SOME REMARKS ON GENERIC STRUCTURES

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ABSTRACT. We show that the $\aleph_0$-categorical structures produced by Hrushovski's predimension construction with a control function fit neatly into Shelah's SOP_n hierarchy: if they are not simple, then they have SOP_3 and $NSOP_4$. We also show that structures produced without using a control function can be undecidable and have SOP.
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INTRODUCTION

In this note we collect together some observations about generic structures constructed using two variations on Hrushovski's method of predimensions.

Before describing the results, we recall briefly some details of the construction method. The original version of this is in [5], where it is used to provide a counterexample to Lachlan's conjecture, and [7], where it is used to construct a non-modular, supersimple $\aleph_0$-categorical structure. The book [13] is a very convenient reference for this (see Section 6.2.1). Generalisations and reworkings of the method (particularly relating to simple theories) are also to be found in [2], [9], [10].

We work with a relational language $L = \{R_i : i \in I\}$ with finitely many relations of each arity. Recall that if B, C are L-structures with a common substructure A then the *free amalgam* $B \coprod_A C$ of B and C over A is the L-structure whose domain is the disjoint union of B and C over A and whose atomic relations are precisely those of B together with those of C. We suppose that $\overline{\mathcal{K}}$ is a univeral class of L-structures which is closed under free amalgamation, that is, if $B, C \in \overline{\mathcal{K}}$ have a common substructure A, then $B \coprod_A C \in \overline{\mathcal{K}}$. Suppose further that the R_i are realised by tuples of distinct elements in structures in $\overline{\mathcal{K}}$ (this is not essential if the language is finite). Denote by $\mathcal{K}$ the finite

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structures in $\overline{\mathcal{K}}$. Note that our assumptions imply that there are only finitely many isomorphism types of structures in $\mathcal{K}$ of any given size.

Now let $(\alpha_i : i \in I)$ be a sequence of non-negative real numbers. Define the *predmension* $d_0(A) = |A| - \sum_i \alpha_i |R_i[A]|$, for $A \in \mathcal{K}$. If $A \subseteq B \in \mathcal{K}$ write $A \leq B$ to mean $d_0(A) < d_0(B')$ for all $A \subset B' \subseteq B$ and say that A is *self-sufficient* in B. It should be emphasised that this notion is *fundamentally different* from the one used in Hrushovski's construction of new strongly minimal sets [6]: we shall say more about this in Section 2.

For structures in $\mathcal{K}$, one has:
- If $X \subseteq B$ and $A \leq B$, then $X \cap A \leq X$;
- If $A \leq B \leq C$, then $A \leq C$.

Consequently, for each $B \in \mathcal{K}$ there is a closure operation given by $\mathrm{cl}_B(X) = \bigcap\{A : A \leq B, X \subseteq A\}$ for $X \subseteq B$.

The relation $\leq$ can be extended to infinite structures so that the above properties still hold: if $M \in \overline{\mathcal{K}}$ and $A \subseteq M$, write $A \leq M$ to mean that $A \cap X \leq X$ for all finite $X \subseteq M$.

In Section 1 we look at the following variation of the construction (from [5] and [7]). We take a *control function*: a continuous, strictly increasing function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with $f(x) \to \infty$ as $x \to \infty$. Consider the class of L-structures $\mathcal{K}_f = \{A \in \mathcal{K} : d_0(X) \geq f(|X|)\ \forall X \subseteq A\}$. For suitable choice of f (call these *good* f), $(\mathcal{K}_f, \leq)$ has the free $\leq$-amalgamation property: if $A_0 \leq A_1, A_2 \in \mathcal{K}_f$ then $A_i \leq A_1 \coprod_{A_0} A_2 \in \mathcal{K}_f$. Thus $(\mathcal{K}_f, \leq)$ is an *amalgamation class*. It follows that there is a countable structure $M_f \in \overline{\mathcal{K}}_f$ which is the union of a chain of finite self-sufficient substructures and satisfies:

($\leq$-*Extension Property*) If $A \leq M_f$ is finite and $A \leq B \in \mathcal{K}_f$, there is an embedding of B over A into M_f whose image is self-sufficient in M_f.

