$\aleph_0$-categorical structures with arbitrarily fast growth of algebraic closure

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1 Introduction

Over the past 25 years, various results have been proved about the growth rates of certain sequences of integers associated to infinite permutation groups. Most of the results concern the number of orbits of the automorphism group of an $\aleph_0$-categorical structure $\mathcal{M}$ on the set of unordered n-subsets or on the set of n-tuples of elements of $\mathcal{M}$. (Recall that by the Ryll-Nardzewski Theorem, if $\mathcal{M}$ is countable and $\aleph_0$-categorical, the number of the orbits of the automorphism group $\mathrm{Aut}(\mathcal{M})$ on the set of n-tuples is finite and equals the number of complete n-types consistent with the theory of $\mathcal{M}$.) Many of the results known about these sequences are collected in the book [Ca90] by P. J. Cameron. One of the oldest (in the realms of 'folklore') is that for any sequence $(k_n)_{n\in\mathbf{N}}$ of natural numbers there is a countable $\aleph_0$-categorical structure $\mathcal{M}$ such that the number of orbits of $\mathrm{Aut}(\mathcal{M})$ on the set of n-tuples from $\mathcal{M}$ is greater than k_n for all n.

These investigations suggested the study of the growth rate of another sequence. Let $\mathcal{M}$ be an $\aleph_0$-categorical structure and X be a finite subset of $\mathcal{M}$. Let $\mathrm{acl}(X)$ be the algebraic closure of X, that is, the union of the finite X-definable subsets of $\mathcal{M}$. Equivalently, this is the union of the finite orbits on $\mathcal{M}$ of $\mathrm{Aut}(\mathcal{M})_{(X)}$, the pointwise stabiliser of X in $\mathrm{Aut}(\mathcal{M})$. Define

$$b_n = b_n(\mathcal{M}) = \max\{|\mathrm{acl}(X)| : X \subseteq \mathcal{M},\ |X| = n\}.$$

In [Ma86, Question 2], H. D. Macpherson posed the following problem:

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Problem: *Given any sequence $\{a_n\}$, is it always possible to find a transitive $\aleph_0$-categorical structure $\mathcal{M}$ such that $b_n \geq a_n$ for every n?*

For 'naturally occuring' examples of $\aleph_0$-categorical structures b_n does not grow faster than doubly exponentially, and in many cases algebraic closure is trivial (i.e. for every finite subset $X \subseteq \mathcal{M}$, $\mathrm{acl}(X) = X$).

Independently, D. M. Evans and H. D. Macpherson gave positive answers to the above question. The structures produced by Evans (see [Ev94], Theorem 4.3) had primitive and not 2-transitive automorphism groups and those of Macpherson had imprimitive automorphism groups. So the question remained as to whether $\mathcal{M}$ in Macpherson's problem could be multiply transitive. We show that it can be (although of course, in a $k+1$-transitive structure, the algebraic closure of k points must be those k points, so we insist only that the algebraic closure grows arbitrarily rapidly from some point onwards). Specifically, we prove the following result (in Section 2).

Theorem 1. *For every sequence $(a_i)_{i\in\mathbf{N}}$ and every $k \in \mathbf{N}$, $k \geq 2$, there exists a $(k-1)$-transitive, countable, $\aleph_0$-categorical k-uniform hypergraph $\mathcal{M}$ such that for each $n > k$, there exists $A_n \subseteq \mathcal{M}$ with $|A_n| = n$ and $|acl(A_n)| \geq a_n$.*

A k-uniform hypergraph is a k-ary relational structure, and one of the difficulties in the proof is that we use a finite language. The construction can be greatly simplified if we use an infinite language (see Theorem 2 in Section 3). All the constructions are modifications of E. Hrushovski's construction of an $\aleph_0$-categorical pseudoplane (see [Hr88], [Wa94], [KL92]).

It would be very interesting to know if similar results can be obtained for stable $\aleph_0$-categorical structures. The Second Author's Ph.D. thesis [Pa95] contains some partial attempts to do this. In contrast, in [St92], P. Steitz showed (using the structure theory of Cherlin, Harrington and Lachlan from [CHL85]) that if T is an $\aleph_0$-categorical, $\aleph_0$-stable theory and $\mathcal{M}$ is its countable model, then both the number $s_n(T)$ of complete n-types consistent with T and the maximal size $b_n(\mathcal{M})$ of the algebraic closure of a n-set of $\mathcal{M}$ are bounded:

$$s_n(T) \leq 2^{mn^2}$$

$$b_n(\mathcal{M}) \leq q^n$$

where m and q are natural numbers depending only on $\mathcal{M}$.

There is another sense, apart from the finiteness of the language, in which Theorem 1 is a stronger result than Theorem 2. For a structure $\mathcal{M}$ and a natural number k define the *k-ary algebraic closure* $\mathrm{acl}^k(X)$ of a subset X to be the smallest subset of $\mathcal{M}$ containing X which is closed under taking algebraic closure of k-element subsets. It is easy to see that the examples constructed to prove Theorem 1 have arbitrarily fast growth rate of k-ary algebraic closure, whereas those constructed (with much less effort) to prove Theorem 2 do not.

