Automorphism groups of countable structures *

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Outline.

Suppose M is a countable first-order structure with a 'rich' automorphism group $\mathrm{Aut}(M)$. We will study $\mathrm{Aut}(M)$ both as a group and as a topological group, where the topology is that of pointwise convergence. This involves a mixture of model theory, group theory, combinatorics, descriptive set theory and topological dynamics. Here, 'rich' is undefined and depends on the context, but examples which we are interested in include: homogeneous structures such as the random graph or the rational numbers as an ordered set; ω-categorical structures; the free group of rank ω. The plan for the lectures is:

Lecture 1: Background. The topology of the symmetric group; automorphism groups; Baire category arguments. Homogeneous structures; amalgamation classes; Fraïssé's theorem and generalisations.

Lectures 2 and 3: A quick tour through some major results. Time permitting, we will look at: The small index property (work of Hodges, Hodkinson, Lascar and Shelah and others); extreme amenability and the Ramsey property (work of Kechris, Pestov and Todorcevic); normal subgroup structure of automorphism groups (work of Lascar, Macpherson - Tent, and Tent - Ziegler).

General background on model theory can be found in standard texts such as [7] or [16]. Introductory material on ω-categoricity can be found in the introduction to [8] (and many other places), and the book [2] focuses on the connections with permutation groups. The notes [1] are a nice introduction to infinite permutation groups. Macpherson's MALOA lectures [14], and the paper [15], give an extensive survey of work on homogeneous structures and their automorphism groups, including much of what is covered in these talks. The introduction to [3] surveys work on classification of homogeneous structures; more recent work can be found amongst the papers on Cherlin's webpage. A slightly different perspective on the material, in terms of Polish group actions, can be found in Kechris' survey [12]. The notes [4] from a previous series of lectures cover a different selection of material.

These notes are quite rough in places and many important references are omitted. I may update them after the talks.

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1 Permutation groups as topological groups

1.1 Notation and terminology

If G is a group acting on a set X and $a \in X$ then the G-orbit which contains a is $\{ga : g \in G\} \subseteq X$. (Note that our groups always act on the left.) This is the equivalence class containing a for the equivanece relation $a \sim b \Leftrightarrow (\exists g \in G)(ga = b)$. If there is a unique G-orbit on X we say that G is transitive on X. If $a \in X$, then let $G_a = \{g \in G : ga = a\}$ be the stabilizer of a in G. There is a canonical bijection, respecting the G-action, betweeen the set of left cosets of G_a in G and the G-orbit containing a, given by

$$gG_a \mapsto ga.$$

In particular, the index of G_a in G is the cardinality of the G-orbit which contains a. (This is sometimes called the *Orbit- Stabilizer Theorem.*)

If G is acting on X, then it also has natural actions on various other sets associated with X. For example, if $n \in \mathbb{N}$, then G acts coordinatewise on the set X^n of n-tuples from X and also on the set of subsets of X of size n.

If $A \subseteq X$ the pointwise stabilizer of A in G is $G_{(A)} = \{g \in G : ga = a \ \forall a \in A\}$.

Exercise: Show that if X is countable and A is a finite subset of X, then $G_{(A)}$ is a subgroup of countable index in G.

Most of the examples of permutation groups we will consider will be automorphism groups of first-order structures, so we review the terminology and notation for this.

Throughout L will denote a first-order language (usually countable). This will always include a symbol for equality, which all structures will interpret as true equality. We will not distinguish between an L-structure M and its domain. If $\bar{a} = (a_1, \ldots, a_n)$ is a finite tuple of elements of M, we might write $\bar{a} \in M$ (rather than $\bar{a} \in M^n$).

If M is an L-structure then $\mathrm{Aut}(M)$ is the automorphism group of M. We think of this as acting on the left: so if $g \in \mathrm{Aut}(M)$ and $a \in M$ then we write ga or $g(a)$ (rather than ag or a^g). We also think of $\mathrm{Aut}(M)$ as acting on M^n via the diagonal action: $g\bar{a} = (ga_1, \ldots, ga_n)$.

If $B \subseteq M$ the pointwise stabilizer of B in $\mathrm{Aut}(M)$ is

$$\mathrm{Aut}(M/B) = \{g \in \mathrm{Aut}(M) : gb = b \ \forall b \in B\}.$$

1.2 The topology on the symmetric group

If X is any non-empty set, the symmetric group $S = \mathrm{Sym}(X)$ is the group of all permutations of X. We regard this as a topological group with open sets being unions of cosets of pointwise stabilizers of finite sets. A permutation group G on X is just a subgroup of $\mathrm{Sym}(X)$ and we give this the relative topology. In other words, the basic open sets in $G \leq \mathrm{Sym}(X)$ are of the form $gG_{(A)}$ for $A \subseteq_{fin} X$ and $g \in G$. Note here that

$$G_{(A)} = \{h \in G : ha = a \forall a \in A\}$$

so

$$gG_{(A)} = \{h \in G : h|A = g|A\}.$$

Note also that each of these basic open sets is also closed as the complement is the union of the other cosets. So G is *totally disconnected*. Also note that if X is countable, then there are countably many of these basic open sets as each is determined by a map between finite subsets of X: so G is *second countable*. In particular, G is *separable* (meaning: there is a countable dense subset).

Lemma 1.1. *Suppose $G \leq \mathrm{Sym}(X)$. Then the closure of G in $\mathrm{Sym}(X)$ is*

$$\bar{G} = \{g \in \mathrm{Sym}(X) : gY = Y \textit{ for all } G\textit{-orbits } Y \textit{ on } X^n, \quad \forall n\}.$$

Proof. First, suppose Y is a subset of X^n. We show that $\{g \in \mathrm{Sym}(X) : gY = Y\}$ is closed. If $gY \not\subseteq Y$ there is some $\bar{y} \in Y$ with $g\bar{y} \notin Y$; so if $g' \in gG_{\bar{y}}$ then $g'Y \not\subseteq Y$. So the complement of $\{g \in \mathrm{Sym}(X) : gY \subseteq Y\}$ is open and therefore this set is closed. Similarly $\{g \in \mathrm{Sym}(X) : gY \supseteq Y\}$ is closed so $\{g \in \mathrm{Sym}(Y) : gY = Y\}$ is closed. Thus the intersection of these over all Y is closed and $\bar{G}$ is therefore contained in this intersection.

It follows that $\{g \in \mathrm{Sym}(X) : gY = Y$ for all G-orbits Y on $X^n, \quad \forall n\}$ is closed and clearly it contains G. So it contains $\bar{G}$.

Finally suppose $g \in \mathrm{Sym}(X)$ preserves the G-orbits on X^n for all n. An open neighbourhood O of g is specified by $g|A$ for some finite $A \subseteq X$. Enumerate A as the tuple $\bar{y}$. As $g\bar{y}$ is in the same G-orbit as $\bar{y}$ there is $h \in G$ with $g\bar{y} = h\bar{y}$. Thus $h \in O$. This shows that $g \in \bar{G}$. $\square$

Corollary 1.2. *A subgroup G of $\mathrm{Sym}(X)$ is closed iff G is the automorphism group of some first-order structure on X.*

Proof. A first-order structure on X is specified by relations and functions on X and the automorphisms are the permutations which preserve these. Note that a permutation preserves a function iff it stabilizes (setwise) its graph. So the automorphsim group is the intersection of the setwise stabilisers of certain subsets of M^n for various n. As in the proof of the lemma, this is a closed subgroup.

Conversely, if $G \leq \mathrm{Sym}(X)$ consider the structure on X which has a relation for each G-orbit on X^n, for each finite n. By the proof of the lemma, the automorphism group of this structure is $\bar{G}$. So if G is closed, the automorphism group is G. $\square$

Remarks 1.3. The structure on X constructed above (with relations the G-orbits on X^n) is sometimes called the *canonical structure* for G on X. Note that if G is oligomorphic (and X is countably infinite) this is an ω-categorical structure (see section 2.3).

We leave the following as an exercise.

Lemma 1.4. *Suppose $G \leq \mathrm{Sym}(X)$. Then G is compact iff G is closed in $\mathrm{Sym}(X)$ and all G-orbits on X are finite.*

If X is countable (say $X = \mathbb{N}$), the topology on $\mathrm{Sym}(X)$ is separable and complete metrizable. To see the latter, consider d given by, for $g_1 \neq g_2$,

$$d(g_1, g_2) = 1/n \text{ where } n \text{ is as small as possible with } g_1 n \neq g_2 n.$$

This is a metric for the topology, but it is not complete. To obtain a complete metric, consider

$$d'(g_1, g_2) = d(g_1, g_2) + d(g_1^{-1}, g_2^{-1}).$$

This is a complete metric for the topology. So if X is countable, then any closed $G \leq \mathrm{Sym}(X)$ is a *Polish group* (a topological group which is separable and complete metrizable).

1.3 Topological arguments

We give some results which illustrate the usefulness of the topology.

Denote by S_∞ the symmetric group of countable degree $\mathrm{Sym}(\mathbb{N})$. Note that $|S_\infty| = 2^{\aleph_0}$.

Theorem 1.5. *Suppose $G \leq S_\infty$ is closed. Then either $|G| = 2^{\aleph_0}$ or there exists a finite $Y \subseteq \mathbb{N}$ with $G_{(Y)} = 1$.*

Proof. This can be proved fairly directly, but here is an argument which uses some results from general topology. Consider the isolated points in G. As G is a topological group, either all points are isolated or no points are isolated (i.e. G is *perfect*). In the first case, the identity element is isolated so there is a basic open set contained in $\{1\}$; the only way this can happen is if $G_{(Y)} = 1$ for some finite Y. In the second case, G is a non-empty perfect complete space, so contains a copy of the Cantor set ([10], I.6.2). In particular $|G| = 2^{\aleph_0}$. $\square$

Definition 1.6. Suppose W is a topological space. A subset $Z \subseteq W$ is *nowhere dense* if its closure $\bar{Z}$ contains no non-empty open subset of W. Equivalently, $W \setminus \bar{Z}$ is dense in W. We say that $Y \subseteq W$ is *meagre* if it is a countable union of nowhere dense sets. Note that by definition, a countable union of meagre sets is meagre. A set X is *comeagre* if its complement is meagre. So this means that X contains the intersection of a countable family of dense open sets.