Equivalently, any $B \in \mathcal{K}_f$ is isomorphic to a self-sufficient substructure of M_f, and isomorphisms between finite self-sufficient substructures of M_f extend to automorphisms of M_f: thus the type of a tuple in M_f is determined by the isomorphism type of its closure.

The structure M_f is unique up to isomorphism and is called the *generic structure* associated to the amalgamation class $(\mathcal{K}_f, \leq)$ (see [8]). Note that in M_f, the closure of a finite set A has size bounded above by $f^{-1}(|A|)$, so the closure is uniformly locally finite and it follows by the Ryll-Nardzewski Theorem and the above remark on types that M_f is $\aleph_0$-categorical.

We shall be particularly concerned with where the theories $Th(M_f)$ can fit in the hierarchy:

$$\text{simple} \Rightarrow NSOP_3 \Rightarrow NSOP_4 \ldots \Rightarrow NSOP.$$

Here $NSOP$ is the negation of the strict order property and $NSOP_n$ is Shelah's strengthening of it from [11] (we repeat the definition in Section 1).

In ([7], Section 4.3), Hrushovski gives an example where M_f is supersimple of SU-rank 1. The point is that by choosing f carefully, the class $\mathcal{K}_f$ is *closed under ITD's* (see Definition 1.4) so one has the independence theorem holding over closed sets (the argument is also given in ([13], 6.2.27) and in more generality in ([2], Theorem 3.6)). In an earlier version of this paper by the first author, it was conjectured that with a suitable choice of good f, one could arrange that M_f would be not simple, but have Shelah's property $NSOP_3$. In fact, we now show that this is not the case (Theorem 1.9) and either M_f is simple, or it has SOP_3. We prove (– see the end of Section 1.3):

Theorem 0.1. *Suppose f is good. Then the following are equivalent:*

(1) *M_f is simple;*
(2) *$\mathcal{K}_f$ is closed under ITD's;*
(3) *M_f has property $NSOP_3$.*

We also prove that if f is good, then M_f has the property $NSOP_4$ (Theorem 1.8).

In Section 2 we consider what happens when the function f is the zero function (so of course does not satisfy $f(x) \to \infty$ as $x \to \infty$). More precisely, define $\mathcal{K}_0$ to be $\{A \in \mathcal{K} : \emptyset \leq A\}$, and similarly $\overline{\mathcal{K}}_0$. Then $(\mathcal{K}_0, \leq)$ and $(\overline{\mathcal{K}}_0, \leq)$ satisfy a strong form of the amalgamation property over $\leq$-substructures (see 6.2.9 of [13], for example):

(*Full $\leq$-Amalgamation Property*) If $A_1, A_2 \in \overline{\mathcal{K}}_0$ have a common substructure A_0 and $A_0 \leq A_1$, then $A_2 \leq A_1 \coprod_{A_0} A_2 \in \overline{\mathcal{K}}_0$.

As before, there is a unique countable structure $M_0 \in \overline{\mathcal{K}}_0$ which is the union of a chain of finite self-sufficient substructures and has the property that if $A \leq M_0$ is finite and $A \leq B \in \mathcal{K}_0$, then there is an embedding of B over A into M_0 whose image is self-sufficient in

M_0. Again we refer to M_0 as the generic structure associated to the amalgamation class $(\mathcal{K}_0, \leq)$. It is not too hard to show that $\mathcal{K}_0$ is closed under ITD's, however this is not enough to guarantee a good model theory for M_0. We look at a particular example where this is the case and show that $Th(M_0)$ is undecidable and has the strict order property (Corollaries 2.3 and 2.5). This answers Question 4.10 in [10] and contradicts claims in Section 4.2 of [7].

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1. Strong order properties and the structures M_f

1.1. **Dimension and the Independence Theorem Diagram.**

Using the notation of the Introduction, we make the following assumption throughout this Section.

Assumption 1.1. Suppose $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is a continuous, increasing function with $f(x) \to \infty$ as $x \to \infty$. Let $\mathcal{K}_f = \{A \in \mathcal{K}_0 : d_0(X) \geq f(|X|)\ \forall X \subseteq A\}$. We *assume* that $(\mathcal{K}_f, \leq)$ is closed under free amalgamation: if $A \leq B_1, B_2 \in \mathcal{K}_f$ then $B_1 \coprod_A B_2 \in \mathcal{K}_f$. Let M_f denote the generic structure for the amalgamation class $(\mathcal{K}_f, \leq)$.

