Bounds in the Theory of Finite Covers

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Abstract

We give an upper bound for the number of conjugacy classes of closed subgroups of the full wreath product $F \operatorname{Wr}_W \operatorname{Sym}(\Omega)$ which project onto $\operatorname{Sym}(\Omega)$. Here, Ω is infinite, W is the set of n-tuples of distinct elements from Ω (for some finite n), F is a finite nilpotent group and the topology on the wreath product is that of pointwise convergence in its imprimitve permutation action. The result addresses a problem which arises in a natural model-theoretic context about classifying certain types of finite covers.
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1 Introduction

This paper is a sequel to [7] and [9], but can be read independently of these. It is motivated by questions arising in model theory concerning finite covers, and can be seen as providing new information about disintegrated totally categorical structures. However, the problems and results can be phrased naturally in terms of permutation groups, and as the methods we adopt are purely group-theoretic, we avoid model-theoretic terminology in most of the paper. A brief commentary on the model theory is given in Section 6.2.

Given an infinite set Ω, a natural number n, and a finite group F acting faithfully on a finite set Δ, consider the symmetric group $G = \operatorname{Sym}(\Omega)$ acting on $W = \Omega^{(n)}$ and form the full wreath product $\Gamma_0 = F \operatorname{Wr}_W G$ acting on $C_0 = \Delta \times W$. There is a natural topology on the symmetric group on any (infinite) set making it into a topological group in which pointwise stabilisers of finite sets give a base of open neighbourhoods of the identity. With this topology, Γ_0 is a closed subgroup of $\operatorname{Sym}(C_0)$ and the natural homomorphism $\rho : \Gamma_0 \to G$ is continuous. In particular, its kernel, K_0, the base group of the wreath product, is closed (and is the product F^W, so is compact and in fact, profinite).

Let $\mathcal{C}$ be the set of closed subgroups $\Gamma \leq \Gamma_0$ with $\rho(\Gamma) = G$. The main purpose of this paper is to give an explicit upper bound on the number of Γ_0-conjugacy classes of subgroups in $\mathcal{C}$ in terms of n and $|F|$, at least in the case where F is *nilpotent*. For general F, it is known that any $\Gamma \in \mathcal{C}$ contains some $\Sigma \in \mathcal{C}$ with $\Sigma \cap K_0$ nilpotent ([7], 5.2 and 5.3), so there may be some possibility to extend our results to the general case.

To get our bounds we first use representation theory of the infinite symmetric group (cf. [9]) to quantify the cohomology calculations in [7] for the case where F is abelian. We then give an inductive argument and a little more cohomological machinery to extend this to the case where F is nilpotent. A key extra ingredient here is a result of J. Krajíček ([14]) on the weight of generating sets for submodules of tabloid modules for the infinite degree symmetric group.

The proof is by induction on the composition length of F, so the base case is where F is $\mathbb{F}_p$, the cyclic group of prime order p, regarded as the additive group of the field with p elements. We regard $K_0 = \mathbb{F}_p^W$ as a topological $\mathbb{F}_p G$-module, and if $\Gamma \in \mathcal{C}$ then $K_\Gamma = \Gamma \cap K_0$ is a closed submodule. We have two steps to describe the possible (conjugacy classes of) Γ. First, bound the number of closed submodules K of K_0. Then for each such K bound the number of Γ_0-conjugacy classes of $\Gamma \in \mathcal{C}$ with $K_\Gamma = K$. The latter number is given by the first (continuous) cohomology group $H^1_c(G, K_0/K)$, and we use dimension-shifting together with the submodule bounds to bound this. This is done in Section 4.

To bound the number of closed submodules of K_0 we consider instead the dual problem: bound the number of $\mathbb{F}_pG$-submodules of the permutation module $\mathbb{F}_p\Omega^{(n)}$. To do this, we use the extension of the representation theory of the finite symmetric groups initiated in [9]. To make inductive arguments work, we need to work with permutation modules $\mathbb{F}_p\Omega^\lambda$ for more general partitions λ. This is done in Section 3.

The passage from the case $F = \mathbb{F}_p$ to arbitrary nilpotent F is done in the proof of Theorem 5.1.

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2 General Results

In this section, we give some general results about modules with finitely many submodules. Throughout, R denotes a ring (with identity 1), G a group, $\mathbb{F}$ a field and A an $\mathbb{F}$-algebra. For R-modules M and N, $\mathrm{Hom}_R(M, N)$ is the group of R-module homomorphisms from M to N and $\mathrm{End}_R(M)$ is $\mathrm{Hom}_R(M, M)$.

Lemma 2.1 *(i) If M is an R-module which has a finite composition series of length m, then any submodule and any factor module of M has a composition series of length at most m.*
(ii) If m_1 and m_2 are the (finite) lengths of every composition series of N and M/N respectively, then M has a finite composition series of length $m_1 + m_2$.
(iii) Let $M_1, M_2, \ldots, M_s$ be R-modules. If M_i has a finite composition series of length m_i (for $i = 1, 2, \ldots, s$), then $M_1 \oplus M_2 \oplus \cdots \oplus M_s$ has a composition series of length $\sum_{i=1}^s m_i$.
(iv) If $\phi_i : M \longrightarrow M_i$, $(i = 1, 2, \ldots, s)$ are R-homomorphisms, then $M/\bigcap_{i=1}^s \ker \phi_i$ is isomorphic to a submodule of $M_1 \oplus M_2 \oplus \cdots \oplus M_s$. $\square$

Definition 2.2 Say that an A-module M is *good* if it has a finite composition series and $\mathrm{End}_A(S) = \{\alpha \ \mathrm{id}_S : \alpha \in \mathbb{F}\}$ for all composition factors S of M, (where id_S denotes the identity map on S).

Lemma 2.3 *Suppose M is a completely reducible good A-module of composition length c. Then $dim_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}G}(A, A)) \leq c^2$.* $\square$

Lemma 2.4 *Let $\mathbb{F}$ be a finite field and M a completely reducible good A-module with composition length c. Then $\mid \mathbb{F} \mid^{c^2}$ is an upper bound for the number of submodules of M.*

Proof: Let $M = S_1 \oplus S_2 \oplus \cdots \oplus S_l$, written according to isomorphism of irreducible submodules, that is $S_i \cong I \oplus I \oplus \cdots \oplus I$ for some irreducible submodule I. Then the number of submodules of M is $\Pi_{i=1}^l n_i$, where n_i is the number of submodules of S_i, by ([5], Lemma 15.2) and the fact that the S_i have no composition factors in common. So it is enough to prove that the number of submodules of I^t is at most $|\mathbb{F}|^{t^2}$. This is done by induction on t, and the base case is of course trivial. Suppose it holds for t and let $S = I^{t+1}$.

Let H be a submodule of S and let T be the projection of H onto the first t coordinates. Suppose first that the kernel of this projection is $\{0\}$. Then there is a unique module homomorphism $\psi : T \to I$ with $H = \{(\bar{x}, \psi(\bar{x})) : \bar{x} \in T\}$. Now, T is isomorphic to I^r for some $r \leq t$ so the number of possibilities for such homomorphisms is at most $|\mathbb{F}|^t$, for fixed T. By inductive hypothesis, the number of possibilities for T is at most $|\mathbb{F}|^{t^2}$, thus the number of such H is at most $|\mathbb{F}|^{t^2+t}$.

If the kernel of the projection of H onto the first t factors is not zero, then H contains the last direct summand of S. By inductive hypothesis, the number of possibilities for such H is at most $|\mathbb{F}|^{t^2}$. Thus, the number of submodules of S is at most $|\mathbb{F}|^{t^2+t} + |\mathbb{F}|^{t^2} \leq |\mathbb{F}|^{(t+1)^2}$. $\square$

Lemma 2.5 *Let $\mathbb{F}$ be a finite field and let M be a good A-module with composition length c. Then $\mid \mathbb{F} \mid^{c^3}$ is an upper bound for the number of submodules of M.*

Proof: The proof is by induction on c. If $c = 1$, then M is irreducible and the result holds. Let $T = J(M)$ be the intersection of all maximal submodules of M. Clearly T is a proper submodule and M/T is completely reducible. By the previous lemma, we may assume that $T \neq \{0\}$.