It's easy to see that the meagre subsets of W form a σ-ideal in the algebra of subsets of W; so we may think of them as 'small' subsets of W.

Theorem 1.7. *(Baire Category Theorem [10], I.8.4) Suppose W is a complete metrizable space. Then every comeagre subset of W is dense in W. Equivalently, the intersection of any countable family of dense open subsets of W is dense in W.*

Corollary 1.8. *Suppose $G \leq S_\infty$ is closed and H is a closed subgroup of G. If $|G : H| \leq \aleph_0$ then H is open in G, that is, $H \geq G_{(A)}$ for some finite set A.*

Proof. Suppose H does not contain $G_{(A)}$ for any finite A. So the complement of H is dense; therefore it is a dense open set. The same is true for each coset of H. If there are only countably many cosets their complements form a countable family of dense open subsets of G with empty intersection. This contradicts BCT. $\square$

We will improve on this result and discuss it further when we talk about the small index property.

2 Fraïssé's Theorem and Extensions

2.1 Amalgamation classes and homogeneous structures

We are interested in (countable) structures with 'large' automorphism groups. One possible interpretation of this is the following.

Definition 2.1. An L-structure M is *homogeneous* if isomorphisms beteween finitely generated substructures extend to automorphisms of M, that is: if $A_1, A_2 \subseteq M$ are f.g. substructures and $f : A_1 \to A_2$ is an isomorphism, then there exists $g \in \mathrm{Aut}(M)$ such that $g|A_1 = f$.

Remarks 2.2.

1. (Warning) Suppose M is any L-structure. For each $n \in \mathbb{N}$ and each $\mathrm{Aut}(M)$-orbit S on M^n, introduce a new n-ary relation symbol R_S into the language. Call the resulting language L^+. We regard M as an L^+-structure M^+ by interpreting a new relation symbol R_S as the orbit S. Then M^+ is a homogeneous L^+-structure and the automorphism group of M^+ is still $\mathrm{Aut}(M)$.

2. If L is a finite relational language, then there are only finitely many isomorphism types of L-structure of any finite size. So if M is a homogeneous L-structure, then $\mathrm{Aut}(M)$ has finitely many orbits on M^n for all $n \in \mathbb{N}$.

3. Let L consist of a single 2-ary relation symbol and consider the L-structure $M = (\mathbb{Q}; \leq)$, the rationals with their usual ordering. This is a homogeneous L-structure (one way to see this: use piecewise linear automorphisms).

Definition 2.3. A non-empty class $\mathcal{A}$ of finitely generated L-structures is a (Fraïssé) *amalgamation class* if:

1. (IP) $\mathcal{A}$ is closed under isomorphisms;

2. (Hereditary Property, HP) $\mathcal{A}$ is closed under f.g. substructures;

3. (Joint Embedding Property, JEP) if $A_1, A_2 \in \mathcal{A}$ there is $C \in \mathcal{A}$ and embeddings $f_i : A_i \to C$ $(i = 1, 2)$;

4. (Amalgamation Property, AP) if $A_0, A_1, A_2 \in \mathcal{A}$ and $f_i : A_0 \to A_i$ are embeddings, there is $B \in \mathcal{A}$ and embeddings $g_i : A_i \to B$ with $g_1 \circ f_1 = g_2 \circ f_2$.

Remarks 2.4.

1. Note that if $\emptyset \in \mathcal{A}$ then JEP follows from AP.

2. As an example, let L consist of a 2-ary relation symbol R and $\mathcal{A}$ the class of all finite graphs (considered as vertex sets with R interpreted as adjacency). This is an amalgamation class. To verify AP, regard f_1, f_2 as inclusions and let B be the disjoint

union of A_1 and A_2 over A_0 with edges $R^{A_1} \cup R^{A_2}$. Take g_1, g_2 to be the natural inclusions. We refer to B as the *free amalgam* of A_1, A_2 over A_0 (and sometimes denote it by $A_1 \coprod_{A_0} A_2$).

Definition 2.5. Suppose M is an L-structure. The *age* of M, $\mathrm{Age}(M)$ is the class of structures isomorphic to some f.g. substructure of M.

Theorem 2.6. *(Fraïssé's Theorem)*

1. *If M is a homogeneous L-structure, then* $\mathrm{Age}(M)$ *is an amalgamation class.*
2. *Conversely, if $\mathcal{A}$ is an amalgamation class of countable L-structures, with countably many isomorphism types, then there is a countable homogeneous L-structure M with $\mathcal{A} = \mathrm{Age}(M)$.*
3. *Suppose $\mathcal{A}$ is as in (2) and M is a countable homogeneous L-structure with age $\mathcal{A}$. Then M has the property that if $A \subseteq M$ is f.g. and $f : A \to B$ is an embedding with $B \in \mathcal{A}$, then there is an embedding $g : B \to M$ with $g(f(a)) = a$ for all $a \in A$. This property determines M up to isomorphism amongst countable structures with age $\mathcal{A}$.*

Definition 2.7. In the above, the structure M is determined up to isomorphism by $\mathcal{A}$ and is referred to as the *Fraïssé limit*, or *generic structure* of $\mathcal{A}$. The property in (3) is sometimes called the *Extension Property*

Examples 2.8. We give some examples of amalgamation classes and homogeneous structures. In each case, the language is the 'natural' language for the structures.

1. The class of all finite graphs is an amalgamation class. The Fraïssé limit is the *random graph.*
2. If $n \geq 3$, let K_n denote the complete graph on n vertices. Consider the class of all finite graphs which do not embed K_n. This is an amalgamation class (free amalgamation gives AP) and the Fraïssé limit is sometimes called the generic K_n-free graph.
3. As with graphs, the class of all finite directed graphs is an amalgamation class. We can use a similar idea to (2) to construct continuum many homogeneous directed graphs. Recall that a tournament is a directed graph with the property that for every two vertices a, b, one of $(a, b), (b, a)$ is a directed edge. There is an infinite set $\mathcal{S}$ of finite tournaments with the property that if A, B are distinct elements of $\mathcal{S}$ then A does not embed in B. If $\mathcal{T}$ is a subset of $\mathcal{S}$, consider the class of finite directed graphs which do not embed any member of $\mathcal{T}$. This is an amalgamation class (use free alamgamation); call the Fraïssé limit $H(\mathcal{T})$. It is easy to see that the elements of $\mathcal{S}$ which are in $\mathrm{Age}(H(\mathcal{T}))$ are the elements of $\mathcal{S} \setminus \mathcal{T}$. So the $H(\mathcal{T})$ are all non-isomorphic. These are called the *Henson digraphs.*
4. The class of all finite linear orders is an amalgamation class (but we cannot use free amalgamation). The Fraïssé limit is isomorphic to $(\mathbb{Q}; \leq)$.

5. The class of all finite partial orders is an amalgamation class.

6. The class of all finite groups is an amalgamation class. The generic structure is Philip Hall's universal locally finite group.

7. (Cherlin, [3]) Let L consist of 3 binary relation symbols G, R, B and consider the class of finite L-structures where R, G, B are symmetric, and for every pair of elements, exactly one of R, G, B holds. So these are complete graphs where each edge is coloured R, G or B. Consider the subclass of structures which omit the triangles:

$$RBB, GGB, BBB.$$

This is an amalgamation class. Amalgamation can be performed using only R, G edges, but a single edge colour will not suffice.

Proof of Theorem 2.6: We sketch a few details of the proof of Fraïssé 's Theorem.

1. Suppose M is a homogeneous L-structure. We show that $\mathrm{Age}(M)$ is an amalgamation class and that M has the Extension Property in 2.6(3). It is easy to see that $\mathrm{Age}(M)$ has IP, HP and JEP, so we verify AP.

Use the notation in the Definition. Without loss we can assume that $A_1, A_2 \subseteq M$ and $f_1 : A_0 \to A_1$ is the inclusion map. Thus $f_2 : A_0 \to A_2$ is an embedding between subsets of M. Call the image B_0. So we have (from f_2) an isomorphism $A_0 \to B_0$. By homogeneity this extends to an automorphism h of M. Let B be the substructure generated by $A_1 \cup h^{-1}(A_2)$, let $g_1 : A_1 \to C$ be inclusion and $g_2 : A_2 \to C$ be $h^{-1}|A_2$. If $a \in A_0$ then $g_2(f_2(a)) = a = g_1(f_1(a))$, as required.

The proof of EP is similar. There is some embedding $k : B \to M$. Let $A' = k(A)$. Then k gives an isomorphism $A \to A'$, which extends to an automorphism h of M. Let $g = h^{-1} \circ k : B \to M$. Then $g(a) = a$ for all $a \in A$, as required.

2. Suppose M, M' are countable L-structures with age $\mathcal{A}$ and which have EP. Suppose $A \subseteq M$ and $A' \subseteq M'$ are f.g. substructures and $k : A \to A'$ is an isomorphism. Using a back-and-forth argument, we can show that that k extends to an isomorphism between M and M'. This shows that any two countable structures with EP are isomorphic, and that any countable structure with EP is homogeneous.

3. To finish the proof, it therefore remains to show that if $\mathcal{A}$ is an amalgamation class of countable L-structures with countably many isomorphism types, then there is a countable structure M with age $\mathcal{A}$ which has EP.

Note first that if $A, B \in \mathcal{A}$, then there are countably many embeddings $A \to B$. We build M inductively as the union of a chain of structures in $\mathcal{A}$:

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots.$$

When doing this we ensure that:

- if $C \in \mathcal{A}$, then C embeds into some A_i;

- if A is a f.g. substructure of A_i and $f : A \to B \in \mathcal{A}$, then there is $j > i$ such that there is an embedding $g : B \to A_j$ with $g(f(a)) = a$ for all $a \in A$.

