Automorphism groups and Ramsey properties of sparse graphs.

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Joint work with Jan Hubička and Jaroslav Nešetřil

THEMES:

- Automorphism groups of nice model-theoretic structures acting on compact Hausdorff spaces.
- Connection with structural Ramsey theory (Kechris - Pestov - Todorčević Correspondence)
- Sparse graphs constructed using Hrushovski amalgamations exhibit interesting new phenomena.

THEOREM A: There is a countable ω-categorical structure M with the property that if $H \leq \mathrm{Aut}(M)$ is (extremely) amenable, then H has infinitely many orbits on M^2.

NOTE: By the Ryll-Nardzewski Theorem, $\mathrm{Aut}(M)$ has finitely many orbits on M^n for all $n \in \mathbb{N}$.

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Amalgamation classes and Fraïssé limits.

L a 1st-order relational language and M a countable L-structure.
$\mathrm{Age}(M)$: class of isomorphism types of finite substructures.

M is *homogeneous* if all isomorphism between finite substructures of M extend to automorphisms of M. In this case $\mathcal{C} = \mathrm{Age}(M)$ satisfies:

AMALGAMATION PROPERTY (AP): If $f_1 : A \to B_1$ and $f_2 : A \to B_2$ are embeddings between elements of $\mathcal{C}$, the there is $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ with $g_1 \circ f_1 = g_2 \circ f_1$.

Conversely: if $\mathcal{C}$ is a countable class of isomorphism types of finite L-structures which is closed under taking substructures, has the joint embedding property and
$\mathcal{C}$ has AP,
then there is a countable, homogeneous structure $M(\mathcal{C})$ with $\mathrm{Age}(M(\mathcal{C})) = \mathcal{C}$. It is unique up to isomorphism.

$\mathcal{C}$ is an *amalgamation class* and $M(\mathcal{C})$ is its *Fraïssé limit*.

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EXAMPLE:

$\mathcal{G}$ the class of all finite graphs; $M(\mathcal{G})$ is the Random Graph.

VARIATION: Can also work with a distinguished notion of embedding / substructure, $(\mathcal{C}; \leq)$.
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Ramsey classes

$L^{\leq}$: relational language with $\leq$.
$\mathcal{A}$: a class of finite $L^{\leq}$-structures closed under substrs and satisfying JEP and where $\leq$ is a linear ordering.

DEFINITION: Say that $\mathcal{A}$ is a Ramsey class if whenever $A \subseteq B \in \mathcal{A}$, there is $B \subseteq C \in \mathcal{A}$ such that if

$$\gamma : \binom{C}{A} \to \{0, 1\}$$

is a 2-colouring of the copies of A in C, there is $B' \in \binom{C}{B}$ (a copy of B in C) such that γ is constant on $\binom{B'}{A}$.

EXAMPLES: (1) $L = \{\leq\}$. Take $\mathcal{A} =$ finite linear orders.
(2) (Nešetřil - Rödl) The class $\mathcal{G}^{\leq}$ of linearly ordered finite graphs.

THEOREM: (Nešetřil) If $\mathcal{A}$ is a Ramsey class, then $\mathcal{A}$ has the amalgamation property.
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Automorphism groups.

Ω infinite set (usually countable); $\mathrm{Sym}(\Omega)$ symmetric group.

$G \leq \mathrm{Sym}(\Omega) \subseteq \Omega^\Omega$ pointwise convergence topology.

Basic open sets: $\{g \in G : g|A = \gamma\}$, $A \subseteq \Omega$ finite and $\gamma : A \to \Omega$.

G is a topological group.
$\mathrm{Sym}(\Omega)$ complete metrizable if Ω is countable.

Lemma

$G \leq \mathrm{Sym}(\Omega)$ is closed iff $G = \mathrm{Aut}(M)$ for some 1st order structure M with domain Ω.

INTERESTING EXAMPLES: M countable homogeneous, or ω-categorical.

REMARK: If $G \leq \mathrm{Sym}(\Omega)$ is closed there is a *homogeneous* structure M with $\mathrm{Aut}(M) = G$ (but the language may have to be infinite).

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Topological Dynamics

G a topological group.

G-flow: compact, Hausdorff, non-empty space X with a continuous G-action.

Definition

1. G is *amenable* if every G-flow X supports a G-invariant Borel probability measure.
2. G is *extremely amenable* if every G-flow has a fixed point.

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G-flows

$G = \mathrm{Aut}(M)$. Some G-flows:

1. Take G-invariant $\Delta \subseteq M^n$; consider $Y = \{0,1\}^\Delta$ as a G-flow. Also consider G-invariant, closed subspaces X of Y.
2. G-invariant, closed subspaces of $S(M)$, Stone space over M.