Let $X \leq M$, $X_1 = X \cap T$, $X_2 = X + T$ and $\phi_X \in \mathrm{Hom}_A(X_2/T, M/X_1)$ be given by $\phi_X(x+T) = x + X_1$. Then X is the union of the cosets in the image of ϕ_X, so is determined by X_1, X_2 and ϕ_X. By induction, the number of possibilities for X_1 is at most $|\mathbb{F}|^{(c-1)^3}$. As X_2/T is completely reducible (of composition length at most $c-1$) the image of ϕ_X lies in the socle of M/X_1, and this has composition length at most $c - 1$. Thus the number of possibilities for ϕ_X is at most $|\mathbb{F}|^{(c-1)^2}$. By the previous lemma the number of possibilities for X_2 is at most $|\mathbb{F}|^{(c-1)^2}$. The inductive step follows readily. $\square$

We will use the following to prove that our permutation modules M^λ are good. It is of course just a version of Schur's Lemma.

Lemma 2.6 *Let M be an absolutely irreducible $A-$module and $\varphi \in End_A(M)$ have a non-zero invariant finite dimensional subspace M_0* (i.e $\varphi(M_0) \subseteq M_0$)*. Then $\varphi = \alpha id_M$ for some $\alpha \in \mathbb{F}$, where $id_M : M \longrightarrow M$ is the identity map.*

Proof: Let $\mathbb{E}$ be an algebraically closed field containing $\mathbb{F}$. Then φ extends to an endomorphism $\tilde{\varphi}$ of $M \otimes_{\mathbb{F}} \mathbb{E}$ and $M_0 \otimes_{\mathbb{F}} \mathbb{E}$ is a finite dimensional, non-zero, $\tilde{\varphi}$-invariant $\mathbb{E}$-subspace. As $\mathbb{E}$ is algebraically closed, $\tilde{\varphi}$ restricted to $M_0 \otimes_{\mathbb{F}} \mathbb{E}$ has an eigenvalue α in $\mathbb{E}$. Thus, by irreducibility of $\tilde{M} = M \otimes_{\mathbb{F}} \mathbb{E}$, we have $\ker(\tilde{\varphi} - \alpha \mathrm{id}_{\tilde{M}}) = \tilde{M}$. It follows easily that $\alpha \in \mathbb{F}$ and $\varphi = \alpha \mathrm{id}_M$. $\square$

3 Modules for the symmetric group

3.1 Tabloid modules

In this section Ω is an infinite set and $G = \mathrm{Sym}(\Omega)$. Following the notation of ([9], Section 1.2) we work with partitions $\lambda = (\infty - n, \lambda_2, \ldots, \lambda_r)$ where $n \in \mathbb{N}$, and $(\lambda_2, \ldots, \lambda_r)$ is a partition of n. We denote by Ω^λ the set of λ-tabloids (as in the finite case these are ordered partitions of Ω into sets of size $\infty - n, \lambda_2, \ldots, \lambda_r$), and M^λ the $\mathbb{F}G$-permutation module with Ω^λ as $\mathbb{F}$-basis. We will occasionally embelish various notations with a subscript to indicate the field over which we are working.

Note that there are natural bijections between the set $\Omega^{(n)}$ of n-tuples of distinct elements from Ω and $\Omega^{(\infty-n,1,\ldots,1)}$, and between $[\Omega]^n$, the set of subsets of size n from Ω, and $\Omega^{(\infty-n,n)}$. Moreover these bijections commute with the G-actions on these sets, and we shall use these bijections to make identifications between these sets, without further comment.

If $\lambda_2 \geq \ldots \geq \lambda_r > 0$ we say that λ is a *proper* partition. For any λ there is a unique proper partition $\hat{\lambda}$ obtained by rearranging the λ_i and there is a canonical isomorphism $M^\lambda \to M^{\hat{\lambda}}$.

Next we give an analogue to ([11], Definition 3.2) for infinite partitions.

Definition 3.1 Let $\lambda = (\infty - n, \lambda_2, \ldots, \lambda_r)$ and $\mu = (\infty - m, \mu_2, \ldots, \mu_s)$ be proper partitions of an infinite set Ω, where $\sum_{i=2}^r \lambda_i = n$ and $\sum_{i=2}^s \mu_i = m$. If $\sum_{i>k}^s \mu_i \leq \sum_{i>k}^r \lambda_i$ for all $k \geq 1$, we say that μ *dominates* λ, and write $\mu \trianglerighteq \lambda$.

We next recall certain key homomorphisms with domain M^λ. If $i < r$ and $1 \leq \delta \leq \lambda_{i+1}$ define $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ where $\mu_i = \lambda_i + \delta$, $\mu_{i+1} = \lambda_{i+1} - \delta$ and all other μ_j equal the corresponding λ_j. The following is elementary, but crucial to all inductions.

Lemma 3.2 *With the above notation $\hat{\mu} \trianglerighteq \lambda$.* $\square$

Continuing with this notation, we define an $\mathbb{F}G$-homomorphism $\hat{\psi}_{i,\delta} : M^\lambda \to M^\mu$ by first defining its effect on the tabloid basis of M^λ and extending it linearly. If $\{t\} \in \Omega^\lambda$ then $\hat{\psi}_{i,\delta}(\{t\})$ is the sum of all μ-tabloids which can be obtained by tranferring δ elements from the $i+1$-th row of t to the i-th row. If we follow this by the canonical isomorphism we obtain $\psi_{i,\delta} : M^\lambda \to M^{\hat{\mu}}$.

Part of the importance of these homomorphisms is that they characterise the *Specht submodule* S^λ of M^λ. Recall that this is defined as the submodule of M^λ generated by the λ-*polytabloids* (see [11, 12] for definitions, and [9] for the straightforward extension to the infinite case). The following is an immediate consequence of ([9], Theorem 2.4):

Lemma 3.3 *Let $\lambda = (\infty - n, \lambda_2, \ldots, \lambda_r)$ be a proper partition of an infinite set Ω. Then*

$$S^\lambda = \bigcap_{i=1}^{r-1} \bigcap_{\delta=1}^{\lambda_{i+1}} \ker \psi_{i,\delta}. \square$$

We denote by $(.,.)$ the G-invariant bilinear form on M^λ which has Ω^λ as an orthonormal basis, and let $\perp$ denote orthogonality with respect to this form. Define $D^\lambda = S^\lambda / S^\lambda \cap (S^\lambda)^\perp$. Recall that James' Submodule Theorem ([12]) holds for our infinite partitions:

Theorem 3.4 ([9], Theorem 2.1.) *If U is a submodule of M^λ then either $U \geq S^\lambda$ or $U \leq (S^\lambda)^\perp$. In particular, D^λ is either $\{0\}$ or irreducible.* $\square$

Next we give the characterisation of $(S^\lambda)^\perp$ analogous to the characterisation of S^λ in Lemma 3.3. Again, this is a straightforward extension of the finite case ([13]). For $1 < l < r$ and $0 \leq \delta \leq \lambda_l$ define $\nu = (\nu_1, \ldots, \nu_r)$ where $\nu_l = \lambda_l + \delta$, $\nu_{l+1} = \lambda_{l+1} - \delta$, and $\nu_j = \lambda_j$ for all other j. As before:

Lemma 3.5 *With the above notation $\hat{\nu} \trianglerighteq \lambda$.* $\square$

Continuing with this notation, we define an $\mathbb{F}G$-homomorphism $\hat{\phi}_{i,\delta} : M^\nu \to M^\lambda$ by first defining its effect on the tabloid basis of M^ν and extending it linearly. If $\{t\} \in \Omega^\nu$ then $\hat{\phi}_{i,\delta}(\{t\})$ is the sum of all λ-tabloids which can be obtained by tranferring δ elements from the i-th row of t to the $i+1$-th row. If we precede this by the inverse of the canonical isomorphism we obtain $\phi_{i,\delta} : M^{\hat{\nu}} \to M^\lambda$. The following then follows immediately from ([9], Theorem 2.7):

Lemma 3.6 *Let $\lambda = (\infty - n, \lambda_2, \ldots, \lambda_r)$ be a proper partition of a set Ω with $r > 2$. Then $(S^\lambda)^\perp = \sum_{i=2}^{r-1} \sum_{\delta=1}^{\lambda_{i+1}} \mathrm{Im}\ \phi_{i,\delta}$. If $r = 2$, then $(S^\lambda)^\perp = \{0\}$.* $\square$

The following is the basis for our induction:

Definition 3.7 Let $\lambda = (\infty - n, \lambda_2, \ldots, \lambda_r)$ be a proper partition of a set Ω. Define $L(\lambda)$ to be the maximal length of a chain of proper partitions $\mu^1 \triangleright \mu^2 \triangleright \ldots \triangleright \mu^L \triangleright \lambda$.

Henceforth, we shall work with proper partitions unless we explicitly state otherwise.