Note that there are countably many tasks to perform here; as we have a countable number of steps at our disposal, it therefore suffices to show that any one of these can be performed. A task of the first form can be performed using JEP. For the second, suppose the construction has reached stage $k > i$. At the next stage we can take A_{k+1} which solves the amalgamation problem $A \to A_k$ (inclusion), $f : A \to B$. Specifically, using AP we obtain $h : A_k \to A_{k+1}$ (which can be taken as inclusion), and $g : B \to A_{k+1}$ with $g(f(a)) = h(a) = a$ for all $a \in A$, as required. □

2.2 An extension of Fraïssé's Theorem

We give a generalization of Fraïssé 's Theorem 2.6. Further generalizations are possible (though the basic structure of the proof is always the same). For example a general category-theoretic version of the Fraïssé construction can be found in [5] and Section 2.6 of [13].

We shall work with a class $\mathcal{K}$ of finitely generated L-structures and a distinguished class of f.g. substructures $A \sqsubseteq B$, pronounced 'A is a *nice* substructure of B' (the terminology is not standard). If $B \in \mathcal{K}$, then an embedding $f : A \to B$ is a $\sqsubseteq$-embedding if $f(A) \sqsubseteq B$. We shall assume that $\sqsubseteq$ satisfies:

(N1) If $B \in \mathcal{K}$ then $B \sqsubseteq B$ (so isomorphisms are $\sqsubseteq$-embeddings);

(N2) If $A \sqsubseteq B \sqsubseteq C$ (and $A, B, C \in \mathcal{K}$), then $A \sqsubseteq C$ (so if $f : A \to B$ and $g : B \to C$ are $\sqsubseteq$-embeddings, then $g \circ f : A \to C$ is a $\sqsubseteq$-embedding).

We say that $(\mathcal{K}, \sqsubseteq)$ is an *amalgamation class* if:

- $\mathcal{K}$ is closed under isomorphisms and has countably many isomorphism types (and countably many embeddings between any pair of elements);
- $\mathcal{K}$ is closed under $\sqsubseteq$-substructures;
- $\mathcal{K}$ has the JEP for $\sqsubseteq$-embeddings;
- $\mathcal{K}$ has AP for $\sqsubseteq$-embeddings: if A_0, A_1, A_2 are in $\mathcal{K}$ and $f_1 : A_0 \to A_1$ and $f_2 : A_0 \to A_2$ are $\sqsubseteq$-embeddings, there is $B \in \mathcal{K}$ and $\sqsubseteq$-embeddings $g_i : A_i \to B$ (for $i = 1, 2$) with $g_1 \circ f_1 = g_2 \circ f_2$.

Remarks 2.9. 1. If $\sqsubseteq$ is just 'substructure' then this is the same as what was previously defined as a (Fraïssé) amalgamation class.

2. The notion $A \sqsubseteq B$ is only defined when B is finitely generated and it will be convenient to extend this to the situation where B is the union of a $\sqsubseteq$-chain of f.g. substructures. We can do this as follows.

> Suppose M is a countable L-structure and there are f.g. $M_i \subseteq M$ (with $i \in \mathbb{N}$) such that $M = \cup_{i\in\mathbb{N}} M_i$ and $M_1 \sqsubseteq M_2 \sqsubseteq M_3 \sqsubseteq \ldots$. Then for f.g. $A \subseteq M$ we define $A \sqsubseteq M$ to mean that $A \sqsubseteq M_i$ for some $i \in \mathbb{N}$. Note that *a priori* this depends on the choice of M_i, though the notation does not reflect this.

A condition on $(\mathcal{K}, \sqsubseteq)$ which guarantees that this *does not* depend on the choice of the M_i is:

(N3) Suppose $A \sqsubseteq B \in \mathcal{K}$ and $A \subseteq C \subseteq B$ with $C \in \mathcal{K}$. Then $A \sqsubseteq C$.

Indeed, suppose this holds and we also write M as the union of a $\sqsubseteq$-chain

$$M_1' \sqsubseteq M_2' \sqsubseteq M_3' \sqsubseteq \ldots.$$

Suppose $A \sqsubseteq M_i$. There exist j, k such that

$$M_i \subseteq M_j' \subseteq M_k.$$

As $M_i \sqsubseteq M_k$ and $M_j' \in \mathcal{K}$, (N3) implies that $M_i \sqsubseteq M_j'$, so $A \sqsubseteq M_j'$.

The generalisation of the amalgamation construction is:

Theorem 2.10. *Suppose $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class of finitely generated L-structures and $\sqsubseteq$ satisfies (N1) and (N2). Then there is a countable L-structure M and f.g. substructures $M_i \in \mathcal{K}$ (for $i \in \mathbb{N}$) such that:*

1. *$M_1 \sqsubseteq M_2 \sqsubseteq M_3 \sqsubseteq \ldots$ and $M = \cup_{i\in\mathbb{N}} M_i$;*
2. *every $A \in \mathcal{K}$ is isomorphic to a $\sqsubseteq$-substructure of M;*
3. *(Extension Property) if $A \sqsubseteq M$ is f.g. and $f : A \to B \in \mathcal{K}$ is a $\sqsubseteq$-embedding then there is a $\sqsubseteq$-embedding $g : B \to M$ such that $g(f(a))$ for all $a \in A$.*

Moreover, M is determined up to isomorphism by these properties and if $A_1, A_2 \sqsubseteq M$ are f.g. and $h : A_1 \to A_2$ is an isomorphism, then h extends to an automorphsim of M (which can be taken to preserve $\sqsubseteq$).

Note that in the above $\sqsubseteq$ is with respect to the chain of M_i. When we apply the result here, the $\sqsubseteq$ will satisfy $(N3)$ so this dependence is irrelevant. We refer to the property in the 'Moreover' part as $\sqsubseteq$-*homogeneity* and say that M is the *generic structure* of the class $(\mathcal{K}, \sqsubseteq)$.

Proof of Theorem 2.10: This is very similar to the proof of Theorem 2.6 so we will only give an outline.

Existence of M: Build the M_i inductively ensuring that:

- if $C \in \mathcal{K}$ there is an i and a $\sqsubseteq$-embedding $f : C \to M_i$;

- if $A \sqsubseteq M_i$ is f.g. and $A \sqsubseteq B \in \mathcal{K}$ then there is $j \geq i$ and a $\sqsubseteq$-embedding $g : B \to M_j$ with $g(a) = a$ for all $a \in A$.

To perform tasks of the first type, we use JEP; for the second type we can use AP as in the proof of 2.6. There are only countably many tasks to perform, so we can arrange that all are completed during the construction of the M_i.

Uniqueness and $\sqsubseteq$-homogeneity: Suppose $M'_1 \sqsubseteq M'_2 \sqsubseteq \ldots$ is a $\sqsubseteq$-chain whose union M' also satisfies (1-3). Write $\sqsubseteq'$ for $\sqsubseteq$ in M' with respect to the M'_i. As in the proof of 2.6, one uses the Extension Property to show that

$$\mathcal{S} = \{f : A \to A' : f \text{ an isomorphism and } A \sqsubseteq M,\ A' \sqsubseteq' M' \text{ f.g.}\}$$

is a back-and-forth system (which is non-empty because of JEP).

It follows that if $f : A \to A'$ is in $\mathcal{S}$ then there is an isomorphism $h : M \to M'$ which extends f. Moreover, the back-and-forth construction of h will ensure that $h(M_i) \sqsubseteq' M'$ and $h^{-1}(M'_i) \sqsubseteq M$ for all i, so $h(B) \sqsubseteq' M' \Leftrightarrow B \sqsubseteq M$ (for f.g. $B \subseteq M$). $\square$

Examples 2.11. (1) (2-out digraphs) Let $\mathcal{K}$ consist of the set of finite directed graphs where every vertex has at most 2 directed edges coming out of it. For $A \subseteq B \in \mathcal{K}$ write $A \sqsubseteq B$ if whenever $a \in A$ and $a \to b$ is a directed edge in B, then $b \in A$. Then $\sqsubseteq$ satisfies N1, N2, N3 and $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class (where the amalgamation is just free amalgamation).

(2) (Free groups) Let $\mathcal{K}$ be the class of finitely generated free groups. For f.g. $A \leq B \in \mathcal{K}$ write $A \sqsubseteq B$ to mean that A is a free factor of B. This clearly satisfies N1, N2 and N3 also holds (cf. Magnus, Karrass, Solitar, Ex 2.4.31). Moreover $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class and the generic structure is the free group of rank ω.

(3) (Hrushovski construction) If A is a finite graph, let $\delta(A)$ be twice the number of vertices minus the number of edges in A. Let $\mathcal{K}$ consist of those A with $\delta(X) \geq 0$ for all $X \subseteq A$. If $A \subseteq B \in \mathcal{K}$ write $A \sqsubseteq B$ to mean $\delta(A) \leq \delta(B')$ whenever $A \subseteq B' \subseteq B$. This satisfies N1, N2, N3 and $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class (where the amalgamation can be taken as free amalgamation).

As with Theorem 2.6, there is a converse statement. We omit the proof.

Theorem 2.12. *Suppose M is a countable L-structure and $(\mathcal{K}, \sqsubseteq)$ (satisfying (N1), (N2)) is such that $M = \cup_{i \in \mathbb{N}} M_i$ for $M_i \in \mathcal{K}$ with $M_i \sqsubseteq M_{i+1}$. Suppose also that $\mathcal{K}$ is the class of isomorphism types of $\sqsubseteq$-substructures of M and that M is $\sqsubseteq$-homogeneous (with respect to the $\sqsubseteq$-chain). Then Then $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class.* $\square$

Remarks 2.13. Suppose $\mathcal{K}$ in Theorem 2.10 has only finitely many isomorphism types of structure of each finite size. Suppose also that there is a function $F : \mathbb{N} \to \mathbb{N}$ such that if $B \in \mathcal{K}$ and $A \subseteq B$ with $|A| \leq n$, then there is $C \sqsubseteq B$ with $A \subseteq C$ and $|C| \leq F(n)$. Then the generic structure M is ω-categorical.