Thus $M_f \in \overline{\mathcal{K}}_f$ is a countably infinite structure with the $\leq$-extension property (for $(\mathcal{K}_f, \leq)$). It is $\aleph_0$-categorical; self-sufficient closure in M_f is equal to algebraic closure and the type of a tuple in M_f is determined by the quantifier-free type of its closure. The same is true of any structure elementarily equivalent to M_f, and we occasionally make use of a highly saturated and strongly homogeneous elementary extension N_f of M_f.

Remarks 1.2. Sufficient conditions on f under which Assumption 1.1 holds are well-known. For example if all the α_i are integers and we assume f is piecewise linear with right-derivative f' which is a decreasing function, then the condition $0 < f'(x) \leq 1/x$ will guarantee that $\mathcal{K}_f$ is an amalgamation class, as in Hrushovski's original paper [5]. A more general, but similar, sufficient condition can be found in

([2]; 3.2). For a given f, $n \in \mathbb{N}$ and $e \in \mathbb{R}^{\geq 0}$ with $e \geq f(n)$, the question of whether there is a structure $A \in \mathcal{K}_f$ with $|A| = n$ and $d_0(A) = e$ is quite delicate (cf. [3], for example), so a straightforward necessary and sufficient condition on f for $\mathcal{K}_f$ to be closed under free amalgamation should probably not be expected. It should also be noted that only the values taken by f at the natural numbers are relevant to the construction of M_f and the above condition on the derivative of f could equally well be expressed as saying that $0 < f(n+1) - f(n) \leq \frac{1}{n}$ and $f(n+1) - f(n)$ is decreasing for all natural numbers n. It is however convenient to regard f as a continuous function defined on all of $\mathbb{R}^{\geq 0}$ in order to use notation such as $f^{-1}(m)$. We can of course do this by taking the piecewise linear function which extends an f defined at the natural numbers.

Any structure B in $\overline{\mathcal{K}}_f$ carries a notion of *dimension* d^B associated to the predimension d_0 and a notion of d^B*-independence*. If $X, Y \subseteq B$ are finite, write $d^B(X) = d_0(\mathrm{cl}_B(X))$ and $d^B(X/Y) = d^B(X \cup Y) - d^B(Y)$. For general Y let $d^B(X/Y) = \inf\{d^B(X/Y_0) : Y_0 \subseteq Y \text{ finite}\}$. Say that X, Z are *d-independent* over Y (in B) if $d^B(X/YZ) = d^B(X/Y)$ and $\mathrm{cl}_B(XY) \cap \mathrm{cl}_B(YZ) = \mathrm{cl}_B(Y)$. If the ambient structure B is clear from the context (for example if we are working in M_f or N_f) then we omit it from the notation. More details of properties of these notions can be found in the references given in the Introduction. In particular we note the following (from [2], Lemma 2.3):

Lemma 1.3. *Suppose $X \leq Y \leq B$ are finite sets and c is a tuple of elements in B. Then $d^B(c/Y) = d^B(c/X)$ if and only if:*

(i) $\mathrm{cl}_B(cX) \cap Y = X$;
(ii) $\mathrm{cl}_B(cX)$ *and* Y *are freely amalgamated over* X;
(iii) $d_B(cY) = d_0(\mathrm{cl}_B(cX) \cup Y)$. □

As already remarked in 1.2, the condition that $\mathcal{K}_f$ be an amalgamation class can be enforced by an assumption about the growth rate of f. A stronger assumption on the growth rate (see Theorem 3.6(ii) in [2], for example) also implies that M_f is simple: if the growth rate of f is sufficiently slow the independence theorem holds over finite closed sets in M_f. As in [9] we phrase the latter as a condition on $\mathcal{K}_f$ in the following way.

Definition 1.4. Let D be an L-structure with substructures D_i, D_{ij} for $0 \leq i < j \leq 3$ (we allow transposition of the indices in D_{ij}). We say that $(D; D_i, D_{ij})$ is an *independence theorem diagram* (ITD) in $\mathcal{K}_f$ if the following hold:

- $D_{ij} \in \mathcal{K}_f$;
- $D_{0j} = D_j$;
- $D_i, D_j \leq D_{ij}$;
- $D_i \cap D_j = D_0$;
- $D_{ij} \cap D_{jk} = D_j$;
- D_i and D_j are d-independent over D_0 in D_{ij};
- Any instance of an L-relation R_i on D is contained entirely within some D_{ij}.