Lemma 3.8 *If $n \in \mathbb{N}$ and $\lambda = (\infty - n, \lambda')$ where λ' is a partition of n, then $n \leq L(\lambda) \leq n^2$.* $\square$

Theorem 3.9 *Let $\lambda = (\infty - n, \lambda_2, \ldots, \lambda_r)$ be a partition of an infinite set Ω. Then:*
(a) any composition factor of M^λ is of the form D^μ for some $\mu \trianglerighteq \lambda$;
(b) M^λ has a finite composition series of length at most $2^{L(\lambda)[L(\lambda)+4]/2}$.
(c) The composition length of M^λ is at most 2^{n^4}.

Proof: There is a total order (of order type ω^*) on the partitions, which contains the partial ordering $\trianglelefteq$. We prove parts (a) and (b) of the result by downwards induction on this ordering. Part (c) then follows from the previous lemma. The base step is trivial, so we assume as inductive hypothesis that the result is true for partitions $\mu \triangleright \lambda$, and deduce it for λ.

By Lemma 3.2 and Lemma 3.3, there are proper partitions $\mu^1, \mu^2, \ldots, \mu^s$, where $s \leq n$ and $\mathbb{F}\mathrm{Sym}(\Omega)$-homomorphisms $\psi_{i,\delta} : M^\lambda \longrightarrow M^{\mu^i}$ such that $S^\lambda = \bigcap_{i=1}^s \ker \psi_i$ and $\mu^i \triangleright \lambda$. From the definition of $L(\lambda)$, we get $L(\mu^i) \leq L(\lambda) - 1$. Therefore by inductive hypothesis M^{μ^i} has a finite composition series of length $m_1 \leq 2^{[L(\lambda)-1][L(\lambda)+3]/2}$, and all the composition factors are of the form D^ν for various $\nu \rhd \lambda$. So M^λ/S^λ, which is isomorphic to a sub-module of $M^{\mu^1} \oplus M^{\mu^2} \oplus \cdots \oplus M^{\mu^s}$ by Lemma 2.1(iv), has composition series of length $m_2 \leq s2^{[L(\lambda)-1][L(\lambda)+3]/2} \leq n2^{[L(\lambda)-1][L(\lambda)+3]/2}$ with the composition factors satisfying (a). Also we have $t \leq n$ and $\nu^1, \ldots, \nu^t$ with $\nu^i \rhd \lambda$ and $(S^\lambda)^\perp$ is isomorphic to a submodule of a homomorphic image of $M^{\nu^1} \oplus M^{\nu^2} \oplus \cdots \oplus M^{\nu^t}$, by Lemma 3.6. So $(S^\lambda)^\perp$ has a finite composition series of length $\leq n2^{[L(\lambda)-1][L(\lambda)+3]/2}$, with factors satisfying (a). Then $S^\lambda \cap (S^\lambda)^\perp$ has a finite composition series of length $\leq n\ 2^{[L(\lambda)-1][L(\lambda)+3]/2}$ by Lemma 2.1(i). Since $D^\lambda = S^\lambda/(S^\lambda \cap (S^\lambda)^\perp)$ is irreducible (or zero) by Theorem 3.4, it follows that S^λ has a composition series of length $m_3 \leq n2^{[L(\lambda)-1][L(\lambda)+3]/2} + 1$. So M^λ has a finite composition series of length $m_2 + m_3 \leq 2n2^{[L(\lambda)-1][L(\lambda)+3]/2} + 1$. Thus, (using Lemma 3.8) $m_2 + m_3 \leq 2L(\lambda)2^{[L(\lambda)-1][L(\lambda)+3]/2} + 1 \leq 2^{L(\lambda)[L(\lambda)+4]/2}$. Moreover all composition factors are of the form D^ν for various $\nu \unrhd \lambda$.□

Corollary 3.10 *Let $m, n \in \mathbb{N}$ with $m \geq n$ and $\lambda = (\infty - n, \lambda_2, \ldots, \lambda_r)$ be an infinite partition. Let $w = (w_1, \ldots, w_m) \in \Omega^{(m)}$, and G_w the stabiliser of w in $\mathrm{Sym}(\Omega)$. Then the composition length of $\mathbb{F}\Omega^\lambda$ as an $\mathbb{F}G_w$-module is at most $(mn)^n 2^{n^4}$.*

Proof: Let $\Omega' = \Omega \setminus \{w_1, \ldots, w_m\}$. The G_w-orbit containing a particular λ-tabloid is determined by the intersections of its rows with $\{w_1, \ldots, w_m\}$. So a crude upper bound on the number of G_w-orbits on Ω^λ is:

$$\left(\begin{array}{c} m \\ n \end{array} \right) r^n \leq m^n r^n \leq (mn)^n.$$

The action of G_w on such an orbit is equivalent to the action of $\mathrm{Sym}(\Omega')$ on $(\Omega')^\mu$ for some $\mu \unrhd \lambda$ and as an $\mathbb{F}G_w$-module, $\mathbb{F}\Omega^\lambda$ decomposes as a direct sum of such modules $\mathbb{F}\Omega^\mu$, with one summand for each orbit. The result follows, by Theorem 3.9. □

Theorem 3.11 *Each M^λ is a good $\mathbb{F}G$-module.*

Proof: By Theorem 3.9 it is enough to prove that any $\mathbb{F}G$-endomorphism of D^λ is given by multiplication by some scalar in $\mathbb{F}$. We do this using Lemma 2.6, so the proof divides into two steps.

(i) First we show that $D^\lambda_{\mathbb{F}}$ is absolutely irreducible. By James' Submodule Theorem, $D^\lambda_{\mathbb{E}}$ is irreducible (or zero) for any field $\mathbb{E}$. So it is enough to show that if $\mathbb{E}$ is an extension field of $\mathbb{F}$ then $D^\lambda_{\mathbb{F}} \otimes \mathbb{E}$ is isomorphic to $D^\lambda_{\mathbb{E}}$. Note that by exactness of $- \otimes \mathbb{E}$ we have:

Claim 1. For any $\mathbb{F}$-vector spaces $A \supseteq B$ with a group G acting on them, there is an $\mathbb{E}G-$isomorphism $(A/B) \otimes_{\mathbb{F}} \mathbb{E} \cong_{\mathbb{E}G} (A \otimes_{\mathbb{F}} \mathbb{E})/(B \otimes_{\mathbb{F}} \mathbb{E})$.

Identify $M^\lambda_{\mathbb{F}}$ with a subset of $M^\lambda_{\mathbb{E}}$ (via $x \mapsto x \otimes 1_{\mathbb{E}}$). We now show that that $\langle S^\lambda_{\mathbb{F}} \cap (S^\lambda_{\mathbb{F}})^{\perp \mathbb{F}} \rangle_{\mathbb{E}} = S^\lambda_{\mathbb{E}} \cap (S^\lambda_{\mathbb{E}})^\perp$. Note firstly that $\langle S^\lambda_{\mathbb{F}} \rangle_{\mathbb{E}} = S^\lambda_{\mathbb{E}}$ and $\langle S^\lambda_{\mathbb{F}} \cap (S^\lambda_{\mathbb{F}})^\perp \rangle_{\mathbb{E}} \subseteq S^\lambda_{\mathbb{E}} \cap (S^\lambda_{\mathbb{E}})^\perp$. For the converse, suppose that $V_{\mathbb{F}}$ is an arbitrary vector space over $\mathbb{F}$. Let $\{x_i : i \in I\}$ be an $\mathbb{F}-$basis of $V_{\mathbb{F}}$. Then $\{x_i \otimes 1_{\mathbb{E}} : i \in I\}$ is an $\mathbb{E}-$basis for $V_{\mathbb{E}} = V_{\mathbb{F}} \otimes \mathbb{E}$. If $V_{\mathbb{F}}$ has a bilinear form $(\ ,\ )_{\mathbb{F}}$, then $V_{\mathbb{E}}$ has a bilinear form $(\ ,\ )_{\mathbb{E}}$ inherited from that of $V_{\mathbb{F}}$, given by $(x_i \otimes 1_{\mathbb{E}}, x_j \otimes 1_{\mathbb{E}})_{\mathbb{E}} = (x_i, x_j)_{\mathbb{F}}$.

Claim 2. Let $U = \{u \in V_{\mathbb{F}} : (u, v)_{\mathbb{F}} = 0 \quad \forall v \in V_{\mathbb{F}}\}$. Then $(V_{\mathbb{E}})^{\perp \mathbb{E}} = \langle U \rangle_{\mathbb{E}}$.