To see this we note that $\mathrm{Aut}(M)$ has finitely many orbits on M^n and apply the Ryll-Nardzewski Theorem, Theorem 2.16. Indeed, by $\sqsubseteq$-homogeneity there are finitely many orbits on $\{\bar{c} \in M^{F(n)} : \bar{c} \sqsubseteq M\}$ and any $\bar{a} \in M^n$ can be extended to an element of this set.

2.3 ω-categoricity

The following material will not be covered in the talks, but is included as background.

Suppose L is a first-order language. By the cardinality of L we mean the cardinality of the set of L-formulas. We shall usually work with countable languages. Recall that a *closed* L-formula (or *L-sentence*) is an L-formula without free variables. If M is an L-structure then a closed formula σ makes an assertion about M which is either true or false (written $M \models \sigma$ and $M \not\models \sigma$ respectively). The *theory* of M, denoted by $Th(M)$, is the set of closed formulas which are true in M.

Of course, if M is finite, then $Th(M)$ determines M (up to isomorphism). However, if M is infinite, then, by the Löwenheim - Skolem Theorems, $Th(M)$ will have at least one model of every cardinality greater than or equal to the cardinality of L.

Definition 2.14. Suppose L is a countable language and M is a countably infinite L-structure. We say that M (or $Th(M)$) is *ω-categorical* if every countable model of $Th(M)$ is isomorphic to M.

Proposition 2.15. *Suppose L is countable relational language and M is a countably infinite homogeneous L-structure. Suppose further that for each $n \in \mathbb{N}$, there are finitely many isomorphism types of substructures of M with n elements. Then M is ω-categorical.*

Proof. This will follow from the Ryll-Nardzewski Theorem below, but it's perhaps instructive to give a direct proof. For simplicity we do this when the language has only finitely many relation symbols.

First, note that $Th(M)$ specifies the age of M: for each n we have a closed formula (of the form $(\forall x_1 \ldots x_n) \ldots$) specifying what the isomorphism type of an n-set can be; moreover we have formulas (of the form $(\exists x_1 \ldots x_n) \ldots$) saying that all these are represented.

Second, note that $Th(M)$ also specifies the Extension Property. For each $A \subseteq B \in \mathrm{Age}(M)$ we have in $Th(M)$ the closed formula:

$$(\forall \bar{x})(\exists \bar{y})(\Delta_A(\bar{x}) \to \Delta_{A,B}(\bar{x}, \bar{y}))$$

where $\bar{x}$ is a tuple of variables of length $|A|$ and $\Delta_A(\bar{x})$ is the basic diagram of A, indicating the isomorphism type of A; similarly $\bar{x}\bar{y}$ has length $|B|$ and $\Delta_{A,B}(\bar{x}\bar{y})$ is the basic diagram of B where the variables $\bar{x}$ pick out the substructure A.

It follows that if M' is a model of $Th(M)$ then M' has the same age as M and has the extension property. So if M' is countable, then it is isomorphic to M. $\square$

We recall some model-theoretic terminlogy. Suppose M is an L-structure and $\theta(x_1, \ldots, x_n)$ an L-formula with free variables amongst $x_1, \ldots, x_n$. Let

$$\theta[M] = \{(a_1, \ldots, a_n) \in M^n : M \models \theta(a_1, \ldots, a_n)\}.$$

This is called a *$\emptyset$-definable* subset of M^n. We say that formulas $\theta(\bar{x})$ and $\psi(\bar{x})$ are *equivalent* (modulo $Th(M)$) if they define the same subset of M^n. Equivalently $(\forall \bar{x})(\theta(\bar{x}) \leftrightarrow \psi(\bar{x})) \in Th(M)$.

More generally if $C \subseteq M$, a C-definable subset of M^n is of the form

$$\eta[M, \bar{c}] = \{\bar{a} \in M^n : M \models \eta(\bar{a}, \bar{c})\}$$

for some L-formula $\eta(\bar{x}, \bar{y})$ and tuple $\bar{c}$ of elements of C. The $\bar{c}$ here are called *parameters*, and $\eta(\bar{x}, \bar{c})$ is a formula with parameters from C.

Suppose $\bar{a}$ is an n-tuple of elements of M and $C \subseteq M$. The *type* of $\bar{a}$ over C (in M), written $\mathrm{tp}^M(\bar{a}/C)$ is the set of formulas $\eta(\bar{x}, \bar{c})$ with parameters from C such that $M \models \eta(\bar{a}, \bar{c})$. Note that if $g \in \mathrm{Aut}(M/C)$ then $\mathrm{tp}^M(g\bar{a}/C) = \mathrm{tp}^M(\bar{a}/C)$.

Theorem 2.16. *(Ryll-Nardzewski, Svenonius, Engeler) Suppose L is a countable first-order language and M a countably infinite L-structure. Then the following are equivalent:*

1. *M is ω-categorical;*
2. $\mathrm{Aut}(M)$ *has finitely many orbits on M^n for all $n \in \mathbb{N}$;*
3. *For each $n \in \mathbb{N}$, every n-type of $Th(M)$ is principal;*
4. *For each $n \in \mathbb{N}$ there are only finitely many equivalence classes of L-formulas with n free variables (modulo $Th(M)$).*

Remarks 2.17.

1. For a proof, see for example ([**?**], 4.4.1). One way to organise the proof is

$$(2) \Rightarrow (4) \Rightarrow (3) \Rightarrow (1) \Rightarrow (3) \Rightarrow (2).$$

All but one of these are either straightforward or an application of compactness. The exception is (1) $\Rightarrow$ (3) which uses the Omitting Types Theorem.

2. A type is *principal* if it contains a formula which implies all of the other formulas in it.
3. It is clear that (2) $\Rightarrow$ (1) gives Proposition 2.15.
4. We say that a group G acting on a set X is *oligomorphic* if G has finitely many orbits on X^n for all $n \in \mathbb{N}$.

Example 2.18. We give an example of how amalgamation constructions can sometimes be used to produce ω-categorical structures (and oligomorphic groups) with prescribed properties.

Suppose $(k_n : n \in \mathbb{N})$ is a given sequence of natural numbers. We construct an ω-categorical structure M such that for every $n \in \mathbb{N}$, the number of orbits of $\mathrm{Aut}(M)$ on M^n is at least k_n. Consider a language L which has k_n n-ary relation symbols for each n. Let $\mathcal{A}$ consist of finite L-structures C such that for each relation symbol R, if $C \models R(c_1, \ldots, c_n)$, then $c_1, \ldots, c_n$ are distinct. This is an amalgamation class (use free amalgamation) and for each n there are only finitely many, but at least k_n, isomorphism types of structures of size n in $\mathcal{A}$. So the Fraïssé limit M is ω-categorical and has the required property.

Corollary 2.19. *Suppose M is ω-categorical.*

1. *Two n-tuples are in the same* $\mathrm{Aut}(M)$*-orbit iff they have the same type over* $\emptyset$ *in* M.

2. *The* $\emptyset$*-definable subsets of* M^n *are precisely the* $\mathrm{Aut}(M)$*-invariant subsets of* M^n*, that is, unions of* $\mathrm{Aut}(M)$*-orbits on* M^n.

3. *If* $C \subseteq M$ *is finite, then the* C*-definable subsets of* M^n *are precisely the* $\mathrm{Aut}(M/C)$*-invariant sets.*

Proof. (1) The direction $\Rightarrow$ is true in general; the other direction is part of the proof of (3) $\Rightarrow$ (2) in the above.

(2) It is clear that a $\emptyset$-definable subset of M^n is $\mathrm{Aut}(M)$-invariant, so is a union of $\mathrm{Aut}(M)$-orbits on M^n. It follows that it is enough to show that each such $\mathrm{Aut}(M)$-orbit X is definable. But this follows from (1) and the fact that types are principal.

(3) Expand the language to the language $L(C)$ by adding new constants for the elements of C. Regard M as an $L(C)$-structure $(M;C)$ in the obvious way and note that a C-definable subset of M^n is the same thing as a $\emptyset$-definable subset of the $L(C)$-structure $(M;C)$. The automorphism group of the latter is $\mathrm{Aut}(M/C)$ and as C is finite, this has finitely many orbits on n-tuples for all n. So $(M;C)$ is ω-categorical and (3) follows from (2). $\square$

The following characterization of homogeneous structures amongst the ω-categorical structures is worth noting. Recall that an L-structure M (or its theory $Th(M)$) is said to have *quantifier elimination* (QE) if for every $n \geq 1$, every L-formula $\theta(x_1, \ldots, x_n)$ is equivalent (modulo $Th(M)$) to a quantifier-free formula $\eta(x_1, \ldots, x_n)$.

Theorem 2.20. *Suppose M is an ω-categorical L-structure. Then $Th(M)$ has quantifier elimination iff M is homogeneous.*

Proof. ($\Rightarrow$:) Suppose A_1, A_2 are f.g. substructures of M and $f : A_1 \to A_2$ is an isomorphism. So by ω-categoricity, A_1, A_2 are finite. Let $\bar{a}_1$ enumerate A_1 and $\bar{a}_2 = f(a_2)$. Then $\bar{a}_1$, $\bar{a}_2$ satisfy the same quantifier-free formulas in M. By QE, it follows that $\mathrm{tp}^M(\bar{a}_1/\emptyset) = \mathrm{tp}^M(\bar{a}_2/\emptyset)$. By Corollary 2.19(1), there is $g \in \mathrm{Aut}(M)$ with $g\bar{a}_1 = \bar{a}_2$. Thus g extends f, as required.

($\Leftarrow$:) If $\bar{a}, \bar{a}'$ are tuples in M with the same quantifier free type, then $\bar{a} \mapsto \bar{a}'$ extends to an isomorphism $f : A \to A'$ between the substructures generated by $\bar{a}, \bar{a}'$. By homogeneity, there is an automorphism of M extending f and so $\bar{a}, \bar{a}'$ have the same type over $\emptyset$ in M. So quantifier-free types determine types (over $\emptyset$) in M; as all types are principal it follows that every formula is equivalent to a quantifier free formula (in M). $\square$

We conclude this subsection with some comments on *algebraic closure*.