This notion is also relevant to the question of whether the structures in Theorem 1 can be taken to be stable. In [Ev00, Lemma 5.2], it is observed that k-ary algebraic closure in the stable $\aleph_0$-categorical structures in [Hr88] grows linearly.

Almost all of the material in this paper is taken from the Ph. D. Thesis [Pa95] of the Second Author at the University of East Anglia. A preliminary version of the paper has been available since 1995. Since then, the construction of unstable structures in [Hr88] has acquired an additional significance. In [Hr97], Hrushovski pointed out that these constructions give rise to simple, unstable $\aleph_0$-categorical structures which are not modular. To ensure that the structures we produce in

Theorems 1 and 2 are simple we require an extra condition on the function f used in the construction. If $f(3t) \leq f(t)+1$ (for $t \geq k$) then the argument in [Hr97] shows that $\mathcal{M}$ is supersimple, of SU-rank k in the case of the structures for Theorem 1, and SU-rank 1 for those of Theorem 2 (more detail can be found in [Ev00, Theorem 3.6]). It seems clear that our arguments can be adjusted to accommodate this extra condition.

We conclude the Introduction with some definitions and notation which will be used in the rest of the paper.

If L is a first-order language, then a countable L-structure $\mathcal{M}$ is said to be *$\aleph_0$-categorical* if its theory has, up to isomorphism, a unique countably infinite model.

A permutation group G on an infinite set Ω is said to be *oligomorphic* if, for every n, it has only a finite number of orbits on Ω^n, the set of the n-tuples of elements of Ω.

The Theorem of Ryll-Nardzewski, Svenonius and Engeler states that a countable first-order structure $\mathcal{M}$ is $\aleph_0$-categorical if and only if its automorphism group $\mathrm{Aut}(\mathcal{M})$ is oligomorphic (see [CK90]).

If $\mathcal{M}$ is $\aleph_0$-categorical and A is a finite subset of $\mathcal{M}$, then the *algebraic closure* $\operatorname{acl}(A)$ of A is the union of the finite orbits of the pointwise stabilizer $(\operatorname{Aut}(\mathcal{M}))_{(A)}$ of A (by Ryll-Nardzewski Theorem, $\operatorname{acl}(A)$ is finite).

The *age* of a structure $\mathcal{M}$ is the class of all structures (with the same language as $\mathcal{M}$) which are isomorphic to finite substructures of $\mathcal{M}$.

A structure $\mathcal{M}$ is *homogeneous* if any isomorphism between finite substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$.

The Theorem of Fraïssé states that a class $\mathcal{C}$ of finite structures over a relational language is the age of some countable homogeneous relational structure if and only if it is closed under isomorphisms and under taking substructures, has only countably many non-isomorphic members and has the amalgamation property. Moreover, two countable homogeneous structures with the same age are isomorphic. Hrushovski's construction is a modification of Fraïssé's, working with a restricted notion of amalgamation (self-sufficient amalgamation). More details can be found in [Hr88], [Wa94], [KL92] and [Ev94], but we hope that our account here is self-contained.

A *k-uniform hypergraph* is a pair (V, H) where V is a set of points called vertices and H is a set of k-subsets of V, called (hyper)edges.

For the rest of the paper, k is fixed ($k \geq 2$) and a k-uniform hypergraph will simply be called a graph.

The reader could refer to [Ca90], [Ev94] and [Ho93] for more background information on $\aleph_0$-categorical structures and infinite permutation groups.

2 Proof of Theorem 1

2.1 The Construction

The construction from [Hr88] makes use of a continuous, non-decreasing function f on the non-negative reals. We assume that the graph of f is piecewise linear, interpolated between the points $(x, f(x))$ for integers x. We define a class of finite (k-uniform hyper)graphs as follows.

Suppose A is a finite graph. Let $x(A)$ be the number of vertices and $e(A)$ be the number of edges of A. Let $y(A) = kx(A) - e(A)$. Let $\mathcal{H} = \{(x, y) : y \geq f(x)\}$ and let $\mathcal{C}_0$ be the class of finite graphs A such that $(x(A), y(A)) \in \mathcal{H}$ and $\mathcal{C}$ the class of the graphs A in $\mathcal{C}_0$ such that for each $A' \subseteq A$, $A' \in \mathcal{C}_0$.

In our construction, we require that f satisfies the following additional properties. The existence of such functions is shown in the proof of Theorem 1.

$\mathcal{F}_1$) For each $m < k$, $f(m) = km$; $k(k-1) < f(k) < k^2$;

$\mathcal{F}_2$) For each $m > k$, $f(m+1) - f(m) \leq \frac{1}{m+1}$;

$\mathcal{F}_3$) The area below f is convex;

$\mathcal{F}_4$) The function f is divergent;

$\mathcal{F}_5$) The function f is increasing;

$\mathcal{F}_6$) For each $m \geq k$, $m \geq \frac{1}{k}f(m+1) + \frac{4}{k^2(k-1)}(f(m+1))^2$.

So f is a slow-growing convex function which tends to infinity. First, we make a few comments on these properties. From $\mathcal{F}_1$ and $\mathcal{F}_2$ it follows that for every $m > k$ we have $f(m) \leq km$. $\mathcal{F}_6$ is a technical property which will be used to prove Lemma 5.