Note that there is no assumption here that $D \in \mathcal{K}_f$. Indeed, we say that $\mathcal{K}_f$ is *closed under ITD's* if whenever $(D; D_i, D_{ij})$ is an ITD in $\mathcal{K}_f$, then $D \in \mathcal{K}_f$.

Theorem 1.5. *Suppose that $\mathcal{K}_f$ is closed under ITD's. Then $Th(M_f)$ is simple.* □

The original proof of this is in [7]. Variations on the original proof can be found in [2] (cf. the proof of Theorem 3.6 there: the condition in the above is exactly the assumption (P5) on M_f in [2]), and in [13].

The following lemma from the proof of Theorem 3.6(ii) of [2] will be useful. The notation $X \leq^* Y$ means $d_0(X) \leq d_0(Y_1)$ for all Y_1 with $X \subseteq Y_1 \subseteq Y$.

Lemma 1.6. *If $(D; D_i, D_{ij})$ is an independence theorem diagram in $\mathcal{K}_f$ then:*

(i) $D_{ij} \leq D$;
(ii) $D_{ij} \leq D_{ij} \cup D_{jk}$;
(iii) $D_{ij} \cup D_{jk} \leq^* D$. □

In the rest of this section we will be interested in the situation where the hypotheses of Theorem 1.5 do not hold: in particular, we will prove a strong version of the converse (Theorem 1.9).

1.2. **Strong order properties.** Recall the following from ([11], Definition 2.5).

Definition 1.7. Suppose T is a complete first-order theory and $n \geq 3$ is an integer. Say that T has *strong order property n* (SOP_n) if there exists a formula $\phi(\bar{x}, \bar{y})$ and an infinite sequence of distinct tuples $(\bar{a}_i : i < \omega)$ in some model N of T such that

(a) $N \models \phi(\bar{a}_i, \bar{a}_j)$ for $i < j < \omega$;

(b) $N \models \neg\exists \bar{x}_0 \ldots \bar{x}_{n-1}(\phi(\bar{x}_0, \bar{x}_1) \wedge \phi(\bar{x}_1, \bar{x}_2) \wedge \cdots \wedge \phi(\bar{x}_{n-1}, \bar{x}_0))$.

The negation of this property is denoted by $NSOP_n$.

Allowing the formula to have parameters changes nothing. Also, we may take the sequence $(\bar{a}_i : i < \omega)$ to be indiscernible (over whatever parameters). Condition (b) simply says that there are no directed n-cycles in the directed graph determined by the relation $\phi(\bar{x}, \bar{y})$.

As mentioned in the introduction, these properties form a hierarchy. We shall show that if $Th(M_f)$ is not simple, then it fits very neatly into this hierarchy. The notation is as in Assumption 1.1:

Theorem 1.8. *The theory $Th(M_f)$ has the property $NSOP_4$. In particular, M_f does not have the strict order property.*

Theorem 1.9. *If $Th(M_f)$ is not simple, then it has SOP_3.*

Proof of Theorem 1.8. Work in a big model N_f of $Th(M_f)$ and suppose $(a_i : i < \omega)$ is an infinite indiscernible sequence of tuples in N_f (over a finite parameter set, which we may assume to be $\emptyset$). Let $p(x_0, x_1)$ be the complete type of (a_0, a_1) in N_f. To show that $Th(M_f)$ is $NSOP_4$ it will be enough to show that

$$p(x_0, x_1) \cup p(x_1, x_2) \cup p(x_2, x_3) \cup p(x_3, x_0)$$

is consistent.

We now follow the notation and some of the arguments from [2] very closely. The structure M_f is the special case $y(B) = |B|$ of the examples in ([2], Section 3). The conditions on f in ([2], 3.1) are irrelevant by our current assumptions on f, so ([2], Theorem 3.6(i)) holds, and (M_f, d_0) has properties (P1-P4, P6, P7) of [2]. The notation $d(c/S)$ is as defined above (and also defined at the start of Section 2.5 (and on p. 259) of [2]) and acl denotes algebraic closure in N_f.