Proof of claim 2: Without loss of generality let $\{x_i : i \in I_0 \subseteq I\}$ be an $\mathbb{F}-$basis for U, and assume that $w = \sum_{i \in I_0} \alpha_i (x_i \otimes 1_{\mathbb{E}}) + \sum_{j \in I_1} \beta_j (x_j \otimes 1_{\mathbb{E}}) \in (V_{\mathbb{E}})^{\perp \mathbb{E}}$, where $I_1 = I \setminus I_0$ and $\alpha_i, \beta_j \in \mathbb{E}$. To prove our claim we may assume each $\alpha_i = 0$ and show that $\beta_j = 0$ for all $j \in I_1$. Suppose not and $\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_r} \neq 0$. As $(w, x_j \otimes 1_{\mathbb{E}})_{\mathbb{E}} = 0$ for all $j \in I$, we have a collection of linear equations

$$\beta_{i_1}(x_{i_1}, x_j)_{\mathbb{F}} + \beta_{i_2}(x_{i_2}, x_j)_{\mathbb{F}} + \cdots + \beta_{i_r}(x_{i_r}, x_j)_{\mathbb{F}} = 0, \ \ j \in I$$

in $\beta_{i_1}, \cdots, \beta_{i_r}$ with coefficient in $\mathbb{F}$ which has a non-trivial solution in $\mathbb{E}^r$. So there is a non-zero solution $\beta^{'}_{i_1}, \cdots, \beta^{'}_{i_r}$ in $\mathbb{F}^r$. Thus $w = \sum_{j\in J} \beta^{'}_j (x_j \otimes 1_{\mathbb{E}}) \in (V_{\mathbb{E}})^{\perp}$, where $J = \{i_1, i_2, \cdots, i_r\}$ and so $\sum_{j\in J} \beta^{'}_j x_j \in (V_{\mathbb{F}})^{\perp}$ as $\beta^{'}_j \in \mathbb{F}$ which is a contradiction to the choice of the basis of $V_{\mathbb{F}}$. Hence $\beta_j = 0$ for all $j \in I_1$ and therefore $(V_{\mathbb{E}})^{\perp} = \langle U\rangle_{\mathbb{E}}$. This is the end the proof of claim 2.

From claim 2 by replacing $V_{\mathbb{F}}$ by $S^{\lambda}_{\mathbb{F}}$ and $U = S^{\lambda}_{\mathbb{F}} \cap (S^{\lambda}_{\mathbb{F}})^{\perp}$, we get $S^{\lambda}_{\mathbb{E}} \cap (S^{\lambda}_{\mathbb{E}})^{\perp} = \langle S^{\lambda}_{\mathbb{F}} \cap (S^{\lambda}_{\mathbb{F}})^{\perp}\rangle_{\mathbb{E}}$. Now by using claim 1, we get $D^{\lambda}_{\mathbb{F}} \otimes \mathbb{E} \cong_{\mathbb{E}G} S^{\lambda}_{\mathbb{E}} / \langle S^{\lambda}_{\mathbb{F}} \cap (S^{\lambda}_{\mathbb{F}})^{\perp}\rangle_{\mathbb{E}} = S^{\lambda}_{\mathbb{E}} / S^{\lambda}_{\mathbb{E}} \cap (S^{\lambda}_{\mathbb{E}})^{\perp} = D^{\lambda}_{\mathbb{E}}$, as required.

(ii) Next, suppose $D^{\lambda} \neq \{0\}$ and let $\phi \in \mathrm{End}_{\mathbb{F}G}(D^{\lambda}_{\mathbb{F}})$.

Claim 3. There is a $\phi-$invariant non-zero, finite dimensional subspace of D^{λ}.

Proof of claim 3: Let X be a finite subset of Ω. If this is large enough, there is some non-zero $x \in D^{\lambda}$ which is fixed by $G_{(X)}$, the pointwise stabiliser of X in G. By Corollary 3.10, the fixed-point set of $G_{(X)}$ on D^{λ} is finite dimensional. Moreover, it is clearly ϕ-invariant.

The result now follows by Lemma 2.6.□

Suppose now that $\mathbb{F}$ is a finite field, $n \in \mathbb{N}$ and $\lambda = (\infty - n, \lambda')$. As a corollary to Theorem 3.11, Theorem 3.9 and Lemma 2.5 we have the following:

Corollary 3.12 *(i) The number of submodules of M^{λ} is bounded above by $\mid \mathbb{F} \mid^{\alpha}$, where $\alpha = 2^{3n^4}$. (ii) Let $\lambda^1, \lambda^2, \cdots, \lambda^r$ be partitions of Ω with $\lambda^i = (\infty - n_i, \nu^i)$. Let $m = \sum_i n_i$. The number of submodules of $M^{\lambda^1} \oplus M^{\lambda^2} \oplus \cdots \oplus M^{\lambda^r}$ is at most $\mid \mathbb{F} \mid^{\beta}$, where $\beta = 2^{3m^4}$.* □

The final result we need from the representation theory of the symmetric group is due to J. Krajíček ([14]), following earlier work of M. Ajtai ([2]). To state it, we need some more terminology.

Definition 3.13 Suppose μ is a partition of the infinite set Ω. Let $x = \sum_{T\in\Omega^{\mu}} a_T T \in M^{\mu}$. We shall say that a tabloid T is in the *support* of x if $a_T \neq 0$. Suppose any element in the support of x has the elements of all its rows apart from the first contained in the finite set A. Abusing terminology, we say that the support of x is contained in A, and define the minimum size of such an A to be the *weight* of x.

The following is a part of ([14], Theorem 3.4) but expressed in our notation:

Theorem 3.14 *For any submodule U of M^{λ} where $\lambda = (\infty - n, 1^n)$, there is a generating set H of U such that any vector in H has weight at most*

$$n^{2^{n+1}}.$$

Proof: In the proof of ([14], Theorem 3.4), Krajíček gives the bound

$$2^{(n+1)^2} n^{2^n+2}$$

and for $n \geq 4$ this is dominated by the above expression (which we have chosen for its slightly simpler form). For $n < 4$ Krajíček's proof can be inspected to show that the bound we give is rather extravagant. □

Corollary 3.15 *Suppose $\lambda = (\infty - n, \lambda')$, where λ' is a partition of n, and $U \leq M^{\lambda}$. Then there exists a homomorphism $\phi : (\mathbb{F}_p \Omega^{(m)})^r \to M^{\lambda}$ with image U, where $m = n^{2^{n+1}}$ and $r = 2^{n^4}$.*

Proof: Let $\mu = (\infty - n, 1^n)$. There is a natural surjection $M^{\mu} \to M^{\lambda}$ (which does not increase the weight of a vector). Thus we may assume that $\lambda = \mu$. By Theorem 3.14 there are generators $x_1, \ldots, x_s$ of U each of weight at most m. By Theorem 3.9 we may take $s \leq r$ here. The result follows: ϕ takes some generator of the i-th summand whose support contains the support of x_i to x_i. □

3.2 Duality

We now return to the problem originally stated in the Introduction, where F is abelian. In this case, we are concerned with closed G-invariant subgroups of F^W, where $W = \Omega^{(n)} = \Omega^{(\infty - n, 1^n)}$. (The group action is given by ${}^g f(w) = f(g^{-1}w)$ for $f \in F^W$, $w \in W$ and $g \in G$.) As this is a compact (continuous) G-module, Pontriagin duality gives an inclusion-reversing correspondence between the closed G-submodules of F^W and the submodules of its dual $(F^W)^*$. Moreover the duality sends exact sequences of compact G-modules to exact sequences of (discrete) G-modules, so the topological G-composition factors of F^W are the duals of the G-composition factors of its dual.

Any finite abelian group is self-dual, so in fact the dual of F^W is the permutation module FW (with coefficients in F), which in our case is one of the tabloid modules. So if $F = \mathbb{F}_p$ we have dual versions of the various results for the M^λ: the same bounds on the topological G-composition length and number of closed submodules hold, and all topological composition factors are of the form $(D^\mu)^*$ for various $\mu \trianglerighteq \lambda$.

The reader is referred to ([6], Section 6.3) for more details. See also the remarks before Lemma 6.4.

4 Cohomology Groups

We refer the reader to ([6], Sections 6 and 7) for more background to the material in this section.

If Ω is any set then there is a natural topology on $\mathrm{Sym}(\Omega)$ which makes it into a topological group. The open sets are unions of cosets of pointwise stabilizers of finite subsets of Ω. We then make any permutation group G on Ω into a topological group by giving it the relative topology.