If $B \in \mathcal{C}$ we say that a subgraph A of B is *self-sufficient* in B (and write $A \leq B$) if, for every finite graph B' such that $A \subset B' \subseteq B$, we have $y(A) < y(B')$. If M is infinite and $A \subseteq M$ is finite, $A \leq M$ means that $A \leq B$ for all finite B with $A \subseteq B \subseteq M$.

We now want to prove that the class $\mathcal{C}$ is a self-sufficient amalgamation class. The following are taken directly from [Hr88]: proofs can also be found in [Wa94], for example.

Lemma 1 *Suppose $C \in \mathcal{C}$.*
(i) If $A, B \subseteq C$ then $y(A \cup B) \leq y(A) + y(B) - y(A \cap B)$.
(ii) If $A \leq C$ and $X \subseteq C$ then $A \cap X \leq X$.
(iii) If $A \leq B \leq C$ then $A \leq C$. □

Definition 1 Let B_1, B_2 be graphs and $A \subseteq B_1, B_2$. The *free amalgam* of B_1 and B_2 over A is the graph E whose universe is the disjoint union over A of the vertices of B_1 and B_2 and whose edges are just those of B_1 and those of B_2.

Then $x(E) = x(B_1) + x(B_2) - x(A)$ and $e(E) = e(B_1) + e(B_2) - e(A)$, so $y(E) = y(B_1) + y(B_2) - y(A)$.

Lemma 2 (Amalgamation) *Let $A, B_1, B_2 \in \mathcal{C}$ with $A \leq B_1$ and $A \leq B_2$. Let E be the free amalgam, as above. Then $E \in \mathcal{C}$ and $B_1, B_2 \leq E$.*

Proof. Let $x_0 = x(A)$, $e_0 = e(A)$, $y_0 = y(A) = kx(A) - e(A)$, $a = (x_0, y_0)$ and $x_i = x(B_i)$, $e_i = e(B_i)$, $y_i = y(B_i) = kx(B_i) - e(B_i)$, $b_i = (x_i, y_i)$, with $i = 1, 2$.

To prove that $E \in \mathcal{C}$, we want to prove that $b_1 + b_2 - a$ lies above the graph of the function f and that every subgraph of E lies in $\mathcal{C}_0$. However it is enough to prove the first part; in fact, if $E' \subseteq E$, then E' is the free amalgam over $A \cap E'$ of $B_1 \cap E'$ and $B_2 \cap E'$ and, by Lemma 1 $A \cap E' \leq B_i \cap E'$ $(i = 1, 2)$.

If $x_1 = x_0$ or $x_2 = x_0$, we have $A = B_1$ or $A = B_2$ and the lemma is trivial. So we can assume $x_1 - x_0 > 0$, $x_2 - x_0 > 0$. From now on, we assume that $x_1, x_2 \geq k$ (the other cases can be handled easily by $\mathcal{F}_1$). Assume, without loss of generality, that

$$\frac{y_2 - y_0}{x_2 - x_0} \geq \frac{y_1 - y_0}{x_1 - x_0}.$$

The slope of the line from a to b_2 is $(y_2 - y_0)/(x_2 - x_0)$ and

$$\frac{y_2 - y_0}{x_2 - x_0} \geq \frac{y_1 - y_0}{x_1 - x_0} \geq \frac{1}{x_1 - x_0} \geq \frac{1}{x_1} \geq f(x_1) - f(x_1 - 1) \geq f(x_1 + 1) - f(x_1),$$

where the last two inequalitiea follow from properties $\mathcal{F}_2$ and $\mathcal{F}_3$.

So the slope of to line from a to b_2 is greater or equal to the slope of the (right) tangent line to the graph of f at the point $(x_1, f(x_1))$ and, by $\mathcal{F}_3$, $b_1 + (b_2 - a)$ lies above the graph of f. Therefore $E \in \mathcal{C}$.

To prove that $B_1 \leq E$, consider B' such that $B_1 \subset B' \subseteq E$. Then B' is the free amalgam over A of B_1 and $B'_2 = B_2 \cap B'$. So $y(B') = y(B_1) + y(B'_2) - y(A)$. As $A \leq B_2$ we have $y(A) < y(B'_2)$, thus $y(B') > y(B_1)$. $\square$

Lemma 3 *There is a countable graph $\mathcal{M}$ with the following properties:*

1. *The class of the finite self-sufficient substructures of $\mathcal{M}$ is $\mathcal{C}$.*

2. *If $A \leq \mathcal{M}$, A finite, $B \in \mathcal{C}$ and $\alpha : A \to B$ is a self-sufficient embedding (i.e. $\alpha A \leq B$), then there exists a self-sufficient embedding $\gamma : B \to \mathcal{M}$ with $\gamma\alpha(a) = a$ for each $a \in A$.*

Moreover,

(i) If $X \subseteq \mathcal{M}$ there is $Y \leq \mathcal{M}$ with $X \subseteq Y$ and $|Y| \leq f^{-1}(y(X))$.

(ii) $\mathcal{M}$ is unique up to isomorphism, and any isomorphism between self-sufficient subgraphs of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.

(iii) $\mathcal{M}$ is $\aleph_0$-categorical.