Definition 2.21. Suppose M is an L-structure and $A \subseteq M$. The *algebraic closure* $\mathrm{acl}(A)$ of A in M is the union of the finite A-definable subsets of M. In general, $\mathrm{acl}(A)$ contains the substructure generated by A and acl is a closure operation on M.

Lemma 2.22. *Suppose M is ω-categorical. Then* acl *is a uniformly locally finite closure operation on M: there is a function $\alpha : \mathbb{N} \to \mathbb{N}$ such that if $A \subseteq M$ is finite, then $|\mathrm{acl}(A)| \leq \alpha(|A|)$.*

Proof. By the previous corollary, $\mathrm{acl}(M)$ is the union of the finite $Aut(M/A)$-orbits on M, so is finite. Note that if $g \in \mathrm{Aut}(M)$ then $\mathrm{acl}(gA) = g\mathrm{acl}(A)$. So as there are only finitely many orbits on finite sets of any given size, there is a uniform bound on the size of the algebraic closures of these sets. $\square$

In particular, if M is ω-categorical there is a uniform bound on the size of n-generator substructures, for all $n \in \mathbb{N}$ (of course, if L is a relational language, this is not saying very much).

We say that algebraic closure in M is *trivial* if $\mathrm{acl}(A) = A$ for all finite $A \subseteq M$. If M is homogeneous, this can be expressed as a condition on its age:

Lemma 2.23. *Suppose L is a countable relational language and M is a homogeneous L-structure which is ω-categorical. Then algebraic closure in M is trivial iff $\mathcal{A} = \mathrm{Age}(M)$ satisfies:*

(Strong Amalgamation Property) if $A_0, A_1, A_2 \in \mathcal{A}$ and $f_i : A_0 \to A_i$ are embeddings, there is $B \in \mathcal{A}$ and embeddings $g_i : A_i \to B$ with $g_1 \circ f_1 = g_2 \circ f_2$ and $g_1(A_1) \cap g_2(A_2) = g_1(f_1(A_0))$.

Proof. First, suppose $\mathcal{A}$ has the strong amalgamation property. Let $B \subseteq M$ be finite and $c \notin M$. We have to show that c is in an infinite $\mathrm{Aut}(M/B)$-orbit, so we show that for every $n \in \mathbb{N}$ there are at least n elements in this orbit. Let C be the substructure $B \cup \{c\}$. By strong amalgamation there is a structure D in $\mathcal{A}$ which consists of n distinct copies of C amalgamated over B; so $D = B \cup \{c_1, \ldots, c_n\}$. We can assume $C \subseteq D$ (say $c = c_1$) and using the Extension Property of M, we can assume that $D \subseteq M$. Then the $B \cup \{c_i\}$ are isomorphic (over B), so by homogeneity, the c_i are in the same $\mathrm{Aut}(M/B)$-orbit.

Conversely, suppose algebraic closure is trivial in M. We modify the proof of AP given in Theorem 2.6. Use the notation in the Definition. Without loss we can assume that $A_1, A_2 \subseteq M$ and $f_1 : A_0 \to A_1$ is the inclusion map. Thus $f_2 : A_0 \to A_2$ is an embedding between subsets of M. Call the image B_0. So we have (from f_2) an isomorphism $A_0 \to B_0$. By homogeneity this extends to an automorphism h of M. So $h^{-1}(A_2) \supseteq A_0$. By Neumann's lemma (after applying an element of $\mathrm{Aut}(M/A_0)$) we can assume that $h^{-1}(A_2) \cap A_1 = A_0$. Let $B = A_1 \cup h^{-1}(A_2)$, let $g_1 : A_1 \to C$ be inclusion and $g_2 : A_2 \to C$ be $h^{-1}|A_2$. If $a \in A_0$ then $g_2(f_2(a)) = a = g_1(f_1(a))$, as required. $\square$

The proof made use of the following (see Corollary 4.2.2 of [7] for a proof):

Theorem 2.24. *(Neumann's Lemma) Suppose G is a group acting on a set X and all G-orbits on X are infinite. Suppose B, C are finite subsets of X. Then there is some $g \in G$ with $B \cap gC = \emptyset$.*

Examples 2.25. We list some examples of 'natural' ω-categorical structures. The first three have trivial algebraic closure, the rest, non-trivial.

1. A pure set $(M; =)$. So the language just has equality; the automorphism group is the full symmetric group $\mathrm{Sym}(M)$.

2. A countable structure $(M; E)$ with an equivalence relation E which has infinitely many classes, all of which are infinite.

3. The countable, dense linear ordering without endpoints $(\mathbb{Q}; \leq)$.

4. The countable atomless boolean algebra $(\mathbb{B}; 0, 1, \wedge, \vee, \neg)$.

5. A vector space $V(\aleph_0, q)$ of dimension $\aleph_0$ over a finite field $\mathbb{F}_q$ with q elements. Note that the usual language for vector spaces over a field K consists of $+, -, 0, \lambda_a (a \in K)$ where λ_a is a function symbol for scalar multiplication by a.

6. Any countable abelian group of finite exponent.

7. A classical (symplectic, orthgonal or Hermitian) space $(V(\aleph_0, q) : +, -, 0, ...)$ over a finite field, where ... consists of the extra structure, such as the bilinear or quadratic form.

Remarks 2.26. Other ways of constructing ω-categorical structures include interpretation and boolean powers. The survey [6] discusses the latter in detail.

3 Selected topics

In this section we look at some results on the small index property, extreme amenability and normal subgroup structure for autmorphism groups. This is not a comprehensive survey and the presentation is often very similar to the lecture notes of Macpherson [14].

3.1 The small index property

We say that a countable structure M (or its automorphism group $\mathrm{Aut}(M)$) has the *small index property* (SIP) if whenever H is a subgroup of $G = \mathrm{Aut}(M)$ of index less than $2^{\aleph_0}$, then H is open. In other words, if $|G : H| < 2^{\aleph_0}$, then there is a finite $X \subseteq M$ with $H \geq G_{(X)}$. (Note that the first formulation makes sense in an arbitrary topological group.)

Remarks 3.1. (1) If $H \leq G$ is open then $|G : H| \leq \aleph_0$.

(2) The SIP implies that we can recover the topology on G from its group-theoretic structure: the open subgroups are precisely the subgroups of small index and the cosets of these form a base for the topology.

(3) For a countable ω-categorical structure M the topological group $\mathrm{Aut}(M)$ determines M *up to biinterpretability* - see the notes [4] for an explanation of this. The following example shows that we cannot expect to recover M completely. Consider M with automorphism group $G = \mathrm{Sym}(M)$. This acts on N, the set of subsets of size 2 from M. Let $G_1 \leq \mathrm{Sym}(N)$ be the set of permutations induced by by this action. It can be shown that this is closed, so we can regard G_1 as the automorphism group of a structure on N. The isomorphism $G \to G_1$ (given by the identity map) is a homeomorphism (check this!). So the structures

M and N have isomorphic topological automorphism groups even though they are different structures.

(4) (Automatic continuity) Suppose M, N are countable structures and M has the SIP. If $\alpha : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ is a homomorphism of groups, then SIP implies that α is continuous (Exercise). If α is an isomorphism, then a result about Polish groups implies that α is a homeomorphism.

(5) (An ω-categorical structure without SIP) The following example is due to Cherlin and Hrushovski. Consider a language L which has a $2n$-ary relation symbol E_n for each $n \in \mathbb{N}$. Let $\mathcal{C}$ be the class of finite L-structures A in which E_n is an equivalence relation on n-tuples of distinct elements of A with at most 2 equivalence classes. This is an amalgamation class; call the generic structure M. So for each n there are two equivalence classes of distinct n-tuples from M. Every permutation of these equivalence classes extends to an automorphism of M. So $G = \mathrm{Aut}(M)$ has a closed normal subgroup G^0 consisting of automorphisms which fix all equivalence classes and the quotient group is topologically isomorphic to the direct product C_2^ω (where C_2 is the cyclic group with 2 elements). Assuming the Axiom of Choice, this has non-open subgroups of index 2.

(6) (Caveat) Sometimes a different definition of SIP in terms of *strong automorphisms* is used.

It is an open question whether the construction in (5) is essentially the only obstruction to the SIP for an ω-categorical structure. More specifically, say that an ω-categorical M is *G-finite* if for every open subgroup $H \leq \mathrm{Aut}(M)$, the intersection of the open subgroups of finite index in H is of finite index in H.

Question 3.2. If M is a countable ω-categorical structure which is G-finite, does M have the SIP?

The SIP was originally proved for $\mathrm{Sym}(\mathbb{N})$ by Dixon, Neumann and Thomas (1986). Their method was subsequently adapted to prove SIP for general linear groups and classical groups over countable fields (Evans, 1986, 1991) and $\mathrm{Aut}(\mathbb{Q}; \leq)$ (Truss, 1989). A different method (applicable to the first two of these) was introduced by Hodges, Hodkinson, Lascar and Shelah (1993) and used to prove SIP for the random graph. This approach was extended to general Polish groups by Kechris and Rosendal (2007) and we follow their presentation (as in [14]).

In the following we consider the action of a topological group G on the direct product G^n by conjugation:

$$(g_1, \ldots, g_n) \stackrel{h}{\mapsto} (hg_1h^{-1}, \ldots, hg_nh^{-1}).$$

If we give G^n the product topology, this is a continuous action, which we refer to as the conjugation action.

Definition 3.3. Suppose G is a Polish group. We say that G has *ample homogeneous generics* (ahg's) if for each $n > 0$, there is a comeagre orbit of G on G^n (with the conjugation action).

Theorem 3.4. *(Kechris - Rosendal ([11], 6.9); Hodges, Hodkinson, Lascar and Shelah) Suppose G is a Polish group with ample homogeneous generics. Then G has the SIP.*

Remarks 3.5. Having ample homogeneous generics is a strong property and implies other properties, including uncountable cofinality and the Bergman property. It does not hold for $\mathrm{Aut}(\mathbb{Q};\leq)$: the property fails for $n=2$ (an observation of Hodkinson).