Definition 4.1 (cf. [6], Definition 6.4.1.) If G is a group and M is a G-module, then a *derivation* from G to M is a map $d : G \longrightarrow M$ which satisfies $d(gh) = d(g) + gd(h)$ for all $g, h \in G$. An *inner derivation* is a derivation of the form $d_a($ for $a \in M)$ where $d_a(g) = ga - a$ for all $g \in G$. The set of all such derivations forms an abelian group $\mathrm{Der}(G, M)$ (with the pointwise addition of functions), and the inner derivations form a subgroup $\mathrm{Inn}(G, M)$. The quotient group is denoted by $H^1(G, M)$, and is referred to as the first cohomology group of G on M. If G is a topological group and M a topological G-module (that is, M is a topological abelian group and the action $G \times M \to M$ is continuous) then the continuous derivations form a subgroup of the group of all derivations, and this clearly contains all the inner derivations. We denote the quotient group of continuous derivations modulo the inner derivations by $H^1_c(G, M)$.

Definition 4.2 If G is a group and M is a G-module then we define the *zero-th cohomology group* $H^0(G, M)$ to be the submodule of $G-$fixed elements of M. This is also denoted by M^G.

The long exact sequence holds for H^0 and H^1_c ([6], Lemma 7.1.2), as well as a version of Shapiro's Lemma ([6], Lemma 7.1.4). The following is also well-known for discrete cohomology groups. We omit the proof that it holds in our context.

Lemma 4.3 *If M is a $G-$module and H is a normal subgroup of G, then there is an action of G/H on $H^1(H, M)$ and an exact sequence*

$$0 \longrightarrow H^1(G/H, M^H) \longrightarrow H^1(G, M) \longrightarrow (H^1(H, M))^{G/H}.$$

If, moreover, G is a topological group and M is a topological G-module, then there is an exact sequence as above in which the H^1 terms are replaced by H^1_c. $\square$

The long exact sequence gives:

Lemma 4.4 *Let $\mathbb{F}$ be a field, G be a topological group and M a compact topological $\mathbb{F}G$-module.*
(i) For any closed submodule N of M

$$|\,(H^1_c(G,M))\,| \leq |\,(H^1_c(G,M/N))\,||\,(H^1_c(G,N))\,|\,.$$

(ii) Let N be a closed submodule of M and suppose that $H^1_c(G,M/N)$ and $H^1_c(G,N)$ are trivial. Then $H^1_c(G,M)$ is trivial.
(iii) Suppose M has a finite topological G-composition series with $\mathcal{S}$ the multiset of composition factors. Then

$$dim_{\mathbb{F}}(H^1_c(G,M)) \leq \sum_{S\in\mathcal{S}} dim_{\mathbb{F}}(H^1_c(G,S)).\square$$

Lemma 4.5 *Suppose S is a Hausdorff topological group and N a compact normal subgroup. Let $\mathcal{H} = \{H \leq S : H \text{ is closed }, H \cap N = 1, NH = S\}$, the set of closed complements to N in S.*
(i) If $H \in \mathcal{H}$, the map $\mu : N \times H \to S$ given by $\mu(n,h) = nh$ is a homeomorphism.
Suppose also that N is abelian.
(ii) If $H \in \mathcal{H}$ there is a bijection $\theta : \mathrm{Der}_c(H,N) \to \mathcal{H}$ such that any $\delta_1, \delta_2 \in \mathrm{Der}_c(H,N)$ are equal modulo an inner derivation if and only if $\theta(\delta_1)$ and $\theta(\delta_2)$ are conjugate in S.
(iii) If $H \in \mathcal{H}$ the number of S-conjugacy classes of elements of $\mathcal{H}$ is $|H^1_c(H,N)|$.

Proof: (i) Clearly μ is a continuous bijection. To show that it is a homeomorphism, it is enough to prove that it is a closed map. Suppose $A \subseteq N$ and $B \subseteq H$ are closed. Then A is a compact subset of S and B is a closed subset of S, so AB is a closed subset of S, by ([10], (4.4)). Thus $\mu(A \times B)$ is closed, and the result follows.
(ii) This is well-known, except that we require the correspondence between *continuous* derivations and *closed* complements to N. First, given $\delta \in \mathrm{Der}_c(H,N)$ let $\theta(\delta) = H_1 = \{\delta(h)h : h \in H\}$. This is a subgroup of S. As δ is continuous, its graph gives a closed subset of $N \times H$. Now H_1 is the image of this subset under μ, so is closed, by (i). Conversely, if $H_1 \in \mathcal{H}$, then by (i) we have a homeomorphism $\mu_1 : N \times H_1 \to S$. The corresponding derivation δ is given by $\delta(h) = -\pi_1(\mu_1^{-1}(h))$, where π_1 is projection to N. This is continuous.
(iii) This follows immediately from (ii). $\square$

We now compute some bounds associated with these cohomology groups for sections of the continous G-modules F^{Ω^λ}, when $F = \mathbb{F}_p$.

Theorem 4.6 *Let $\lambda = (\infty - n, \lambda_2, \ldots, \lambda_r)$ be a partition of an infinite set Ω and $W = \Omega^\lambda$ be the set of all λ-tabloids. Let $G = \mathrm{Sym}(\Omega)$. If K is a closed submodule of $K_0 = \mathbb{F}_p^W$, then the dimension of $H^1_c(G, K_0/K)$ over $\mathbb{F}$ is bounded above by 2^{18n^4}.*

Proof: The composition factors of K_0/K are of the form $(D^\mu)^*$ for various $\mu \trianglerighteq \lambda$, and we have a bound on their number. So (by Lemma 4.4) it is enough to obtain a bound on the $\mathbb{F}$-dimension of $H^1_c(G,(D^\mu)^*)$.

By the long exact sequence we have:

$$\dim(H^1_c(G,(D^\mu)^*)) \leq \dim(H^0(G,(S^\mu)^*/(D^\mu)^*)) + \dim(H^1_c(G,(S^\mu)^*)).$$

The first of these is bounded by the composition length of $\mathbb{F}_p^{\Omega^\mu}$, and this is at most 2^{n^4}. For the second, suppose $\mu = (\infty - m, \mu')$, and let $\nu = (\infty - 2m, 1^{2m})$. As the Specht module is generated by any polytabloid, there is a surjective homomorphism $\varphi : M^\nu \to S^\mu$. Dualizing, we have an embedding $\varphi^* : (S^\mu)^* \to (M^\nu)^*$. So by the long exact sequence again,

$$\dim(H^1_c(G,(S^\mu)^*) \leq \dim(H^0(G,(M^\nu)^*/im\varphi^*)) + \dim(H^1_c(G,(M^\nu)^*)).$$

By Shapiro's lemma

$$H^1_c(G,(M^\nu)^*) = H^1_c(G,\mathbb{F}^{\Omega^{(2m)}}) \cong H^1_c(G_w,\mathbb{F}_p) = \mathrm{Hom}(G_w,\mathbb{F}_p) = \{0\}$$

for any $w \in \Omega^{(2m)}$, as G_w has no proper subgroup of finite index. Also $\dim(H^0(G,(M^\nu)^*/im\varphi^*))$ is bounded by the composition length of M^ν, so by Theorem 3.9:

$$\dim(H^1_c(G,(S^\mu)^*)) \leq 2^{16m^4} \leq 2^{16n^4}.$$

Therefore

$$\dim(H^1_c(G,(D^\mu)^*)) \leq 2^{n^4} + 2^{16n^4}$$

and so

$$\dim(H^1_c(G,K_0/K)) \leq 2^{n^4}(2^{n^4} + 2^{16n^4}) \leq 2^{18n^4}. \square$$

5 The Main Result

We return to the problem stated in the Introduction. Thus, Ω is an infinite set, $n \in \mathbb{N}$, and $\lambda = (\infty - n, \lambda')$ an infinite partition, where λ' is a partition of n. We consider $G = \mathrm{Sym}(\Omega)$ acting on Ω^λ, and some group F of permutations on a finite set Δ. We form the unrestricted wreath product $\Gamma_0 = F \,\mathrm{Wr}_W\, G$ in its imprimitive action on $\Delta \times W$. The projection $\rho : \Gamma_0 \to G$ is continuous with kernel $K_0 = F^W$, and we consider $\mathcal{C} = \{\Gamma \leq \Gamma_0 : \Gamma \text{ closed }, \rho(\Gamma) = G\}$. Clearly Γ_0 acts by conjugation on $\mathcal{C}$. The main result is:

Theorem 5.1 *If F is nilpotent, then the number of Γ_0-conjugacy classes of subgroups in $\mathcal{C}$ is at most $|F|$ to the power $2^{k^4 b_n}$, where k is the composition length of F and $b_n = 6n^{2^{(n+2)^4}}$.*

Lemma 5.2 *(i) The topology on Γ_0 is independent of the permutation action of F.*
(ii) If E is a normal subgroup of F there is a natural, continuous, open epimorphism $\alpha : \Gamma_0 \to (F/E)\,\mathrm{Wr}_W\, G$ with kernel E^W. Thus Γ_0/E^W is isomorphic to $(F/E)\,\mathrm{Wr}_W\, G$, as a topological group.
(iii) If $\Gamma \in \mathcal{C}$ then $\rho|\Gamma : \Gamma \to G$ is continous, open and maps closed subgroups of Γ to closed subgroups of G.