Proof. The construction is standard using Lemmas 1 and 2 and (ii) follows from a back-and-forth argument once we have (i). To obtain (iii) note that properties (1) and (2) are first-order expressible (by (i)); alternatively one can use (i), (ii) to show that the number of n-types over $\emptyset$ is finite, and use the Ryll-Nardzewski theorem.

To see (i), note that if $X \subseteq B$ and $y(X) \geq y(B)$ then $x(B) \leq f^{-1}(y(X))$, by definition of $\mathcal{C}$. Take $Y \supseteq X$ with $y(Y) \leq y(X)$ and Y as large as possible. Then $Y \leq \mathcal{M}$. □

Remark 1 If $\mathcal{M}$ is the above structure, and A is a finite subset of $\mathcal{M}$, then $\mathrm{acl}(A)$ is the smallest B such that $A \subseteq B$ and $B \leq \mathcal{M}$.

2.2 Some graphs in $\mathcal{C}$

Suppose we have a function f with properties $\mathcal{F}_1 - \mathcal{F}_6$ and a structure $\mathcal{M}$ as in Lemma 3. Then for every $n > k$ and m with $n \leq m \leq f^{-1}(kn)$, we construct a graph $A_n \subseteq \mathcal{M}$ of size n such that $|\mathrm{acl}(A_n)| = m$.

Let A_n be the graph with n vertices and no edges. Then $x(A_n) = n$ and $y(A_n) = kn$. Let t be a natural number. We construct a graph A containing A_n in the following way. The vertices of A are $v_{i,j}$, where $1 \leq i \leq t-1$ and $0 \leq j \leq n-1$ and $v_{t,0}, \ldots, v_{t,s}$ where $s \leq n-1$. The set S_i of vertices with first index i will be referred to as the i-th layer of A. So A has t layers and all apart from the final one have n vertices. We now define the edges of A (remember that these are k-element sets of vertices). For $1 \leq i \leq t-1$ and $0 \leq j \leq n-1$ let $B_{i,j} = \{v_{i,j}, v_{i,j+1}, \ldots, v_{i,j+k-1}\}$, where the second index here is read modulo n. So this is a set of k 'consecutive' points in the i-th layer. We refer to such a subset as a *block*. Each block contains k subsets of size $k-1$, and for a vertex $v_{i+1,j}$ we have an edge $\{v_{i+1,j}\} \cup S$ for each such subset $S \subseteq B_{i,j}$.

We identify S_1 with A_n. To compute $y(A)$ consider building A by adding points one at a time in successive layers, starting with S_1. Each time we add a point $v_{i,j}$, we add k new edges, associated to the block $B_{i-1,j}$. Thus adding a new point in this way does not change the y-value of the graph, so $y(A) = y(S_1) = kn$. Let m_0 be the largest integer with $m_0 \leq f^{-1}(kn)$. Clearly if A is to be in $\mathcal{C}$ we must have $x(A) \leq m_0$. We show that the converse is true. The difficulty is, of course, that we must check that $y(H) \geq f(x(H))$ for all subgraphs H of A.

Lemma 4 *If H is a subgraph of A and $|S_i \cap H| = n$ for some $1 \leq i \leq t$, then $y(H) \geq kn$.*

Proof. We refer to $H \cap S_j$ as a layer of H and say that this is *full* if it has n vertices. Without loss of generality, we can assume that the last non-empty layer of H is a full layer. In fact, if it is not, we can consider the subgraph H' of H in which the full layer is the last one and note that $y(H') \leq y(H)$ (as in the computation of $y(A)$). As t has not yet been fixed, we may simplify the notation and assume that the last layer of H is the t-th and that this is full.

As H is a subgraph of A, we can obtain H from A by deleting some of its vertices. We want to calculate the number of edges which are lost when we delete those vertices. So let u_ℓ be the number of vertices we delete in the layer S_ℓ, with $\ell = 1, \ldots, t$. As $u_t = 0$ we have:

$$x(H) = x(A) - \sum_{\ell=1}^{t-1} u_\ell.$$

For $1 \leq \ell \leq t-1$ let r_ℓ be the minimum number of edges between S_ℓ and $S_{\ell+1}$ that we lose when we delete u_ℓ points in S_ℓ. Then $e(H) \leq e(A) - \sum_{\ell=1}^{t-1} r_\ell$, and so

$$y(H) \geq y(A) - k\sum_{\ell=1}^{t-1} u_\ell + \sum_{\ell=1}^{t-1} r_\ell = kn - k\sum_{\ell=1}^{t-1} u_\ell + \sum_{\ell=1}^{t-1} r_\ell.$$

Then we want to show that

$$\sum_{\ell=1}^{t-1} r_\ell \geq k\sum_{\ell=1}^{t-1} u_\ell. \tag{1}$$

Claim: Let u be the number of vertices deleted in S_ℓ. Then if r is the minimum number of edges lost between this layer and $S_{\ell+1}$, we have the following table:

(i)	$u = 0$	$r = 0$;
(ii)	$u = 1$	$r = k(k-1)$;
(iii)	$2 \leq u \leq n-k$	$r \geq uk$;
(iv)	$u = n-(k-1)$	$r = kn - 2$;
(v)	$n-(k-2) \leq u \leq n$	$r = kn$.