Claim: There is a finite set c of parameters such that $(a_i : i < \omega)$ is c-indiscernible and for $i = 1, 2$ we have $d(a_i/ca_0 \ldots a_{i-1}) = d(a_i/c)$ (i.e. a_0, a_1, a_2 are d-independent over c).

The proof is as in paragraphs 2 and 3 of the proof of 2.19(b) in [2], but we repeat the outline here. Extend the indiscernible sequence to an indiscernible sequence $(a_i : i \in \mathbb{Z})$. Let $A_0 = \mathrm{acl}(a_i : i < 0)$.

Then $(a_i : i \geq 0)$ is A_0-indiscernible and d-independent over A_0. By extending the sequence, and then thinning, we may assume that $X = \mathrm{acl}(A_0 a_{i_2}) \cap \mathrm{acl}(A_0 a_{i_0} a_{i_1})$ is constant for $i_0 < i_1 < i_2$ and then that $(a_i : i \in \omega)$ is X-indiscernible. By (P7) there is a finite $C \subseteq X$ such that $d(a_2/a_0 a_1 C) = d(a_2/C)$, and C-indiscernibility gives the d-independence of a_0, a_1, a_2. ($\Box$ Claim)

Note that (as M_f is a generic stucture) $\mathrm{tp}(a_i, a_j/c)$ is determined by the isomorphism type of $E_{ij} = \mathrm{cl}(a_i a_j c)$. Let $C = \mathrm{cl}(c)$, let $E_i = \mathrm{cl}(a_i c)$ and let $A = E_{01} \cup E_{12}$. So by the d-independence of a_0, a_1, a_2 over c we have that A is the free amalgam of E_{01} and E_{12} over E_1. Moreover $E_0 \cup E_2 = A \cap E_{02} \leq A$ and $E_0 \cup E_2$ is the free amalgam of E_0 and E_2 over C. By the latter, there is an isomorphism $\gamma : E_0 \cup E_2 \to E_0 \cup E_2$ over C which interchanges the tuples a_0, a_2.

Consider the embeddings $h_1 : E_0 \cup E_2 \to E_{02}$ given by inclusion and $h_2 : E_0 \cup E_2 \to E_{02}$ given by appyling γ and then inclusion. Let F be the free amalgam obtained from these embeddings and $g_i : E_{02} \to F$ such that $g_1 \circ h_1 = g_2 \circ h_2$. By assumption $F \in \mathcal{K}_f$, so we can assume that (an isomorphic copy of) $F \leq N$. Let $a'_0 = g_1(h_1(a_0)), a'_1 = g_1(a_1), a'_2 = g_1(h_1(a_2))$ and $a'_3 = g_2(a_1)$. Then

$$\mathrm{tp}(a_0, a_1) = \mathrm{tp}(a'_0, a'_1) = \mathrm{tp}(a'_1, a'_2) = \mathrm{tp}(a'_2, a'_3) = \mathrm{tp}(a'_3 a'_0)$$

as required. To see that the types are equal, one simply has to consider closures of the two tuples (- they are even equal over the image of c in F). $\Box$

1.3. **Proof of Theorem 1.9.** Assumption 1.1 continues to hold throughout. By Theorem 1.5, if $Th(M_f)$ is not simple, there is some ITD $(D; D_i, D_{ij})$ in $\mathcal{K}_f$ such that D is *not* in K_f (and therefore not a substructure of M_f). From this, we will construct a sequence $(\bar{a}_i : i < \omega)$ in M_f and a formula ϕ witnessing SOP_3. The idea is that each $\bar{a}_i$ consists of independent copies of D_1, D_2, D_3 and the relation ϕ says that the different copies are related in the same way as in the D_{ij}. That D is not a substructure of M_f then gives that the relation ϕ has no directed triangles. The precise form of the argument is somewhat more complicated and we split it into pieces.

1.3.1. *The Structures E^r.* In the following we will often abuse notation and identify a finite set with some fixed enumeration of the set. For example, if X is a finite L-structure we denote by $\mathrm{qftp}(X)$ the quantifier free type of (some fixed enumeration of) X.