Proof: (i), (ii) are easy. For (iii), see for example, ([6], Lemma 1.4.2). $\square$

Proof of Theorem: The proof is by induction on k. In the base case, $F = \mathbb{F}_p$ is cyclic of prime order p, and $K_0 = \mathbb{F}_p^W$ is a continuous G-module. Let $\Gamma \in \mathcal{C}$. Then $K_\Gamma = \Gamma \cap K_0$ is a closed G-submodule of K_0. So by the dual version of Corollary 3.12, there are at most $p^{2^{3n^4}}$ possibilities for K_Γ. Now consider $\mathcal{C}_\Gamma = \{\Gamma_1 \in \mathcal{C} : K_{\Gamma_1} = K_\Gamma\}$. There is a one-to-one correspondence between this and the set of closed complements to K_0/K_Γ in Γ_0/K_Γ given by $\Gamma_1 \mapsto \Gamma_1/K_\Gamma$, and this correspondence commutes with Γ_0-conjugation. Thus, by Lemma 4.5, the number of Γ_0-conjugacy classes in $\mathcal{C}_\Gamma$ is $|H^1_c(G, K_0/K_\Gamma)|$ (because $(\Gamma/K_\Gamma)/(K/K_\Gamma) \cong \Gamma/K \cong G$, as topological groups). By Theorem 4.6, this is at most $p^{2^{18n^4}}$. The base case then follows immediately.

For the inductive step, let $E \leq Z(F)$ be cyclic of prime order p, and let $Z_0 = E^W \leq \Gamma_0$. By Lemma 5.2, there is a continuous, open epimorphism $\Gamma_0 \to \overline{\Gamma}_0 = (F/E)\,\mathrm{Wr}_W\, G$ with kernel Z_0, and $\Gamma_0/Z_0 \cong \overline{\Gamma}_0$. So by inductive hypothesis the number of Γ_0-conjugacy classes of $\Sigma \in \mathcal{C}$ with $\Sigma \geq Z_0$ is at most $|(F/E)|^{2^{(k-1)^4 b_n}}$.

Note that as Z_0 is central in K_0, conjugation in Γ_0 makes Z_0 into a continuous G-module. Fix some $\Gamma_1 \in \mathcal{C}$ and let $K_1 = \Gamma_1 \cap Z_0$. Note that the latter is a normal subgroup of Γ_0 (as it is centralised by K_0 and normalised by Γ_1) and a closed G-submodule of Z_0. There are therefore at most $p^{2^{3n^4}}$ possibilities for K_1, by Corollary 3.12. Let $\Sigma = \Gamma_1 Z_0$. We have already bounded the possibilities for this. We next count Σ-conjugacy classes of $\Gamma_2 \leq \Sigma$, $\Gamma_2 \in \mathcal{C}$ with $\Gamma_2 \cap Z_0 = K_1$.

Let $\Pi = \Sigma/K_1$ and $\tilde{\ }: \Sigma \to \Pi$ the natural map (which is of course continuous and open). Then $\tilde{\Gamma}_1$ is a closed subgroup of Π, $\tilde{Z}_0$ is a compact normal subgroup of Π and $\tilde{\Gamma}_1 \cap \tilde{Z}_0 = 1$, $\tilde{\Gamma}_1 \tilde{Z}_0 = \Pi$. So by Lemma 4.5, $H^1_c(\tilde{\Gamma}_1, \tilde{Z}_0)$ counts the number of Π-classes of closed complements to $\tilde{Z}_0$ in Π, and this is the same as the number of Σ-classes of $\Gamma_2 \in \mathcal{C}$ with $\Gamma_2 \leq \Sigma$ and $\Gamma_2 \cap Z_0 = K_1$.

Let $T_1 = \Gamma_1 \cap K_0$. This is a compact normal subgroup of Γ_1. Let $T = \tilde{T}_1$. Then $\tilde{\Gamma}_1/T \cong \Gamma_1/T_1 \cong G$. By Lemma 4.3 we have an exact sequence

$$H^1_c(\tilde{\Gamma}_1/T, \tilde{Z}_0^T) \to H^1_c(\tilde{\Gamma}_1, \tilde{Z}_0) \to H^1_c(T, \tilde{Z}_0)^{\tilde{\Gamma}_1/T}.$$

Now, as Z_0 is central in K_0, T acts trivially on $\tilde{Z}_0$, so the last term here is the set of continuous group homomorphisms

$$\gamma : T \to \tilde{Z}_0$$

such that $\gamma({}^g t) = g\gamma(t)$ for all $t \in T$ and $g \in \tilde{\Gamma}_1$. We denote this by $\mathrm{Hom}^c_G(T, \tilde{Z}_0)$. Thus

$$|H^1_c(\tilde{\Gamma}_1, \tilde{Z}_0)| \leq |H^1_c(G, Z_0/K_1)||\,\mathrm{Hom}^c_G(T, \tilde{Z}_0)|.$$

As $Z_0 \cong \mathbb{F}_p^W$, the first factor is bounded above by $p^{2^{18n^4}}$, by Theorem 4.6. The following bounds the remaining factor.

Claim A: $|\,\mathrm{Hom}^c_G(T, \tilde{Z}_0)|$ is at most p to the power $s2^{3((k(mn)^n 2^{n^4}+1)^4}$, where $m = n^{2^{n+1}}$ and $s = 2^{n^4}$.

First, we embed (as continuous G-modules) $\tilde{Z}_0$ into $(\mathbb{F}_p^{\Omega^{(m)}})^s$ for some $m, s \in \mathbb{N}$. Thus we require an exact sequence

$$0 \to K_1 \to Z_0 \xrightarrow{\varphi^*} (\mathbb{F}_p^{\Omega^{(m)}})^s.$$

Dually, we require

$$(\mathbb{F}_p\Omega^{(m)})^s \xrightarrow{\varphi} Z_0^* \to K_1^* \to 0.$$

Now, $Z_0^* = \mathbb{F}_p\Omega^\lambda$ and m, s have to be chosen so that φ has image a prescribed submodule of this. By Corollary 3.15 we may take $m = n^{2^{n+1}}$ and $s = 2^{n^4}$.

Note that

$$\mathrm{Hom}^c_G(T, \tilde{Z}_0) \subseteq \mathrm{Hom}^c_G(T, (\mathbb{F}_p^{\Omega^{(m)}})^s) \cong (\mathrm{Hom}^c_G(T, \mathbb{F}_p^{\Omega^{(m)}}))^s$$

so our next step is to bound $\mathrm{Hom}^c_G(T, \mathbb{F}_p^{\Omega^{(m)}})$.

Take a fixed $w \in \Omega^{(m)}$ and let $U_w = \{f \in \mathbb{F}_p^{\Omega^{(m)}} : f(w) = 0\}$. For $\phi \in \mathrm{Hom}^c_G(T, \mathbb{F}_p^{\Omega^{(m)}})$ put $T_{w,\phi} = \phi^{-1}(U_w)$ and $T_{1w} = \bigcap_\phi T_{w,\phi}$. Then $T/T_{w,\phi}$ is an abelian group of order at most p since $T/T_{w,\phi} \hookrightarrow \mathbb{F}_p^{\Omega^{(m)}}/U_w \cong \mathbb{F}_p$. Also T/T_{1w} is an elementary abelian $p-$group since T/T_{1w} embeds into $\prod_\phi T/T_{w,\phi}$. We seek to bound the size of this.

Let $\Gamma_i^w = \{g \in \Gamma_i : \rho(g)w = w\}$ (for $i = 0, 1$). So $T_1 \leq \Gamma_1^w \leq \Gamma_0^w$. Consider Γ_0^w as a topological group with Γ_1^w acting as a group of operators (by conjugation).