Proof of the claim:
(i) Trivial.
(ii) The point we delete belongs to k blocks and to $(k-1)$ of the k edges of each block.
(iv) If we have only $(k-1)$ vertices left in the layer, we can have at most 2 edges left.
(v) If we have less than $(k-1)$ vertices left in the layer, we cannot form any edge.
(iii) We split this case into two parts.

(a) Suppose that, after deleting the u vertices, there is at least one whole block of points left. We can assume this block to be $B_{\ell,n-k}$. We will prove the statement by induction on u. Suppose $u = 2$. By (ii), if we delete the vertex $v_{\ell,j}$, we lose $k(k-1)$ edges. If we now delete $v_{\ell,j+1}$, we lose 1 (new) edge for each of the $(k-1)$ blocks $B_{\ell,j-k+2}, \dots B_{\ell,j}$ and $(k-1)$ edges of the block $B_{\ell,j+1}$. In total, we lose $(k-1)\cdot 1 + 1\cdot(k-1) = 2(k-1)$ edges. If, instead of $v_{\ell,j+1}$, we delete $v_{\ell,j+2}$, we have only $(k-2)$ blocks in common with $v_{\ell,j}$ and so the number of edges we lose is $(k-2)\cdot 1 + 2\cdot(k-1)$; in general, if, after $v_{\ell,j}$, we delete the point $v_{\ell,j+q}$, we lose $(k-q)\cdot 1 + q\cdot(k-1)$ new edges, when $1 \le q \le k$ and $k(k-1)$ if $q \ge k$. So the minimum value of r is $2(k-1)$, reached for $q = 1$. Then, for $u = 2$

$$r = k(k-1) + 2(k-1) = (2+k)(k-1) = 2k + (k^2 - k - 2) \ge 2k = uk.$$

Now consider what happens when we delete $u+1$ vertices of S_ℓ. Do this in increasing order of the second index. By inductive hypothesis, the number r_u of edges lost when we have deleted the first u points is $\ge uk$. Let $v_{\ell,j}$ be the vertex with maximal j amongst the u points. Now delete the vertex $v_{\ell,j+p}$ (note that $0 < p < n-k-j$). In this case, if $p \le k-1$, we lose $(k-1)$ edges of $B_{\ell,j+p}$ and at least one further edge of $B_{\ell,j}$; if $p \ge k$, we lose $k-1$ edges of each of the k blocks $B_{\ell,j+p}, \dots, B_{\ell,j+p-(k-1)}$. Then

$$r \ge uk + k - 1 + 1 = k(u+1).$$

This proves case (iii)(a).

(b) Suppose now that every block of S_ℓ has lost at least one point. Every remaining point $v_{\ell,i}$ in S_ℓ is joined at most to k points of $S_{\ell+1}$ (the vertices $v_{\ell+1,i-k+1}, \dots v_{\ell+1,i}$). Moreover, as every remaining block of S_ℓ has at most $(k-1)$ points, there is at most one edge to which both $v_{\ell,i}$ and $v_{\ell+1,i-s}$

$(s = 0, \ldots k-1)$ belong. So the number $kn - r$ of edges left is at most $k(n-u)$. Then $kn - r \le k(n-u)$, i.e. $r \ge ku$.

This concludes the proof of the claim.

As in each case of the claim $r \ge uk$, Equation (1) holds and the lemma is proved. $\square$

Corollary 1 *Suppose $x(A) \le m_0$. If H is a subgraph of A and H has a full layer, then $H \in \mathcal{C}_0$.*

Proof. $f(x(H)) \le f(x(A)) \le kn \le y(H)$. $\square$

Lemma 5 *If $x(A) \le m_0$ then $A \in \mathcal{C}$.*

Definition 2 A graph $B \in \mathcal{C}_0$ is *tight* if $f(x(B)+1) > y(B)$.

Proof of Lemma 5: Suppose A is not in the class $\mathcal{C}$. Then there are subgraphs of A which are not in $\mathcal{C}_0$. Let H' be minimal with respect to this property. Let x be a vertex in the last layer of H' and consider the graph $H = H' \setminus \{x\}$. By minimality of H', H and all its subgraphs are in $\mathcal{C}_0$ and so in $\mathcal{C}$.

Claim: H is tight.

Proof of the claim: First, we note that $x(H) = x(H') - 1$ and $e(H) = e(H') - \varepsilon$ where:

$\varepsilon = k$, if all the block corresponding to x (i.e. the block of the previous layer to which x is joined) is in H';

$\varepsilon = 1$, if $(k-1)$ points of the block corresponding to x are in H';

$\varepsilon = 0$, if less than $(k-1)$ points of the block corresponding to x are in H'.

So $y(H) = k(x(H') - 1) - (e(H') - \varepsilon) = y(H') - k + \varepsilon$. If we can prove that $k = \varepsilon$, then $y(H') = y(H)$ and so $f(x(H)) \le y(H)$ and $f(x(H)+1) = f(x(H')) > y(H') = y(H)$, which proves that H is tight. Now $f(x(H')) - f(x(H)) = f(x(H)+1) - f(x(H))$, $y(H) \ge f(x(H))$ and $y(H') < f(x(H'))$. By $\mathcal{F}_2$,

$$1 \ge \frac{1}{x(H)+1} \ge f(x(H)+1) - f(x(H)) > y(H') - y(H)$$

and $y(H') - y(H) = k - \varepsilon$. Then $k - \varepsilon < 1$ and so $k = \varepsilon$. This concludes the proof of the claim.