The following is a useful way of showing the existence of ahg's.

Suppose $(\mathcal{K},\sqsubseteq)$ is an amalgamation class of f.g. L-structures and $\sqsubseteq$ satisfies N1, N2, N3 (as in Section 2.2). Let M be the generic structure of the class and $G=\mathrm{Aut}(M)$. Denote by $\mathcal{A}(M)$ the set of f.g. $A\sqsubseteq M$. For the rest of this section, by a *partial automorphism* of M we mean an isomorphism $f : A_1 \to A_2$ where $A_i \in \mathcal{A}(M)$. Note that every partial automorphism of M extends to an automorphism of M.

Theorem 3.6. *With the above notation, suppose that the following two conditions hold:*

(i) (Amalgamation property for partial automorphisms) Suppose $A \sqsubseteq B_i \in \mathcal{A}(M)$ (for $i=1,2$). Then there is $g\in G_{(A)}$ with the following property. If f_1, f_2 are in $\mathrm{Aut}(B_1)$ and $\mathrm{Aut}(gB_2)$ respectively and stabilize A, and $f_1|A = f_2|A$, then $f_1\cup f_2$ extends to an automorphism of M.

(ii) (Extension property for partial automorphisms) If $f_1,\ldots,f_n$ are partial automorphisms of M then there is $B\in\mathcal{A}(M)$ containing their domains and images and $g_i\in\mathrm{Aut}(B)$ such that $f_i\subseteq g_i$ for all $i\leq n$.

Then $G=\mathrm{Aut}(M)$ has ample homogeneous generics.

Proof. (Sketch) Say that $(g_1,\ldots,g_n)\in G^n$ is *generic* if

(a) The set of $A\in\mathcal{A}(M)$ such that $g_i(A)=A$ for all $i\leq n$ is cofinal in $\mathcal{A}(M)$; and

(b) Suppose $A\in\mathcal{A}(M)$ and $g_i(A)=A$ for all i; let $A\sqsubseteq B\in\mathcal{A}(M)$ and $h_i\in\mathrm{Aut}(B)$ extend $g_i|A$. Then there is $\alpha\in G_{(A)}$ such that $\alpha g_i\alpha^{-1}\supseteq h_i$ (for $i\leq n$).

We claim that: (1) the set of generics in G^n is comeagre; and (2) any two generics in G^n lie in the same G-orbit.

For (2) Suppose $(g_1,\ldots,g_n)$ and $(h_1,\ldots,h_n)$ are generic. We can build an element of G which cojugates one to the other by using a back-and-forth argument and the following obervation:

Claim: If $A\sqsubseteq B\in\mathcal{A}(M)$, and A is invariant under the h_i, g_i, and $g_i|A = h_i|A$ for all i, there is $\beta\in G_{(A)}$ with $\beta g_i\beta^{-1}|B = h_i|B$.

To see the claim, note that using (a) for the h_i, we may assume $h_i(B)=B$ for all i. Now use (b) for the g_i.

To show (1) we write the set of generics in G^n as a countable intersection of dense open sets. For $A\in\mathcal{A}(M)$ let

$$X(A)=\{(g_1,\ldots,g_n)\in G^n : \exists B\in\mathcal{A}(M) \text{ with } A\sqsubseteq B \text{ and } g_iB=B\ \forall I\leq n\}.$$

It is easy to see that $X(A)$ is open. It follows from (ii) that $X(A)$ is dense. Note that

$$\bigcap_{A \in \mathcal{A}(M)} X(A)$$

is the set of elements of G^n which satisfy (a).

Suppose $A \sqsubseteq B \in \mathcal{A}(M)$ and $h_1, \ldots, h_n \in \mathrm{Aut}(B)$ satisfy $h_i A = A$. Let

$$Y(A, B, \bar{h}) = \{(g_1, \ldots, g_n) \in G^n : \text{ if } (\forall i)(g_i|A = h_i|A) \text{ then } (\exists \alpha \in G_{(A)})\ (\forall i)(\alpha g_i \alpha^{-1}|B = h_i)\}.$$

The intersection of these consists of elements of G^n which satisfy (b).

Each $Y(A, B, \bar{h})$ is easily seen to be open. For denseness, consider a basic open set specified by partial automorphisms $(f_1, \ldots, f_n)$. By (ii) we can assume these all have the same domain and image C and we can also assume harmlessly that they extend the h_i. Using (i), there is $\beta \in G_{(A)}$ and automorphisms g_i such that $g_i \supseteq f_i \cup \beta h_i \beta^{-1}$. Then $(g_1, \ldots, g_n)$ is in the required open set and is in $Y(A, B, \bar{h})$. This shows the denseness and so gives (1). □

It is worth noting that (i), (ii) can be translated directly into properties of the class $(\mathcal{K}; \sqsubseteq)$ as in the following examples.

Examples 3.7. (1) Let $\mathcal{K}$ be the class of finite graphs (and $\sqsubseteq$ is just embedding). So the generic structure M is the random graph. For the amalgamation property for automorphisms, note that if D is the free amalgmation of B_1 and B_2 over A and $f_i \in \mathrm{Aut}(B_i)$ stabilize A and have the same restriction to A, then their union is an automorphism of D. The Extension Property for this class is a theorem of Hrushovski - see ([14], Lemma 3.13) for a short proof and further references. It follows that the random graph has the SIP. The result generalises to other free amalgamation classes: see [14] for details and references.

(2) Let $\mathcal{K}$ be the class of finitely generated free groups and $\sqsubseteq$ dentote being a free factor. The free product with amalgamation gives the amalgamation property for partial automorphisms. For the Extension Property we can take B to be any free factor of M which contains the domains and images of the f_i. So the free group of rank ω has the SIP (Bryant and Evans, 1997).

3.2 Extreme amenability and structural Ramsey theory

We discuss some results from the paper of Kechris, Pestov and Todorcevic [9].

Definition 3.8. Suppose G is a topological group.

(1) A *G-flow* is a non-empty, compact (Hausdorff) space Y with a continuous G-action $G \times Y \to Y$.

(2) We say that G is *extremely amenable* if whenever Y is a G-flow, there is a G-fixed point in Y.

Remarks 3.9. (1) Suppose $G \leq \mathrm{Sym}(X)$ is closed and $H \leq Y$ is open. Then the left coset space $Z = G/H$ is discrete and for $k \in \mathbb{N}$, the space $Y = \{1, \ldots, k\}^Z$ of functions $f : Z \to \{1, \ldots, k\}$ with the product topology is a G-flow (the action is $(gf)(z) = f(g^{-1}z)$). Note that here, as G is transitive on Z, the only fixed points are the constant functions. We think of Y as the space of colourings of Z with $\leq k$ colours.

(2) The group G is *amenable* if every G-flow has an invariant finitely additive probability measure.

(3) An alternative way of expressing extreme amenability is that the *universal minimal G-flow $M(G)$* is a point.

(4) If $G \leq \mathrm{Sym}(X)$ then $\{0,1\}^{X^2}$ is a G-flow, as is every closed G-invariant subset of this. We can think of this as the set of all binary relations on X. Let

$$LO(X) = \{R \in \{0,1\}^{X^2} : \ R \text{ is a linear order on } X\}.$$

This is a closed, G-invariant subset. So we obtain:

Lemma 3.10. *If $G \leq \mathrm{Sym}(X)$ is extremely amenable, then there is a G-invariant linear order on X.*

So, for example, $\mathrm{Sym}(X)$ is not extremely amenable.

Theorem 3.11. *(Kechris, Pestov, Todorcevic, 2005) Suppose $G \leq \mathrm{Sym}(X)$ is closed. The following are equivalent:*

(1) G is extremely amenable;

(2) Suppose H is an open subgroup of G and $Z = G/H$. If $c : Z \to \{1, \ldots, k\}$ and $A \subseteq Z$ is finite, there is $g \in G$ and $i \leq k$ such that $c(ga) = i$ for all $a \in A$.

(3) G preserves a linear ordering on X and G has the Ramsey property.

I will first discuss the equivalence of (1) and (2) here and then say what the Ramsey property is.

(1) $\Rightarrow$ (2): Consider the G-flow $\{1, \ldots, k\}^Z$. Let Y be the closure in this of the G-orbit $\{gc : g \in G\}$. This is a G-flow, so must contain a G-fixed point. So it contains a constant function $f_i(z) = i$ (for some $i \leq k$). In other words, f_i is in the closure of $\{gc : g \in G\}$. This translates into the condition in (2).

(2) $\Rightarrow$ (1): This is a bit harder, but not excessively so. The proof shows that to decide whether G is extremely amenable, it suffices to consider G-flows whch are closed subflows of $\{1, \ldots, k\}^{G/H}$ (for $H \leq G$ open and $k \in \mathbb{N}$). In fact, we can restrict to k here and take H from a base of open neighbourhoods of 1.

Example 3.12. Let $X = \mathbb{Q}$ and $G = \mathrm{Aut}(\mathbb{Q})$. We verify (2). Take $H = G_{(C)}$ where C is a subset of $\mathbb{Q}$ of size n. Then we can identify Z with the set of n-tuples $b_1 < b_2 < \cdots < b_n$ from $\mathbb{Q}$, or, indeed, the set $[\mathbb{Q}]^n$ of subsets of $\mathbb{Q}$ of size n.

So we can think of a function $c : Z \to \{1, \ldots, k\}$ as a k-colouring of $[\mathbb{Q}]^n$. By the classical Ramsey Theorem there is an infinite $Y \subseteq \mathbb{Q}$ such that c is constant on $[Y]^n$. Given a finite

$A \subseteq Z$ let S be the elements of $\mathbb{Q}$ appearing in tuples in A. So S is a finite subset of $\mathbb{Q}$ and we can find $g \in G$ with $gS \subseteq Y$. Then c is constant on gA, as required for (2). It follows:

Corollary 3.13. *(Pestov)* $\mathrm{Aut}(\mathbb{Q}; \leq)$ *is extremely amenable.*

We now define the Ramsey property.