Claim B: Γ_0^w has a topological Γ_1^w-composition series of length at most $c_1 + 1$ where $c_1 = k(mn)^n 2^{n^4}$.

Claim C: $|T/T_{1w}| \leq p^{c_1}$.

Proof of Claim B: Take a chief series $1 = F_0 < F_1 < \cdots < F_k = F$ for F such that $F_i/F_{i-1} \leq Z(F/F_{i-1})$ (and $p_i = |F_i/F_{i-1}|$ is prime). Let $Z^i = F_i^{\Omega^\lambda} \leq K_0$. Then $1 = Z_0 < Z_1 < \cdots < Z^k = K_0 < \Gamma_0^w$ is a Γ_1^w-invariant series for Γ_0^w. Moreover, $\Gamma_0^w/K_0 \cong G_w$ and Γ_1^w acts on Z^i/Z^{i-1} in the same way as G_w acts on $\mathbb{F}_{p_i}^{\Omega^\lambda}$. The claim follows by the dual version of Corollary 3.10.

Proof of Claim C: By Claim B, T_1 has a topological Γ_1^w-composition series of length at most c_1, so the same is true of T. Each $T_{w,\phi}$ is a closed, Γ_1^w-invariant subgroup of index at most p in T. The claim therefore follows from Claim B by a Jordan-Hölder Theorem.

We now return to the proof of Claim A. Let $T^1 = \bigcap_w T_{1w}$. Then $T^1 \trianglelefteq \tilde{\Gamma}_1$ and $\hat{T} = T/T^1$ is abelian. Thus conjugation in $\tilde{\Gamma}_1$ makes $\hat{T}$ into a continuous G-module. Each $\hat{T}_w = T_{1w}/T^1$ is an open G_w-invariant subgroup of $\hat{T}$ and G permutes these transitively. Moreover (as G_w has no proper subgroup of finite index) G_w acts trivially on the finite quotient $\hat{T}/\hat{T}_w \cong T/T_{1w}$, and all of these groups are isomorphic to some finite abelian group A. It follows that (as a continuous G-module) $\hat{T}$ is isomorphic to a closed submodule of $A^{\Omega^{(m)}}$ and so, by Claim C, to a closed submodule of $(\mathbb{F}_p^{\Omega^{(m)}})^{c_2}$ for some $c_2 \leq c_1$.

By construction, T^1 is in the kernel of all $\phi \in \mathrm{Hom}^c_G(T, \mathbb{F}_p^{\Omega^{(m)}})$, so $\mathrm{Hom}^c_G(T, \mathbb{F}_p^{\Omega^{(m)}})$ can be considered as $\mathrm{Hom}^c_G(\hat{T}, \mathbb{F}_p^{\Omega^{(m)}})$. By the previous paragraph, the graph of any continuous $G-$homomorphism from T/T_1 to $\mathbb{F}_p^{\Omega^{(m)}}$ corresponds to a closed submodule of $(\mathbb{F}_p^{\Omega^{(m)}})^{c_2} \times \mathbb{F}_p^{\Omega^{(m)}} = (\mathbb{F}_p^{\Omega^{(m)}})^{c_2+1}$. By the dual version of Corollary 3.12 the number of these is bounded by p to the power $2^{3((c_2+1)m)^4}$. Thus $|\,\mathrm{Hom}^c_G(T, \tilde{Z}_0)|$ is bounded above by p to the power

$$s2^{3((k(mn)^n 2^{n^4}+1)^4},$$

and this gives Claim A.

We now put all of this together. By Claim A and what immediately preceded it, the number of Σ-classes of $\Gamma_2 \in \mathcal{C}$ with $\Gamma_2 \leq \Sigma$ and $\Gamma_2 \cap Z_0 = K_1$ is bounded by p to the power

$$2^{18n^4} + s2^{3((k(mn)^n 2^{n^4}+1)^4}.$$

Multiplying this by the number of possibilities for K_1 and the number of possibilities for (Γ_0-classes of) Σ we obtain the bound:

$$|(F/E)|^{2^{(k-1)^4 b_n}} p^A$$

where

$$A = 2^{3n^4} + 2^{18n^4} + s2^{3((k(mn)^n 2^{n^4}+1)^4)}.$$

Substituting in for m and s we obtain

$$2^{3n^4} + 2^{18n^4} + 2^{n^4} 2^{3((kn^{n(2^{n+1}+1)} 2^{n^4}+1)^4)},$$

and, very crudely, we can bound this by 2 to the power

$$6k^4 n^{2^{(n+2)^4}} = k^4 b_n.$$

The inductive step follows. □

Remarks 5.3 **1.** All of the machinery we have developed to handle the case $F = \mathbb{F}_p$ will work for arbitrary finite abelian F. Thus the argument in the base step of the above proof can be used to show that if F is abelian, then we have the much better bound of

$$|F|^{2^{18n^4+1}}.$$

The difference arises from the fact that in this case we do not need to use Krajíček's result.
2. By similar reasoning, the bound in Theorem 5.1 ought to hold with k being the nilpotency class of F, rather than the composition length. We will not pursue this as it does not significantly affect the overall form of the bound.

6 Commentary

6.1 Finite covers

As is well-known, a subgroup of $\mathrm{Sym}(W)$ is closed if and only if it is the group of automorphisms of some first-order structure with domain W. Thus we make the following definition (cf. [6]).

A *permutation structure* is a pair $\langle W; G\rangle$ where G is a closed subgroup of $\mathrm{Sym}(W)$. We refer to G as the automorphism group and write $G = \mathrm{Aut}(W)$. When we do this we refer to 'the permutation structure W,' the appropriate automorphism group being understood. If A, B are subsets of W (or of some set on which $\mathrm{Aut}(W)$ is acting naturally) then $\mathrm{Aut}(B/A)$ denotes the group of permutations of B which extend to automorphisms of W fixing every element of A.

Suppose C, W are permutation structures. A finite-to-one surjection $\pi : C \to W$ is a *finite cover* of W if the set of fibres of π is preserved by $\mathrm{Aut}(C)$ and the induced map $\rho : \mathrm{Aut}(C) \to \mathrm{Sym}(W)$ given by $\rho(g)(w) = \pi(g(\pi^{-1}(w)))$ has image $\mathrm{Aut}(W)$. Fix such a finite cover. Its *kernel* is the kernel of ρ, namely $\mathrm{Aut}(C/W)$. If $w \in W$ let $C(w) = \pi^{-1}(w)$. The *binding group* at w is $B(w) = \mathrm{Aut}(C(w)/W)$ and this is a normal subgroup of the *fibre group* $F(w) = \mathrm{Aut}(C(w)/w)$. If $\mathrm{Aut}(W)$ is transitive on W, then these are isomorphic as w varies. There is a continuous epimorphism $\chi_w : \mathrm{Aut}(W/w) \to F(w)/B(w)$ ([6], Lemma 2.1.1). Thus if $\mathrm{Aut}(W/w)$ has no proper open subgroup of finite index, then $F(w) = B(w)$.

Suppose now that $\mathrm{Aut}(W)$ is transitive. Let $\Gamma = \mathrm{Aut}(C)$ and $K_0 = \prod_{w\in W} F(w) \leq \mathrm{Sym}(C)$. Then Γ normalises K_0. Let $\Gamma_0 = K_0\Gamma$, pick $w_0 \in W$ and let $X = C(w_0)$, $F = F(w_0)$. We can identify C with $X \times W$ so that Γ_0 is the wreath product $F \,\mathrm{Wr}_W\, \mathrm{Aut}(W)$ (this is a particular case of the theorem of Krasner and Kaloujnine). Thus we may regard $\mathrm{Aut}(C)$ as a closed subgroup of this wreath product.

The *cover problem* is, given W and data $(F(w), B(w), \chi_w)$, to determine the possible finite covers with this data. Here 'determine' means up to isomorphism of permutation structures (see [6]). By the above, it follows that if we can determine (up to conjugacy) the closed subgroups of the wreath product which project onto $\mathrm{Aut}(W)$, then we have a solution to this problem. So in the case where W is $\langle\Omega^\lambda; \mathrm{Sym}(\Omega)\rangle$ and the fibre groups are nilpotent, our main theorem gives an explicit finite bound on the number of isomorphism types of finite covers of W.

It would be more satisfactory to have such a result expressed in terms of the binding group. Note that we already have this if $\lambda = (\infty - n, 1^n)$ as the stabiliser of an n-tuple of elements from Ω is isomorphic to the symmetric group, and this has no proper subgroup of finite index. Thus in this case, the fibre group equals the binding group.