Also $y(H) < kn$, because

$$kn = y(A) \geq f(x(A)) \geq f(x(H)+1) > y(H') = y(H).$$

Let $h_i = |H \cap S_i|$; n_2 the number of points $v_{i+1,j}$ of H with the corresponding block $B_{i,j}$ contained in H; n_1 the number of points of H having all but 1 point of the corresponding block contained in H; n_0 the number of points of H having less than $k-1$ points of the corresponding block contained in H. Let s_i be the number of points of H in the i-th layer not having a complete block in the $(i-1)$-th layer and $\ell = \max(i : h_i \neq 0)$. Note that in counting n_0, n_1, n_2 and s_i we do not consider the points in the first layer.

First, we show that, if $2 \leq i \leq \ell$, then $h_i > 0$, i.e. H has no empty layer apart, possibly, from the first.

Suppose this is not true. By definition $h_\ell > 0$ so H can be written as the free amalgam over the empty set of two non-empty subgraphs H_1 and H_2, with $H_1, H_2 \in \mathcal{C}$. Then $x(H) = x(H_1) + x(H_2)$ and $y(H) = y(H_1) + y(H_2)$. Suppose $x(H_2) \geq x(H_1) \geq k$. By $\mathcal{F}_2$:

$$\begin{array}{rcl}
f(x(H)+1) - f(x(H)) & \leq & \frac{1}{x(H)+1} = \frac{1}{x(H)-(-1)} \\
f(x(H)) - f(x(H)-1) & \leq & \frac{1}{x(H)} = \frac{1}{x(H)-0} \\
 & \vdots & \\
f(x(H_2)+1) - f(x(H_2)) & \leq & \frac{1}{x(H_2)+1} = \frac{1}{x(H)-[x(H_1)-1]}
\end{array}$$

If we sum these inequalities, we obtain:

$$f(x(H)+1) - f(x(H_2)) \leq \sum_{i=-1}^{x(H_1)-1} \frac{1}{(x(H)-i)} < \sum_{i=-1}^{x(H_1)-1} \frac{1}{x(H)-[x(H_1)-1]} =$$

$$[x(H_1)+1]\frac{1}{x(H)-x(H_1)+1} = \frac{x(H_1)+1}{x(H_2)+1} \leq 1.$$

Then $f(x(H)+1) < f(x(H_2))+1 \leq y(H_2)+1 \leq y(H_2)+y(H_1) = y(H)$ and, as H is tight, this is a contradiction. If $x(H_1) < k$ then one easily obtains a contradiction to tightness.

Note that by Lemma 4, H has no full layer. Moreover, by construction of A we may clearly assume that $h_1 \neq 0$. Then for each i with $1 \leq i \leq \ell$, we have $0 < h_i < n$. Note that $\sum_{i=2}^{\ell} s_i = n_0 + n_1$. For $i = 1, \ldots, \ell - 1$, the number of points of H in the $(i+1)$-th layer which have a complete block in the previous layer is $h_{i+1} - s_{i+1}$. Thus as the i-th layer is not full,

$h_i \geq (h_{i+1} - s_{i+1}) + k - 1$, with equality occurring when the h_{i+1} points with complete block are consecutive (mod n). Rearranging we have:

$$h_{i+1} - h_i \leq s_{i+1} - (k-1) \text{ for each } i = 1, \ldots, \ell - 1.$$

We can sum and obtain, for $1 \leq j \leq \ell - 1$,

$$h_{j+1} - h_1 \leq \sum_{i=2}^{j+1} s_i - j(k-1), \tag{2}$$

i.e.

$$h_{j+1} \leq h_1 + \sum_{i=2}^{j+1} s_i - j(k-1) \leq h_1 + n_0 + n_1 - j(k-1).$$

So

$$\sum_{j=1}^{\ell-1} h_{j+1} \leq (\ell-1)(h_1 + n_0 + n_1) - \frac{\ell(\ell-1)}{2}(k-1),$$

and therefore

$$x(H) \leq h_1 + (\ell-1)(h_1 + n_0 + n_1) - \frac{\ell(\ell-1)}{2}(k-1).$$

Now

$$y(H) = k(h_1 + n_0 + n_1 + n_2) - kn_2 - n_1 = kh_1 + kn_0 + (k-1)n_1 \tag{3}$$

and as $k - 1 \geq k/2$, we have $y(H) \geq (k/2)(h_1 + n_0 + n_1)$. So

$$x(H) \leq h_1 + \frac{2y(H)}{k}(\ell-1) - \frac{\ell(\ell-1)}{2}(k-1). \tag{4}$$

We want an inequality independent of ℓ. By (2) with $j = \ell - 1$,

$$h_\ell - h_1 \leq \sum_{i=2}^{\ell} s_i - (\ell-1)(k-1) = n_0 + n_1 - (\ell-1)(k-1)$$

thus

$$(\ell-1) \leq \frac{1}{k-1}(n_0 + n_1 - h_\ell + h_1) \leq \frac{1}{(k-1)}(n_0 + n_1 + h_1) \leq \frac{1}{(k-1)}\frac{2y(H)}{k}. \tag{5}$$