Suppose that $(D; D_i, D_{ij})$ is a fixed ITD in $\mathcal{K}_f$.

Definition 1.10. Let r be a positive integer. We define the L-structure E^r to have domain:

$$E^r = D_0 \cup \bigcup \{A_i, B_i, C_i : 1 \le i \le r\} \cup \bigcup \{Z_{ij}, Z'_{ij}, Z''_{ij} : 1 \le i < j \le r\}$$

and such that the following conditions hold:

(1) The intersection of any two sets from $\{A_i, B_i, C_i : i \le r\}$ is D_0.
(2) We have the following isomorphisms:
 - $\mathrm{qftp}(A_i B_j Z_{ij}/D_0) = \mathrm{qftp}(D_1 D_2 D_{12}/D_0)$
 - $\mathrm{qftp}(B_i C_j Z'_{ij}/D_0) = \mathrm{qftp}(D_2 D_3 D_{23}/D_0)$
 - $\mathrm{qftp}(C_i A_j Z''_{ij}/D_0) = \mathrm{qftp}(D_1 D_3 D_{13}/D_0)$.

 So in particular $A_i, B_j \le Z_{ij}$, $B_i, C_j \le Z'_{ij}$, and $C_i, A_j \le Z''_{ij}$ for $i < j$.
(3) The intersection of any two sets from $\{Z_{ij}, Z'_{ij}, Z''_{ij} : i < j \le r\}$ is D_0 if the index sets are disjoint, or the appropriate member of $\{A_i, B_i, C_i\}$ if the index sets intersect in i.
(4) The only instances of L-relations R_k in E^r are those occurring within each Z_{ij}, Z'_{ij} or Z''_{ij}.

The main aim of this subsection is to show that $E^r \in \mathcal{K}_f$. Before doing that we prove a preliminary lemma.

Let $\mathbf{A}^r = \bigcup_{i \le r} A_i$, $\mathbf{B}^r = \bigcup_{i \le r} B_i$, $\mathbf{C}^r = \bigcup_{i \le r} C_i$.

Lemma 1.11. *For all natural numbers $r \ge 1$, if $D_0 \le E^r$, then $\mathbf{A}^r, \mathbf{B}^r, \mathbf{C}^r \le E^r$.*

Proof: We prove the lemma for $\mathbf{A}^r$: the other cases follow by symmetry. Let $\mathbf{Z} = B_1 \cup A_r \cup \bigcup_{i<j}^r Z_{ij}$.

Claim 1: $d_0(\mathbf{Z}) = d_0(\mathbf{A}^r) + d_0(\mathbf{B}^r) - d_0(D_0)$.

By definition Z_{ij} is isomorphic to D_{12} so as D_1, D_2 are d-independent in D_{12} we have $d_0(Z_{ij}) = d_0(A_i) + d_0(B_j) - d_0(D_0) = d_0(A_i B_j)$ by Lemma 1.3. Thus

$$\begin{aligned} d_0(\mathbf{Z}) &= d_0(B_1 \cup A_r \cup \bigcup_{i<j\le r} A_i B_j) \\ &= d_0(A_1 \dots A_r B_1 \dots B_r) \\ &= d_0(A_1 \dots A_r) + d_0(B_1 \dots B_r) - d_0(D_0), \end{aligned}$$

as required.

In Claims 2, 3 and 4, cl and d denote cl_Z and d^Z respectively.

Claim 2: $d(\mathbf{A}^r) = d_0(\mathbf{A}^r)$ and $d(\mathbf{B}^r) = d_0(\mathbf{B}^r)$.

Put $A = \mathrm{cl}(\mathbf{A}^r)$ and $B = \mathrm{cl}(\mathbf{B}^r)$. By assumption, $D_0 \leq E^r$ so $D_0 \leq Z$. Moreover $d(\mathbf{A}^r/D_0) \geq d(\mathbf{A}^r/\mathbf{B}^r)$ so

$$\begin{aligned} & d_0(A) - d(D_0) \geq d_0(\mathbf{Z}) - d_0(B) \text{ (as } \mathbf{Z} = \mathrm{cl}_{\mathbf{Z}}(AB)) \\ \text{i.e.}\quad & d_0(A) - d(D_0) \geq d_0(\mathbf{A}^r) - d_0(\mathbf{B}^r) - d_0(D_0) - d_0(B) \text{ (by Claim 1).} \end{aligned}$$

But this holds iff $d_0(A) - d_0(\mathbf{A}^r) \geq d_0(\mathbf{B}^r) - d_0(B)$. Since $d_0(A) \leq d_0(\mathbf{A}^r)$ the left hand side of the equation is less than or equal to 0. Similarly the right side must be greater than or equal to 0 so the only possibility is that we have equality everywhere and $d_0(A) = d_0(\mathbf{A}^r)$, $d_0(B) = d_0(\mathbf{B}^r)$ as required.