Definition 3.14. Suppose $G \leq \mathrm{Sym}(X)$.

(1) A *G-type* σ is a G-orbit on finite subsets of X. If σ, ρ are G-types, write $\rho \leq \sigma$ iff for all $F \in \rho$ there is $F' \in \sigma$ with $F \subseteq F'$.

(2) Suppose $\rho \leq \sigma \leq \tau$ are G-types.

  (i) If $F \in \sigma$ let $\binom{F}{\rho} = \{F' \subseteq F : F' \in \rho\}$.

  (ii) If $k \in \mathbb{N}$ write

$$\tau \to (\sigma)^{\rho}_{k}$$

  to mean that for every $F \in \tau$ and colouring $c : \binom{F}{\rho} \to \{1, \ldots, k\}$ there is $F_0 \in \binom{F}{\sigma}$ which is monochromatic for c (that is, $c|\binom{F_0}{\rho}$ is constant).

(3) We say that G has the *Ramsey property* if for all k and G-types $\rho \leq \sigma$ there is a G-type $\tau \geq \sigma$ such that $\tau \to (\sigma)^{\rho}_{k}$.

Exercise: $G = \mathrm{Aut}(\mathbb{Q}; \leq)$ has the Ramsey property - this is the finite Ramsey theorem.

We now look at the proof of (2) $\Rightarrow$ (3) in the Theorem 3.11 (the proof of (3) $\Rightarrow$ (2) is also reasonably straightforward).

Suppose (2) holds. We have to show that the Ramsey property holds. Suppose not - so there are $k \in \mathbb{N}$ and G-types $\rho \leq \sigma$ such that for no τ do we have $\tau \to (\sigma)^{\rho}_{k}$.

Let $F_0 \in \sigma$. For every finite $E \supseteq F_0$ the set

$$C_E = \{c : \binom{E}{\rho} \to \{1, \ldots, k\} : \text{ no monochrome } F \in \tbinom{E}{\sigma}\}$$

is non-empty. Restriction gives a directed system $C_{E'} \to C_E$ for $E' \supseteq E$. By König's lemma there is therefore $c : \rho \to \{1, \ldots, k\}$ with no monochrome $F \in \sigma$. This contradicts (2).

The investigation of classes of (finite) structures with the Ramsey property is an area of combinatorics known as *structural Ramsey Theory.* A nice summary of some of the results is contained in ([14], Section 4). Note that by Theorem 3.11 we should expect such structures to carry an ordering. The following shows that there is a strong connection with homogeneous structures:

Theorem 3.15. *Suppose $\mathcal{C}$ is a class of finite ordered structures for a finite relational language and $\mathcal{C}$ is closed under substructures and has JEP. If $\mathcal{C}$ is a Ramsey class, then $\mathcal{C}$ has the amalgamation property.*

See [14], 4.8 for references and a proof.

Examples of countable homogeneous ordered structures with extremely amenable automorphism group include ordered versions of: the random graph, the universal homogeneous K_n-free graphs, the Henson digraphs.

3.3 Normal subgroup structure

We begin with some 'classical' results.

Theorem 3.16. *(J. Schreier and S. Ulam, 1933) Suppose X is countably infinite. If $g \in \mathrm{Sym}(X)$ moves infinitely many elements of X, then every element of $\mathrm{Sym}(X)$ is a product of conjugates of g. In particular, $\mathrm{Sym}(X)/FSym(X)$ is a simple group.*

Theorem 3.17. *(A. Rosenberg, 1958). Suppose V is a vector space of countably infinite dimension over a field K. If $FGL(V)$ denotes the elements of $GL(V)$ which have a fixed point space of finite codimension, then $GL(V)/(K^\times.FGL(V))$ is a simple group.*

Theorem 3.18. *(G. Higman, 1954). The non-trivial, proper normal subgroups of $G = \mathrm{Aut}(\mathbb{Q}; \leq)$ are the left-bounded automorphisms, $L = \{g \in G : \exists a\ g|(-\infty, a) = id\}$, the right-bounded automorphisms $R = \{g \in G : \exists a\ g|(a, \infty)\}$ and $B = L \cap R$.*

Theorem 3.19. *(J. Truss, 1985). Let Γ be the countable random graph. Then $\mathrm{Aut}(\Gamma)$ is simple.*

It is tempting to conjecture from these that automorphism groups of 'nice' countable structures should not have any non-obvious normal subgroups. This is false:

Theorem 3.20. *(M. Droste, C. Holland, D. Macpherson, 1988(?)) The automorphism group of a countable, homogeneous semilinear order has $2^{2^{\aleph_0}}$ normal subgroups.*

However, there are some general results. The ones I want to focus on here are all descendants of the following.

Theorem 3.21. *(D. Lascar, 1992) Suppose M is a countable saturated structure with a $\emptyset$-definable strongly minimal set D. Suppose that $M = \mathrm{acl}(D)$. Suppose $g \in G = \mathrm{Aut}(M/\mathrm{acl}(\emptyset))$ is unbounded, i.e. for every $n \in \mathbb{N}$ there is some $X \subseteq D$ with $\dim(gX/X) > n$. Then G is generated by the conjugates of g.*

Remarks 3.22.

1. This implies the results for $\mathrm{Sym}(X)$ and $GL(V)$.
2. The proof uses Polish group arguments.
3. The ideas were used by T. Gardener (1995) to prove an analogue of Rosenberg's result for classical groups over finite fields.
4. The result was used by Z. Ghadernezhad and K. Tent (2012) to prove simplicity of the automorphism groups of certain generalized polygons and so obtain new examples of simple groups with a BN-pair.

Some of the ideas from Lascar's paper were used to prove:

Theorem 3.23. *(D. Macpherson and K. Tent, 2011) Suppose M is a countable, transitive homogeneous relational structure whose age has free amalgamation. Suppose $\mathrm{Aut}(M) \neq \mathrm{Sym}(M)$. Then* *(a)* $\mathrm{Aut}(M)$ *is simple;*

(b) (Melleray) if $1 \neq g \in \mathrm{Aut}(M)$ *then every element of* G *is a product of 32 conjugates of* $g^{\pm 1}$*.*

Note: This implies Truss' result and unpublished results of M. Rubin (1988).

K. Tent and M. Ziegler (2012) generalized this to the case where M has a *stationary independence relation* $\mathop{\smile\!\!\!\!\!\!|}$ and used this to prove:

Theorem 3.24. *Suppose* U *is the Urysohn rational metric space. If* $g \in \mathrm{Aut}(U)$ *is not bounded, then every automorphism of* U *is a product of 8 conjugates of* g*.*

We will now discuss what is meant by a stationary independence relation.

For the rest of this section we use the following notation:

- M is a countable first-order structure;
- $G = \mathrm{Aut}(M)$;
- cl is a G-invariant, finitary closure operation on subsets of M;
- If $X \subseteq_{fin} M$ and a is fixed by $G_{(X)}$, then $a \in \mathrm{cl}(X)$ (where $G_{(X)} = \{g \in G : gx = x \ \forall x \in X\}$).
- $\mathcal{X} = \{\mathrm{cl}(A) : A \subseteq_{fin} M\}$;
- $\mathcal{F}$ consists of all maps $f : X \to Y$ with $X, Y \in \mathcal{X}$ which extend to automorphisms of M. Call these *partial automorphisms.*

Example: Take cl to algebraic closure in M. So, for example, if M is the Fraïssé limit of a free amalgamation class, then $\mathrm{acl}(X) = X$ for all $X \subseteq M$.

In what follows, $\mathop{\smile\!\!\!\!\!\!|}$ is a relation between subsets A, B, C of M: written $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and pronounced 'A is independent from C over B.'

Definition 3.25. *We say that* $\mathop{\smile\!\!\!\!\!\!|}$ *is a stationary independence relation compatible with* cl *if for* $A, B, C, D \in \mathcal{X}$ *and finite tuples* a, b*:*

1. *(Compatibility) We have* $a \mathop{\smile\!\!\!\!\!\!|}_b C \Leftrightarrow a \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}(b)} C$ *and*

$$a \mathop{\smile\!\!\!\!\!\!|}_B C \Leftrightarrow e \mathop{\smile\!\!\!\!\!\!|}_B C \text{ for all } e \in \mathrm{cl}(a, B) \Leftrightarrow \mathrm{cl}(a, B) \mathop{\smile\!\!\!\!\!\!|}_B C.$$

2. *(Invariance) If* $g \in G$ *and* $A \mathop{\smile\!\!\!\!\!\!|}_B C$*, then* $gA \mathop{\smile\!\!\!\!\!\!|}_{gB} gC$*.*
3. *(Monotonicity) If* $A \mathop{\smile\!\!\!\!\!\!|}_B C \cup D$*, then* $A \mathop{\smile\!\!\!\!\!\!|}_B C$ *and* $A \mathop{\smile\!\!\!\!\!\!|}_{B \cup C} D$*.*

4. *(Transitivity) If $A \perp_B C$ and $A \perp_{B \cup C} D$, then $A \perp_B C \cup D$*

5. *(Symmetry) If $A \perp_B C$, then $C \perp_B A$.*

6. *(Existence) There is $g \in G_B$ with $g(A) \perp_B C$.*

7. *(Stationarity) Suppose $A_1, A_2, B, C \in \mathcal{X}$ with $B \subseteq A_i$ and $A_i \perp_B C$. Suppose $h : A_1 \to A_2$ is the identity on B and $h \in \mathcal{F}$. Then there is some $k \in \mathcal{F}$ which contains $h \cup \mathrm{id}_C$ (where id_C denotes the identity map on C).*

Remarks 3.26. 1. For all $a \in M$ and finite X we have $a \perp_X \mathrm{cl}(X)$. Moreover $a \perp_X a$ iff $a \in \mathrm{cl}(X)$.

2. Tent and Ziegler consider this where $\mathrm{acl}(X) = X$ and $\mathrm{cl}(X) = X \ \forall X$. Write $A \perp_B C$ to mean $A \cap C \subseteq B$. This satisfies (1-6), but not necessarily (7).

