \sqsubseteq \mathbb{M}$ to mean: if $a \in A$ and $a \to b$ then $b \in A$.
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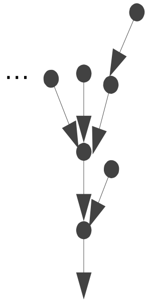

Forking independence

THEOREM. (Shelah) The following properties hold for stable T:

(0) if $g \in \mathrm{Aut}(\mathbb{M})$: $c \mathop{\smile\!\!\!\!\!\!|}_A B \Leftrightarrow gc \mathop{\smile\!\!\!\!\!\!|}_{gA} gB$;

(1) for $A \subseteq B \subseteq C$: $c \mathop{\smile\!\!\!\!\!\!|}_A C \Leftrightarrow c \mathop{\smile\!\!\!\!\!\!|}_A B$ and $c \mathop{\smile\!\!\!\!\!\!|}_B C$;

(2) $c \mathop{\smile\!\!\!\!\!\!|}_A b \Leftrightarrow b \mathop{\smile\!\!\!\!\!\!|}_A c$;

(3) if $c \not\mathop{\smile\!\!\!\!\!\!|}_A B$ there is a finite $B_0 \subseteq B$ with $c \not\mathop{\smile\!\!\!\!\!\!|}_A B_0$;

(4) there is a countable $A_0 \subseteq A$ with $c \mathop{\smile\!\!\!\!\!\!|}_{A_0} A$;

(5) given c and $A \subseteq B$ there is $c' \models \mathrm{tp}(c/A)$ with $c' \mathop{\smile\!\!\!\!\!\!|}_A B$;

(6) $c \mathop{\smile\!\!\!\!\!\!|}_A c \Leftrightarrow c \in \mathrm{acl}(A)$;

(7) given c and $A \subseteq B$ there are $\leq 2^{\aleph_0}$ possibilities for $\mathrm{tp}(c'/B)$ with $c' \models \mathrm{tp}(c/A)$ and $c' \mathop{\smile\!\!\!\!\!\!|}_A B$.

These properties characterise stability and $\mathop{\smile\!\!\!\!\!\!|}$.

This extends to $\mathbb{M}^{eq}$ and we have:

(7′) If A is algebraically closed in $\mathbb{M}^{eq}$ and $B \supseteq A$ then $\mathrm{tp}(c/A)$ has a unique non-forking extension to a type over B.

Triviality and one-basedness

Properties that mean that $\mathop{\smile\!\!\!\!\!\!\;|}$ is 'uncomplicated':

DEFINITION:

(1) T is one-based if whenever $C, B \subseteq \mathbb{M}^{eq}$ are algebraically closed (in $\mathbb{M}^{eq}$) then $C \mathop{\smile\!\!\!\!\!\!\;|}_{C \cap B} B$.

(2) T is trivial if whenever $a \mathop{\smile\!\!\!\!\!\!\;|}_A b$ and $c \not\mathop{\smile\!\!\!\!\!\!\;|}_A a, b$, then $c \not\mathop{\smile\!\!\!\!\!\!\;|}_A a$ or $c \not\mathop{\smile\!\!\!\!\!\!\;|}_A b$.

EXAMPLES:

- ACF_p is neither trivial nor one-based.
- $T_{V(K)}$ is one-based but not trivial: take linearly independent a, b, then $a + b \not\mathop{\smile\!\!\!\!\!\!\;|} a, b$ but $a + b \mathop{\smile\!\!\!\!\!\!\;|} a$ and $a + b \mathop{\smile\!\!\!\!\!\!\;|} b$.
- T_D is one-based and trivial.

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Pseudoplanes

THEOREM: (Pillay, Zilber, Lachlan) T is not one-based iff there is a *complete type definable pseudoplane I* in $\mathbb{M}^{eq}$.

This means: $I = I(x, y)$ is a complete type (over some parameter set) such that:

1. if $\models I(a, b)$ then $a \notin \mathrm{acl}(b)$ and $b \notin \mathrm{acl}(a)$ (over the parameters);
2. if $\models I(a, b_1) \wedge I(a, b_2) \wedge (b_1 \neq b_2)$ then $a \in \mathrm{acl}(b_1, b_2)$;
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IDEA: If $\models I(a, b)$ think of a as a point and b as a line (or curve) and I as incidence. The axioms have a geometric translation.

EXAMPLE: (Free pseudoplane) Let T_U be the undirected version of T_D and $I = \mathrm{tp}(a, b/\emptyset)$ where (a, b) is an edge. This is a type definable pseudoplane, so T_U is not one-based. (It is trivial.)

REMARK: (Hodges) Note that we can view T_U as a reduct of T_D: pass to the definable relation $R(x, y) \vee R(y, x)$. So a reduct of a one-based theory is not necessarily one-based.