EXAMPLE: $G = \mathrm{Sym}(\Omega)$. We have a G-flow:

$$LO(\Omega) = \{R \subseteq \Omega^2 : R \text{ is a linear order on } \Omega\}.$$

COROLLARY: If $H \leq G$ is e.a. then there is an H-invariant linear order on Ω.

Theorem (Pestov, 1998)

$\mathrm{Aut}(\mathbb{Q}; \leq)$ is e.a.

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The Kechris - Pestov - Todorčević Correspondence

Theorem (KPT, 2005)

Suppose M is a countable, homogeneous, linearly ordered relational structures with age $\mathcal{A}$. TFAE:

1. $\mathrm{Aut}(M)$ is extremely amenable.
2. $\mathcal{A}$ is a Ramsey class.

So Ramsey classes correspond to homogeneous structures with e.a. automorphism groups.

EXAMPLE: $\mathcal{G}^{\leq}$ (finite l.o. graphs) is a Ramsey class. Let $\Gamma^{\leq} = M(\mathcal{G}^{\leq})$. Then $\mathrm{Aut}(\Gamma^{\leq})$ is e.a. The graph reduct Γ is the Random Graph and $\mathrm{Aut}(\Gamma^{\leq}) \leq \mathrm{Aut}(\Gamma)$.
Note that $\mathcal{G}^{\leq}$ is a precompact expansion of $\mathcal{G}$: every $A \in \mathcal{G}$ expands to finitely many iso types of structures in $\mathcal{G}^{\leq}$.
Equivalently each $\mathrm{Aut}(\Gamma)$-orbit on Γ^n splits into finitely many $\mathrm{Aut}(\Gamma^{\leq})$-orbits.

The universal minimal flow

A G-flow X is *minimal* if every G-orbit on X is dense.

FACT: (Ellis) There is a unique *universal* minimal G-flow, $M(G)$.

DEF: Let $G = \mathrm{Aut}(M)$. Say $H \leq G$ is *precompact* if for every G-orbit $\Delta \subseteq M^n$, H has finitely many orbits on Δ.

KPT; Nguyen Van Thé

Suppose M is a countable L-structure. If $G = \mathrm{Aut}(M)$ has a precompact e.a. closed subgroup $H = \mathrm{Aut}(N)$, then $M(G)$ can be described. In particular, $M(G)$ is metrizable and has a comeagre orbit. The same is therefore true of every minimal G-flow.

EXAMPLES: (1) $M(\mathrm{Sym}(\Omega)) = LO(\Omega)$.
(2) If Γ is the random graph, then $M(\mathrm{Aut}(\Gamma)) = LO(\Gamma)$.

COROLLARY: $\mathrm{Sym}(\Omega)$ and $\mathrm{Aut}(\Gamma)$ are amenable.

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Question

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  - If M is countable ω-categorical, is there an ω-categorical expansion N of M with $\mathrm{Aut}(N)$ extremely amenable? Equivalently, is there a precompact e.a. closed subgroup of $\mathrm{Aut}(M)$.
- Particularly interesting case: M homogeneous in a finite relational language.
- Why ask the question?
  - Ubiquity of ω-categorical structures with e.a. automorphism groups
  - Ubiquity of Ramsey classes
  - Applications: reducts; complexity of CSP's (Bodirsky, Pinsker et al.)
  - Describing $M(G)$ for G closed, oligomorphic permutation group.
  - Evidence. Work on Ramsey expansions of Fraïssé classes: Nešetřil - Rödl; Jasinski, Laflamme, Nguyen van Thé, Woodrow; ...

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  - If M is countable ω-categorical, is there an ω-categorical expansion N of M with $\mathrm{Aut}(N)$ extremely amenable? Equivalently, is there a precompact e.a. closed subgroup of $\mathrm{Aut}(M)$.
- Particularly interesting case: M homogeneous in a finite relational language.
- Why ask the question?
  - Ubiquity of ω-categorical structures with e.a. automorphism groups
  - Ubiquity of Ramsey classes
  - Applications: reducts; complexity of CSP's (Bodirsky, Pinsker et al.)
  - Describing $M(G)$ for G closed, oligomorphic permutation group.
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Sparse graphs.

DEF: Suppose $k \in \mathbb{N}$. A graph $M = (M; E)$ is *k-sparse* if for all finite $A \subseteq M$ we have $|E[A]| \leq k|A|$.

FACT: If the graph $M = (M; E)$ is k-sparse, then it is *k-orientable*: the edges of M can be directed so that each vertex has at most k directed edges coming out.