Claim 3: $A \cap B = D_0$.

Clearly $\mathbf{Z} = \mathrm{cl}(\mathbf{A}^r\mathbf{B}^r) = \mathrm{cl}(AB)$. We know that:

$$d_0(A \cap B) \leq d_0(A) + d_0(B) - d_0(AB).$$

However since $\mathrm{cl}(AB) = \mathbf{Z}$ we have

$$\begin{aligned} d_0(A) + d_0(B) - d_0(AB) &\leq d_0(A) + d_0(B) - d_0(\mathbf{Z}) \\ &= d_0(\mathbf{A}^r) + d_0(\mathbf{B}^r) - d_0(\mathbf{Z}) \\ &= d_0(D_0) \text{ (by Claim 1).} \end{aligned}$$

Thus $d_0(D_0) \geq d_0(A \cap B)$. As $D_0 \leq A \cap B$ we must have $D_0 = A \cap B$, as required.

Claim 4: $\mathbf{A}^r \leq \mathbf{Z}$.

Let $W_{ij} = Z_{ij} \backslash (B_j \backslash D_0)$. For fixed i the substructure $\bigcup_{i<j} W_{ij}$ is a free amalgam of the W_{ij} over A_i. As $A_i \leq Z_{ij}$ it follows that $A_i \leq W_{ij}$, so $A_i \leq \bigcup_{i<j} W_{ij}$. As $\bigcup_{1\leq i<j\leq r} W_{ij}$ is the free amalgam of these over D_0, it follows that $\mathbf{A}^r \leq \bigcup_{1\leq i<j\leq r} W_{ij}$. Thus $A \cap \bigcup_{1\leq i<j\leq r} W_{ij} = \mathbf{A}^r$. But by Claim 3, $A \cap B = D_0$. Thus $A = \mathbf{A}^r$.

Claim 5: $\mathbf{Z} \leq E^r$

Note that by symmetry of the argument, we also have $\mathbf{B}^r \leq \mathbf{Z}$ and by Claim 1, $\mathbf{A}^r$ and $\mathbf{B}^r$ are d-independent in $\mathbf{Z}$ over D_0. We can make the obvious definition of $\mathbf{Z}'$ and $\mathbf{Z}''$ and again by the symmetry of the situation, $\mathbf{B}^r, \mathbf{C}^r \leq \mathbf{Z}'$ and are d-independent etc. Thus we can view E^r as obtained as an ITD with constituent parts $\mathbf{A}^r, \mathbf{B}^r, \mathbf{C}^r$ (as the D_i

in the definition) and $\mathbf{Z}, \mathbf{Z}', \mathbf{Z}''$ (as the D_{ij}). Claim 5 then follows from Lemma 1.6(i).

The lemma now follows from Claims 4 and 5. □

Theorem 1.12. *For all natural numbers $r \geq 1$ we have $D_0 \leq E^r$ and $E^r \in \mathcal{K}_f$.*

Proof: We prove this by induction on r, the case $r = 1$ being trivial (E^1 is just the free amalgam of A_1, B_1, C_1 over D_0).

For the inductive step, suppose that $D_0 \leq E^r \in \mathcal{K}_f$. By Lemma 1.11 we also know that $\mathbf{A}^r, \mathbf{B}^r, \mathbf{C}^r \leq E^r$. We want to show that $E^r \leq E^{r+1}$ and $E^{r+1} \in \mathcal{K}_f$.

We obtain E^{r+1} from E^r in three stages, adding in turn B_{r+1} and all of its corresponding $Z_{i,r+1}$ to E^r, then C_{r+1} and the $Z'_{i,r+1}$, then A_{r+1} and the $Z''_{i,r+1}$.