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The following is due to A. Pillay ($+$ modification by H. Nübling):

DEFINITION: Suppose $n \geq 1$ is a natural number. Say that T is n-ample if there exist A and $c_0, \ldots, c_n$ in $\mathbb{M}$ such that:

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- Free pseudospace: A. Baudisch and A. Pillay (2000) define a free psudospace: a 3-sorted structure consisting of points, lines, planes. They show that its theory T_{BP} is ω-stable, trivial and 2-ample.
- Free n-space: Two recent papers (K. Tent; 2012) and (A. Baudisch, A. Martin-Pizarro, M. Ziegler; 2012) generalize the construction of a free pseudspace to construct a free n-space: an $(n+1)$-sorted structure consisting of points, lines, planes, ... They show that its theory is ω-stable, trivial, n-ample and not $(n+1)$-ample (so the ampleness hierarchy is strict).

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Avoiding triviality: 1-ampleness

The Hrushovski construction gives non-trivial, 1-ample structures.

Obtain this in a way which relates it to the free pseudoplane.

Define:

$\mathcal{D}$: directed graphs with at most 2 directed edges out of each vertex;

$A \sqsubseteq B$: if $a \to b$ and $a \in A$ then $b \in A$.

Then $(\mathcal{D}, \sqsubseteq)$ has the full amalgamation property: if $A \sqsubseteq B \in \mathcal{D}$ and $A \subseteq C \in \mathcal{D}$ then the free amalgam $E = C \coprod_A B$ is in $\mathcal{D}$ and $C \sqsubseteq E$.

Can form a rich structure N for $(\mathcal{D}, \sqsubseteq)$: if $A \sqsubseteq B \in \mathcal{D}$ are fg and $A \sqsubseteq N$ there is an embedding $g : B \to N$ with $g|A = id$ and $g(B) \sqsubseteq N$.

Write down a theory T such that

1. $N \models T$
2. every sufficiently saturated model of T is rich.

It follows that T is complete and stable. Moreover if $C, B \sqsubseteq N$ then $C \mathop{\smile\!\!\!\!\!\mid}_{C \cap B} B$, so T is one-based and trivial.

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The following connects this with Hrushovski's original 1988 construction.

Let M be the undirected reduct of N and $T^- = Th(M)$.

THEOREM (DE; 2005): T^- is the theory of the (uncollapsed) Hrushovski structure with predimension $\delta(X) = 2|X| - e(X)$. In particular, it is ω-stable, non-trivial, 1-ample and not 2-ample.

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There is a trivial stable theory with a non-trivial reduct which is *n*-ample for all *n*

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Theorem (DE, 2003 ; DE, 2013)

There is a trivial stable theory with a non-trivial reduct which is n-ample for all $n \leq 3$.

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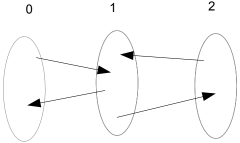
$n = 2$: the trivial structure

$\mathcal{E}_0$: 3-sorted directed graphs; at most 2 edges out of each vertex.



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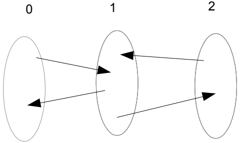

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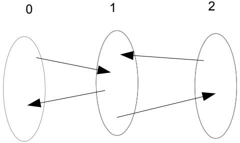

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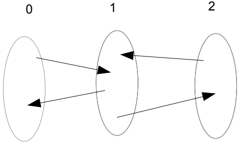

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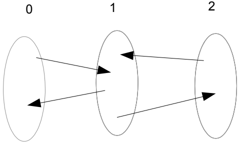

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$\mathcal{E}$ is not closed under substructures. However:

LEMMA: Suppose $B \in \mathcal{E}$ and $A \subseteq B$ is closed under successors of vertices of sorts 0,2. Then $A \in \mathcal{E}$. In particular, if $C \subseteq B$ is finite there is a finite $A \subseteq B$ with $C \subseteq A \in \mathcal{E}$ and $|A| \leq 2|C|$.

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Then $Th(V)$ is axiomatised by the axioms for $\mathcal{E}$ and these $\sigma_{X,A}$.

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REMARK: This is a variation on the 2003 construction; it's not clear whether that construction can be made to work.

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Final Remarks:

1. Similar construction for $n > 2$: the Lemma fails. There is a substitute result in case $n = 3$.
2. Open: Is there a stable T which is not trivial, n-ample for $n > 3$ and which does not interpret an infinite group? (See next slide.)

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Postscript

After the talk, E. Bouscaren and C. Laskowski pointed out that (2) on the previous slide is not the right question. One can take the 'disjoint union' of the n-space of Tent / Baudisch et al. and Hrushovski's s.m. set: the result is not trivial (because of the s.m. set) and n-ample (because of the n-space). Perhaps the correct question is to ask for a stable n-ample T which does not interpret an infinite group and where n-ampleness is witnessed by elements whose types which are orthogonal to all trivial types. This excludes these 'disjoint union' examples, but I do not know whether the examples given for $n = 2, 3$ actually satisfy this condition.