By (4) and (5)

$$x(H) \le h_1 + \frac{2y(H)}{k}\frac{2y(H)}{k(k-1)} - \frac{\ell(\ell-1)}{2}(k-1) \le h_1 + \frac{4(y(H))^2}{k^2(k-1)}.$$

Also, by (3), $y(H) \ge kh_1$ and therefore

$$x(H) \le \frac{y(H)}{k} + \frac{4(y(H))^2}{k^2(k-1)}.$$

But by $\mathcal{F}_6$ and tightness of H we have

$$x(H) \ge \frac{f(x(H)+1)}{k} + \frac{4}{k^2(k-1)}(f(x(H)+1))^2 > \frac{y(H)}{k} + \frac{4}{k^2(k-1)}(y(H))^2$$

and we have a contradiction. $\square$

For the rest of this subsection, $\mathcal{M}$ is as in Lemma 3.

Lemma 6 *Suppose $A \le \mathcal{M}$ is isomorphic to the graph just constructed. Then A is the algebraic closure of A_n.*

Proof. We know that $Y = \mathrm{acl}(A_n) \le A$ as $A \le \mathcal{M}$. On the other hand by Lemma 4, $y(Y) \ge kn = y(A)$. So $Y = A$. $\square$

Corollary 2 *For each $n > k$, there exists $A_n \subset M$ such that $|A_n| = n$ and $|acl(A_n)| = m$ where m is the integer part of $f^{-1}(kn)$.* $\square$

Lemma 7 *The group Aut($\mathcal{M}$) is $(k-1)$-transitive.*

Proof. By Lemma 3(ii) it suffices to observe that any set of $k-1$ points in $\mathcal{M}$ is self-sufficient. This follows from $\mathcal{F}_1$. $\square$

2.3 Proof of Theorem 1

Theorem 1 *For every sequence $(a'_i)_{i\in\mathbf{N}}$, there exists a $(k-1)$-transitive countable $\aleph_0$-categorical k-uniform hypergraph $\mathcal{M}$ such that for each $n > k$, there exists $A_n \subseteq \mathcal{M}$ with $|A_n| = n$ and $|acl(A_n)| \ge a'_n$.*

Proof. By Corollary 2 it is now enough to construct a function f satisfying $\mathcal{F}_1$-$\mathcal{F}_6$ and such that $f^{-1}(kn) \geq a'_n$, for $n > k$.

We define by induction a sequence $(a_i)_{i\in\mathbf{N}}$ with $a_i \geq a'_i$ for each $i > k$ and a function (defined on the integers) such that $f(a_i) = ki$, for each $i > k$.

Let $a_i = i$, $f(i) = ik$ for $i < k$, $f(k) = k^2 - 1$. Suppose $j \geq k+1$ and that we have constructed $a_1, \ldots, a_{j-1}$ and $f(m)$ for $m \leq a_{j-1}$. Let r be an integer parameter and define a space $P(r) \subseteq \mathbf{R}^r$ by $(p_1, \ldots, p_r) \in P(r)$ iff

$\mathcal{P}_1)$ $p_i > 0$;

$\mathcal{P}_2)$ $p_i > p_{i+1}$;

$\mathcal{P}_3)$ $p_i < 1/(a_{j-1} + i)$;

$\mathcal{P}_4)$ $p_1 < f(a_{j-1}) - f(a_{j-1} - 1)$.

Each of the sets defined by a condition $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ is convex, thus so is $P(r)$. In particular, $P(r)$ is connected. Define a function $\gamma : P \to \mathbf{R}$ by

$$\gamma(p_1, \ldots p_r) = \sum_{i=1}^{r} p_i.$$

The function γ is continuous and 0 is a limit point of $\mathrm{Im}\gamma$. Moreover, as

$$\sum_{i=1}^{\infty} \min \left(\frac{1}{a_{j-1} + i} \ , \ f(a_{j-1}) - f(a_{j-1} - 1) \right)$$

is divergent, there exists $\overline{r}$ such that, for each $r \geq \overline{r}$, we have $\sum_{i=1}^{r} p_i > k$ for some $(p_1, \ldots, p_r) \in P(r)$. For such an r, by the Intermediate Value Theorem, there exists $(\alpha_1, \ldots, \alpha_r) \in P(r)$ such that

$$\sum_{i=1}^{r} \alpha_i = k.$$

Let r_{j-1} be the smallest such r with

$$a_{j-1} + r \geq \ \max\,(a'_j, (j+1) + [4(j+1)^2]/(k-1)),$$

and let $(\alpha_1, \ldots, \alpha_{r_{j-1}})$ be chosen as above. Let $a_j = a_{j-1} + r_{j-1}$ and, for $a_{j-1} + 1 \leq m \leq a_j$,

$$f(m) = f(a_{j-1}) + \sum_{i=1}^{m - a_{j-1}} \alpha_i.$$

Note that

$$f(a_j) = f(a_{j-1}) + \sum_{i=1}^{a_j - a_{j-1}} \alpha_i = k(j-1) + \sum_{i=1}^{r_{j-1}} \alpha_i = kj$$

and $a_j = a_{j-1} + r_{j-1} \geq a'_j$.