Theorem 6.1 *Let λ' be a partition of n, let $\lambda = (\infty - n, \lambda')$, and $W = \langle\Omega^\lambda; \mathrm{Sym}(\Omega)\rangle$. Suppose B is a nilpotent finite group of composition length k. Then the number of isomorphism types of finite cover of W with binding group B is at most $|B|$ to the power $2^{k^4 b_n}$, where $b_n = 6n^{2^{(n+2)^4}}$.*

Proof: If $\lambda' = 1^n$, then this follows from what we have already. If not, we convert finite covers of W into finite covers of $W_1 = \Omega^{(n)}$ without changing the binding group.

Suppose $\pi : C \to W$ is a finite cover. There is a trivial finite cover $\sigma_1 : W_1 \to W$ (σ_1 just puts the elements of a tuple $w \in \Omega^{(n)}$ into the finite rows of a λ-tabloid in a systematic way). We form the fibred product of these two finite covers of W. Specifically, let $C_1 = \{(c, w_1) \in C \times W_1 : \pi(c) = \sigma_1(w_1)\}$; let π_1 be projection of this to the second coordinate; let $\mathrm{Aut}(C_1) = \{(g, h) \in \mathrm{Aut}(C) \times \mathrm{Aut}(W_1) : \rho(g) = h\}$ (we are using $\mathrm{Aut}(W) = \mathrm{Sym}(\Omega) = \mathrm{Aut}(W_1)$). Then $\pi_1 : C_1 \to W_1$ is a finite cover and the projection $C_1 \to C$ is a trivial finite cover. The binding group in π_1 is B, which is necessarily the same as the fibre group. Moreover, isomorphic finite covers of W_1 give rise (via σ_1) to isomorphic finite covers of W. The result then follows from the first case. □

6.2 The model-theoretic context

Our original reason for studying the problem was to answer a question which arose in model theory. By [3] any disintegrated totally categorical structure is biinterpretable with a finite cover of a Grassmannian of a pure set (i.e. a permutation structure $W = \langle\Omega^{(\infty-n,n)}; \mathrm{Sym}(\Omega)\rangle$). It was known that there are only finitely many isomorphism types of such finite covers once the kernel is specified. Indeed Cherlin and Hrushovski ([4], but see also [1]) prove in a wider context that if one specifies an *arity* (defined in the next section) and a fibre group there are only finitely many finite covers of W. Their arguments (and the arguments in [7]) do not, however, appear to give bounds: this is what we provide here.

As well as the bounds, our result gives information which does not seem to follow from the model-theoretic analysis: the arity is not required in this case, as the number of isomorphism types of cover is finite once the binding group is specified. So in fact, there must be a bound on the arity of these finite covers. It would be very interesting to have such a bound, but all we are able to provide is a bound when the kernel (or binding group) is abelian.

6.3 Arities

Definition 6.2 For $\Gamma \leq \mathrm{Sym}(C)$ and a natural number n, let

$$\mathrm{cl}_n(\Gamma) = \{g \in \mathrm{Sym}(C) : (\forall x \in C^n)(\exists h \in \Gamma)(gx = hx)\}.$$

This is the *n-closure* of Γ. Note that $\mathrm{cl}_{n+1}(\Gamma) \subseteq \mathrm{cl}_n(\Gamma)$ and the closure of Γ in $\mathrm{Sym}(C)$ is $\bigcap_n \mathrm{cl}_n(\Gamma)$.

Suppose Γ is closed. If $\mathrm{cl}_n(\Gamma) = \Gamma$ we say that Γ is *n-ary* and call the least such n the *arity* of Γ (if there is no such n, the arity is ∞). Denote this by $\mathrm{arity}(\Gamma)$.

Lemma 6.3 *Suppose W is such that $\mathrm{Aut}(W/X)$ has no proper open subgroup of finite index for all finite $X \subseteq W$. Let $\pi : C \to W$ be a finite cover with kernel K and write $\Gamma = \mathrm{Aut}(C)$, $\Sigma = \mathrm{Aut}(W)$. Then*

$$\mathrm{arity}(\Gamma) \leq n = max(2, \mathrm{arity}(\Sigma), \mathrm{arity}(K)).$$

Proof: Of course, we may assume $n < \infty$. We show that Γ is n-closed. Let $\Gamma^{(n)} = \mathrm{cl}_n(\Gamma)$. As $\Gamma^{(n)}$ has the same orbits on pairs as Γ, it preserves the set of fibres of π, so there is an induced map $\tau : \Gamma^{(n)} \to \mathrm{Sym}(W)$. It is easy to show (using the definition of n) that the image of τ is Σ. Thus it remains to prove that the kernel L of τ is equal to K.

For finite $X \subseteq W$ let $\Gamma(X) = \mathrm{Aut}(\bigcup_{x\in X} \pi^{-1}(x)/X)$ and $K(X) = \mathrm{Aut}(\bigcup_{x\in X} \pi^{-1}(x)/W)$. Then $K(X)$ is a closed normal subgroup of finite index in $\Gamma(X)$ and exactly as in the proof of ([6], Lemma 2.1.1), there is a continuous epimorphism from $\mathrm{Aut}(W/X)$ onto $\Gamma(X)/K(X)$. So by hypothesis, $\Gamma(X) = K(X)$.

Let $k \in L$. Then for all $c_1, \ldots, c_n \in C$ there is $h \in \Gamma$ with $hc_i = kc_i$ for $i = 1, \ldots, n$. As $\pi(kc_i) = \pi(c_i)$, it follows from the above that we may take $h \in K$. So $k \in \mathrm{cl}_n(K) = K$, as required. □

Henceforth, let $W = \langle \Omega^{(n)}; \mathrm{Sym}(\Omega)\rangle$. We are interested in bounding the arity of of a finite cover $\pi : C \to W$. For simplicity, we assume this is a regular finite cover, i.e. the fibre groups acts regularly on the fibres. The structure W is 2-ary: consider for $i, j = 1, \ldots, n$ the relations $R_{ij}(x, y) \Leftrightarrow x_i = y_j$ where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in W$. So it remains to bound the arities of the possible kernels. We are able to do this in the case where the fibre group is the cyclic of prime order, using Krajíček's result.

Let F be a finite abelian group and consider the two $F\,\mathrm{Sym}(\Omega)$-modules FW and F^W. Recall that these are dual and their submodules related by Pontriagin duality. If K is a closed submodule of F^W let $\mathrm{Ann}(K) = \{\sum a_w w \in FW : \sum a_w f(w) = 0 \text{ for all } f \in K\}$. This is a submodule of FW. Conversely, if L is a submodule of FW define L° to be $\{f \in F^W : f(x) = 0 \text{ for all } x \in L\}$ (where $f(x)$ is evaluated linearly, and we are identifying F and its dual). If $x = \sum a_w w \in FW$ define $\theta(x) = |\{w : a_w \neq 0\}|$.

Lemma 6.4 *Suppose K is a closed submodule of F^W and $x_1, \ldots, x_r$ a generating set for $\mathrm{Ann}(K)$. Then*

$$\mathrm{arity}(K) \leq m = max(\theta(x_i) : i = 1, \ldots, r).$$

Proof: Let $f \in \mathrm{cl}_m(K)$ and write $x_i = \sum a_w w$. There exists $g \in K$ such that whenever $a_w \neq 0$ we have $f(w) = g(w)$. So $f(x_i) = g(x_i) = 0$. Thus $f \in (\mathrm{Ann}(K))^\circ$. But this is equal to K, by duality. Thus $\mathrm{cl}_m(K) = K$, as required. □

It is clear that Krajíček's result gives a bound on the θ for generating sets of submodules of F_pW, and so the above gives a bound on the arities of finite covers of W with fibre groups which are cyclic of prime order. Presumably this can easily be extended to the full abelian case. It would be very interesting to pass to the non-abelian case.

6.4 Problems

Most of the following have already been mentioned in the paper.

1. What happens if the binding group is not nilpotent? Are there still only finitely many possibilities for the kernels, and if so, are there only finitely many isomorphism types of covers?

2. The bounds we give seem rather large: can they be significantly reduced? Is there an approach which does not require Krajíček's result?

3. Obtain bounds on the arity of a finite cover of $\langle \Omega^{(n)}; \mathrm{Sym}(\Omega)\rangle$ (with nilpotent binding group).

4. What happens for finite covers where the base is a Grassmannian of a vector space over a finite field? Here one would need to work with covers of bounded arity.

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