DEF: If M is k-sparse, let

$$X(M) = \{D \subseteq M^2 : (M; D) \text{ is a } k\text{-orientation of } M\} \subseteq \{0,1\}^{M^2}.$$

Note that this is an $\mathrm{Aut}(M)$-flow.

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Theorem A

FACT: (Hrushovski) There is an ω-categorical 2-sparse graph M_F with all vertices of infinite valency.

Theorem A′ (DE, Jan Hubička and Jaroslav Nešetřil)

Suppose M is a countable, k-sparse graph of infinite valency. If $H \leq \mathrm{Aut}(M)$ is amenable, then H has infinitely many orbits on M^2.

COROLLARY: There is no precompact amenable subgroup of $\mathrm{Aut}(M_F)$.

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Proof of Thm A′: Step 1

- Suppose M is a graph with all vertices of infinite valency and $H \leq \mathrm{Aut}(M)$ has finitely many orbits on M^2.
- If $c \in M$ let H_c denote the stabilizer of c in H.
- For $c \in M$ let $\mathrm{cl}(c)$ be the union of the finite H_c-orbits on M.
- There is $n \in \mathbb{N}$ s.t. $|\mathrm{cl}(c)| \leq n$ for all $c \in M$.
- If $b \in \mathrm{cl}(c)$ then $\mathrm{cl}(b) \subseteq \mathrm{cl}(c)$.
- STEP 1: There are adjacent $a, b \in M$ such that b is in an infinite H_a-orbit and a is in an infinite H_b-orbit.

PROOF: Suppose there do not exist such a, b. Then for every edge a, b in M either $a \in \mathrm{cl}(b)$ or $b \in \mathrm{cl}(a)$. Take b with $\mathrm{cl}(b)$ of maximal size. There is $a \notin \mathrm{cl}(b)$ adjacent to b. By assumption, $\mathrm{cl}(a) \supset \mathrm{cl}(b)$: contradiction.

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Proof of Thm A′: step 2

- GIVEN: M is a k-sparse graph, $H \leq \mathrm{Aut}(M)$, and $a, b \in M$ are adjacent and such that a in an infinite H_b-orbit and b is in an infinite H_a-orbit.
- Show H is not amenable.
- Suppose there is an H-invariant probability measure μ on $X(M)$.
- Let $S(ab) = \{D \in X(M) : (a, b) \in D\}$. May assume $p = \mu(S(ab)) > 0$.
- Let $b_1, \ldots, b_n$ be in the same H_a-orbit as b and s_i the characteristic function of $S(ab_i)$. Note $\mu(S(ab_i)) = p$.
- For $D \in X(M)$,

$$\sum_{i \leq n} s_i(D) \leq k \text{ so } \int_{D \in X(M)} \sum_{i \leq n} s_i(D) d\mu(D) \leq k.$$

- So $np \leq k$: contradiction.

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Further results

THEOREM B: Suppose $Y \subseteq X(\mathrm{Aut}(M_F))$ is a minimal $\mathrm{Aut}(M_F)$-subflow. Then all $\mathrm{Aut}(M_F)$-orbits on Y are meagre in Y.

Other things: Find e.a. subgroups of $\mathrm{Aut}(M_F)$ which are maximal e.a. ; likewise for amenable subgroups...

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Open Questions

QUESTION: (Bodirsky, ...) If M is a structure homogeneous for a finite relational language, is there a precompact e.a. subgroup $H \leq \mathrm{Aut}(M)$?

SIDE QUESTION: Is there a homogeneous structure in a finite relational language in which a sparse graph of infinite valency can be interpreted?

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Hrushovski's construction I

- $\mathcal{G}$: class of finite graphs $(A; R)$
- If $C \subseteq A \in \mathcal{G}$ let
$$\delta(C) = 2|C| - |R[C]|.$$
(Predimension of C.)
- If $A \subseteq B \in \mathcal{C}$ write $A \leq_d B$ if $\delta(X) > \delta(A)$ whenever $A \subset X \subseteq B$.
- Note: If $A \leq_d B \leq_d C$ then $A \leq_d C$.

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Hrushovski's construction II

- $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ an increasing function which tends to infinity.
- Let
$$\mathcal{G}_F = \{A \in \mathcal{G} : \delta(Y) \geq F(|Y|) \text{ for all } Y \subseteq A\}.$$

- For suitable F the class $(\mathcal{G}_F, \leq_d)$ has free amalgamation over $\leq_d$-substructures.
- In this case the Fraïssé limit construction gives a countable graph M_F characterised by:
  - M_F is the union of a chain of finite $\leq_d$-subgraphs;
  - every graph in $\mathcal{G}_F$ is isomorphic to a $\leq_d$-subgraph of M_F;
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