Let f be the simplest piecewise linear function which joins the points $(n, f(n))$ $(n \geq k)$. We now want to show that f satisfies $\mathcal{F}_1, \ldots, \mathcal{F}_6$.

$\mathcal{F}_1$. By definition of f.

$\mathcal{F}_2$. By construction and $\mathcal{P}_3$.

$\mathcal{F}_3$. By $\mathcal{P}_2$ and $\mathcal{P}_4$.

$\mathcal{F}_4$. As $f(a_j) = kj$ for $j > k$.

$\mathcal{F}_5$. By $\mathcal{P}_1$.

$\mathcal{F}_6$. By definition of r_{i-1}, for each $i > k$, $a_i \geq (i+1) + [4(i+1)^2]/(k-1)$. Let $n > k$ and let j be such that $a_j \leq n < n+1 \leq a_{j+1}$. Then $f(n+1) \leq f(a_{j+1}) = k(j+1)$, so

$$\frac{f(n+1)}{k} + \frac{4}{k^2(k-1)}[f(n+1)]^2 \leq \frac{k(j+1)}{k} + \frac{4k^2(j+1)^2}{k^2(k-1)} \leq a_j \leq n.$$

$\square$

3 Using an Infinite Language

3.1 The amalgamation class

In the previous section we have used a language with a single k-ary relation symbol. If an infinite language is used, things become a lot simpler. Suppose $k_0 \geq 1$.

Let $\mathcal{L}$ be a language with, for every $k \geq k_0$, a_k k-ary relations R_{ik} $(i = 1, \ldots, a_k)$. We work with structures in which these relations are symmetric and disjoint. So, if $R_{ik}(x_1, \ldots, x_k)$ holds, then $x_1, \ldots, x_k$ are distinct and R_{ik} holds for any permutation of the x_i's and $R_{i'k}(x_1, \ldots, x_k)$ does not hold for $i \neq i'$.

Let X be such a structure with domain a finite set $\{x_1, \ldots, x_s\}$ and some of the relations R_{ik} defined on it. Let $|X| = x(X)$ and define:

$$y(X) = |X| - \sum_{i,k} |R_{ik}[X]|$$

i.e.

$$y(X) = s - \sum_{k=k_0}^{s} \sum_{i=1}^{a_k} |R_{ik}[X]|,$$

where $|R_{ik}[X]|$ denotes the number of subsets of $\{x_1, \ldots, x_s\}$ which satisfy the relation R_{ik}. As before we define a finite structure A to be *self-sufficient* in a structure B if $A \subseteq B$ and $y(A) < y(B')$ for every B' such that $A \subset B' \subseteq B$.

Let f be a continuous function on the non-negative reals whose graph is the piecewise linear hull of the points $(n, f(n))$ for integer n and satisfying:

f_1) For every $n \geq k_0$, $f(n+1) - f(n) \leq 1/n$.

f_2) The function f is divergent.

f_3) The function f is increasing and the sequence $f(n+1) - f(n)$ is non-increasing.

f_4) For every $n \geq k_0$, $f(n + a_{n+1}) \leq n$.

f_5) For every $1 \leq s \leq k_0$, $f(s) = s - 1$.

As before, let $\mathcal{C}_0 = \{X : y(X) \geq f(|X|)\}$ and $\mathcal{C} = \{X : X' \in \mathcal{C}_0$ for each $X' \subseteq X\}$. Lemma 1 still holds and we define the free amalgam as before. Then Lemma 2 holds for this class, and we build a structure $\mathcal{M}$ with properties as in the previous section. Its automorphism group is $(k_0 - 2)$-transitive.

Theorem 2 *For any sequence $\{a_i\}_{i \in \mathbf{N}}$, there exists a $(k_0 - 2)$-transitive structure $\mathcal{M}$ such that for each $k > k_0$, there exists $X \subseteq \mathcal{M}$ with $|X| = n$ and $|acl(X)| \geq a'_n$.*

Proof. As with the previous construction, one can find $a_i \geq a'_i$ and a function f satisfying the conditions f_1 - f_5. Take $\mathcal{M}$ constructed with these as data.

Let X be a set with n points and no relations defined on it. Then $y(X) = n$. Let Y be a structure with domain $X \cup \{x_1, \ldots, x_{a_{n+1}}\}$ and such that the only relations are $R_{i(n+1)}(X \cup \{x_i\})$, for $i = 1, \ldots, a_{n+1}$. Then $|Y| = n + a_{n+1}$ and

$$y(Y) = n + a_{n+1} - a_{n+1} = n = y(X).$$

The structure Y is in $\mathcal{C}$. In fact, if $Y' \subset X$ and $Y' \not\supseteq X$ then $y(Y') = |Y'| \geq f(|Y'|)$. If $Y' \supseteq X$ then $y(Y') = n \geq f(n + a_{n+1}) \geq f(|Y'|)$.

By the properties of $\mathcal{M}$, we can find a structure Z, self-sufficient in $\mathcal{M}$ and isomorphic to Y. Let X' be the copy of X inside Z. Exactly as before, one shows that $\mathrm{acl}(X') = Z$. $\square$

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