3. Suppose M is the Fraïssé limit of a free amalgamation class (of relational structures). Let $\mathrm{cl}(X) = X \ \forall X$. Define $A \perp_B C$ to mean $A \cap C \subseteq B$ and $A \cup B$, $C \cup B$ are freely amalgamated over B. This is a stationary independence relation on M.

4. Suppose M is a countable-dimensional vector space over a countable field K. So $G = GL(M)$. Let cl be linear closure and take $A \perp_B C$ to mean that $\mathrm{cl}(A \cup B) \cap \mathrm{cl}(C \cup B) = \mathrm{cl}(B)$. This gives a stationary independence relation.

Definition 3.27. Say that $g \in G$ *moves almost maximally* if for all $B \in \mathcal{X}$ and $a \in M$ there is a' in the $G_{(B)}$-orbit of a such that

$$a' \underset{B}{\perp} ga'.$$

Example 3.28. Suppose $(M; \mathrm{cl}; \perp)$ is the vector space example. If $g \in G$ does not move almost maximally, then for some finite dimensional subspace B, for all $v \in M$ we have $gv \in \langle v, B \rangle$. Thus g acts as a scalar α on M/B. So $(\alpha^{-1}g - 1)v \in B$ for all v and it follows that g is a scalar multiple of a finitary transformation.

Lemma 3.29. *Suppose $(M; \mathrm{cl}; \perp)$ is the free amalgamation example. Suppose also that $G = \mathrm{Aut}(M)$ is transitive on M and $G \neq \mathrm{Sym}(M)$. If $1 \neq g \in G$, then*

(1) g moves infinitely many points of each $G_{(B)}$-orbit (for each finite $B \subseteq M$); and

(2) there is $h \in G$ such that $[g, h] = g^{-1}h^{-1}gh$ moves almost maximally.

The following is a generalization of the result of Tent and Ziegler (though the proof is almost the same).

Theorem 3.30. *(Evans, Ghadernezhad, Tent, 2013) Suppose M is a countable structure with a stationary independence relation compatible with a closure operation* cl. *Suppose that $G = \mathrm{Aut}(M)$ fixes every element of $\mathrm{cl}(\emptyset)$. If $g \in G$ moves almost maximally, then every element of G is a product of 16 conjugates of g.*

Remarks 3.31. 1. If $\mathrm{cl}(X) = X \ \forall X$, this is proved in the paper of Tent and Ziegler.

2. As observed by Tent and Ziegler, it implies the result of Macpherson and Tent for the free amalgamation example.

3. Following Lascar's paper, the proof uses the topology on G which has a base of open neighbourhoods of the form $\{g \in G : g(x) = f(x) \ \forall x \in X\}$ for $f : X \to Y$ a partial automorphism ($X, Y \in \mathcal{X}$). This is complete metrizable, but not necessarily separable. A trick from Lascar's paper allows one to work in separable closed subgroups and then the proof of Tent and Ziegler works.

4 A short Appendix on Model Theory

4.1 First-order languages and structures

In a first-order language one has an alphabet of symbols and certain finite sequences of these symbol (the formulas of the language) are the objects of interest. The symbols are connectives $\wedge$ (*and*), $\vee$ (*or*), $\neg$ (*not*); quantifiers $\forall$ and $\exists$; punctuation (parentheses and commas); variables; and constant, relation and function symbols, with each of the last two coming equipped with a finite 'arity' specifying how many arguments it has. The number of these constant, relation and function symbols (together with their arities) is referred to as the *signature* of the language.

The *terms* of the language are built inductively. Any variable or constant symbol is a term and if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is also a term (all terms are built in this way).

Now we can build the *formulas* of the language. Again, this is done inductively. If R is an n-ary relation symbol in the language and $t_1, \ldots, t_n$ are terms then $R(t_1, \ldots, t_n)$ is a formula (an *atomic* formula). If ϕ, ψ are formulas and x a variable, then $(\phi) \wedge (\psi)$, $(\phi) \vee (\psi)$, $\neg(\phi)$, $\forall x(\phi)$, $\exists x(\phi)$ are formulas (of higher 'complexity'). A formula not involving any quantifiers is called *quantifier free* or *open*. There is a natural notion of a *free variable* in a formula, and when we write a formula as $\phi(x_1, \ldots, x_m)$ we mean that its free variables are amongst the variables $x_1, \ldots, x_m$. A formula with no free variables is called a *sentence*. For more details the reader could consult ([7], Section 2.1).

If L is a first-order language then an *L-structure* consists of a set M equipped with a constant (that is, a distinguished element of M), n-ary relation (that is, a subset of M^n), and n-ary function $M^n \to M$ for each constant symbol and n-ary relation and function symbol in L. If $\phi(x_1, \ldots, x_m)$ is an L-formula and $a_1, \ldots, a_m \in M$ then one can 'read' $\phi(a_1, \ldots, a_m)$ as a statement about the behaviour of $a_1, \ldots, a_m$ and these constants, relations and functions (interpreting each constant, relation or function symbol as the corresponding constant, relation or function of M), which is either true or false. If it is true, then we write

$$M \models \phi(a_1, \ldots, a_m).$$

All of this can of course be made completely precise (defined inductively on the complexity of ϕ): see ([7], Section 2.1) again. We shall always have $=$ as a binary relation symbol in L and interpret it as true equality in any L-structure.

If Φ is a set of L-sentences and M an L-structure we say that M is a *model* of Φ (and write $M \models \Phi$) if every sentence in Φ is true in M. If there is a model of Φ we say that Φ is *consistent*. The set of L-sentences true in M is called the *theory* of M. Two L-structures M_1 and M_2 are *elementarily equivalent* if they have the same theory. This is written as $M_1 \equiv M_2$. Thus in this case the structures M_1 and M_2 cannot be distinguished using the language L. The following basic result of model theory shows that one should not expect first-order languages to be able to completely describe infinite structures.

Theorem 4.1. (Löwenheim-Skolem) *Let L be a first-order language with signature of cardinality λ. Let μ, ν be cardinals with $\mu, \nu \geq \max(\lambda, \aleph_0)$, and suppose M_1 is an L-structure with cardinality μ. Then there exists an L-structure M_2 elementarily equivalent to M_1 and of cardinality ν.*

The 'upward' part of this result (where $\nu \geq \mu$) follows easily from the fundamental theorem of model theory:

Theorem 4.2. (The Compactness Theorem) *Let L be a first-order language and Φ a set of L-sentences. If every finite subset of Φ is consistent, then Φ is consistent.*

The original version of this is due to Gödel (1931). Proofs (using a method due to Henkin (1949)) can be found in ([7], Theorem 6.1.1). Algebraists may prefer the proof using ultraproducts and the theorem of Łos ([7], Theorem 9.5.1).

If M, N are L-structures with $M \subseteq N$ and the distinguished relations, functions (and constants) of N extend those of M, then we say that M is a *substructure* of N. If also for every L-formula $\phi(x_1, \ldots, x_m)$ and $a_1, \ldots, a_m \in M$ we have

$$M \models \phi(a_1, \ldots, a_m) \Leftrightarrow N \models \phi(a_1, \ldots, a_m)$$

then we say that M is an *elementary* substructures of N (and that N is an elementary extension of M) and write $M \preceq N$. A stronger version of the Löwenheim-Skolem Theorem (4.1) is true: the smaller of M_1, M_2 may be taken to be an elementary substructure of the larger. Proofs can be found in ([7], Corollaries 3.1.5 and 6.1.4).

4.2 Definable sets; types

Suppose L is a first-order language and M an L-structure. Let $n \in \mathbb{N}$. A subset A of M^n is called (parameter) *definable* if there exist $b_1, \ldots, b_m \in M$ and an L-formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ with

$$A = \{\bar{a} \in M^n : M \models \phi(\bar{a}, \bar{b})\}.$$

If the parameters $\bar{b}$ can be taken from the subset $X \subseteq M$ then A is said to be X-definable. The union of the finite X-definable subsets of M is called the *algebraic closure* of X, denoted by $\mathrm{acl}(X)$, and the union of the X-definable singleton subsets of M is the *definable closure* of X, denoted by $\mathrm{dcl}(X)$. It is not hard to check that both of these are indeed closure operations on M.

So the definable subsets of M^n are the ones which can be described using L-formulas (and parameters). Conversely one could take a particular n-tuple $\bar{a} \in M^n$ and a set of parameters $A \subseteq M$ and ask what the language L can say about $\bar{a}$ (in terms of A and M). This gives the notion of the *type* of $\bar{a}$ over A, which by definition is

$$\mathrm{tp}^M(\bar{a}/A) = \{\phi(x_1, \ldots, x_n, b_1, \ldots, b_m) : b_1, \ldots, b_m \in A,\ M \models \phi(\bar{a}, \bar{b})\}$$

(the superscript M is dropped if this is clear from the context). It is sometimes useful to consider the type of $\bar{a}$ (over A) using only certain L-formulas. For example, for the *quantifier free type* of $\bar{a}$ over A one takes only quantifier free ϕ in the above definition. It is also possible to define the type of an infinite sequence of elements of M. The reader can consult ([7], Section 6.3) for further details here.

More generally, a (complete) n-type over A is a set of L-formulas with parameters from A equal to $\mathrm{tp}_N(\bar{a}/A)$ for some elementary extension N of M and some $\bar{a} \in M^n$. There is no reason to suppose, for arbitrary M and A, that this type should be *realised* in M, that is, there exists $\bar{a}' \in M^n$ with $\mathrm{tp}^M(\bar{a}'/A) = \mathrm{tp}^N(\bar{a}/A)$. For example, this would clearly be impossible if $A = M$ and $\bar{a} \notin M^n$. However, it can happen that for some infinite cardinal κ if $|A| < \kappa$ then every complete n-type over A is realised in M: in this case M is called *κ-saturated,* and if $\kappa = |M|$ then M is *saturated.* The reader should consult ([7], Chapter 10) for more on this.

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