Let $\mathbb{B}^{r+1} = \bigcup_{i \leq r} Z_{i,r+1}$. As this is a free amalgam of the $Z_{i,r+1}$ over B_{r+1}, it is in $\mathcal{K}_f$. Using Lemma 1.6(i) and the fact that the A_i are d-independent over D_0, one obtains that $\mathbf{A}^r \leq \mathbb{B}^{r+1}$. Let E^r_* be the free amalgam of $\mathbb{B}^{r+1}$ and E^r over $\mathbf{A}^r$. Then $E^r \leq E^r_*$ and (by Assumption 1.1) $E^r_* \in \mathcal{K}_f$.

For the next step we put $\mathbb{C}^{r+1} = \bigcup_{1 \leq r} Z'{}_{i,r+1}$; since it is a free amalgam it belongs to $\mathcal{K}_f$. Let E^r_{**} be the free amalgam of $\mathbb{C}^{r+1}$ and E^r_* over $\mathbf{B}^r$. As in the previous step we have $E^r_* \leq E^r_{**} \in \mathcal{K}_f$.

Finally let $\mathbb{A}^{r+1} = \bigcup_{1 \leq r} Z''{}_{i,r+1}$. This is in $\mathcal{K}_f$ and E^{r+1} is the free amalgam of $\mathbb{A}^{r+1}$ and and E^r_{**} over $\mathbf{C}^r$. Thus, as in the previous steps, $E^r_{**} \leq E^{r+1} \in \mathcal{K}_f$. As we have $D_0 \leq E^r \leq E^r_* \leq E^r_{**} \leq E^{r+1}$, this completes the inductive step. □

1.3.2. *Witnessing SOP_3.* Suppose now that $\mathcal{K}_f$ satisfies Assumption 1.1 and $(D; D_i, D_{ij})$ is an ITD in $\mathcal{K}_f$ with $D \notin \mathcal{K}_f$. Consider the structures $E^r \in \mathcal{K}_f$ constructed from this ITD as in the previous subsection. Inside E^{2r} we have two substructures isomorphic to E^r:

$$\begin{aligned} X &= \bigcup_{1 \leq i < j \leq r} Z_{ij} \cup Z'_{ij} \cup Z''_{ij} \\ Y &= \bigcup_{r+1 \leq i < j \leq 2r} Z_{ij} \cup Z'_{ij} \cup Z''_{ij}. \end{aligned}$$

We consider these as being enumerated in some fixed way compatible with the isomorphism and let η be an L-formula describing the quantifier-free type of X (with this enumeration). Let $\bar{x}, \bar{y}$ be tuples of variables of length $|E^r|$ and $\bar{z}$ a tuple of variables of length $|E^{2r}|$. Let $\zeta(\bar{x}, \bar{y}, \bar{z})$ describe the quantifier-free type of (X, Y, E^{2r}). Define the formula $\phi_r(\bar{x}, \bar{y})$ to be:

$$\eta(\bar{x}) \wedge \eta(\bar{y}) \wedge (\exists \bar{z})\zeta(\bar{x}, \bar{y}, \bar{z}).$$

Lemma 1.13. *There exists an infinite sequence of distinct tuples $(X_i : i < \omega)$ in M_f such that $M_f \models \phi_r(X_i, X_j)$ for all $i < j$.*

Proof: By saturation of M_f, we may assume that all E^r are $\leq$-subsets of M_f. The lemma follows by taking $X_k = \bigcup_{1+rk \leq i < j \leq (r+1)k} Z_{ij} \cup Z'_{ij} \cup Z''_{ij}$ (suitably enumerated). □

Thus Theorem 1.9 will follow from:

Lemma 1.14. *With the above notation, there is a natural number s such that if $r \geq s$ then*

$$M_f \not\models \exists \sigma\tau\upsilon\, \phi_r(\sigma, \tau) \wedge \phi_r(\tau, \upsilon) \wedge \phi_r(\upsilon, \sigma).$$

Proof. We show that if r is large enough and $M_f \models \phi_r(\sigma, \tau) \wedge \phi_r(\tau, \upsilon) \wedge \phi_r(\upsilon, \sigma)$ then we have a copy of the forbidden D as a substructure of M_f. Before continuing however we require more notation. Without loss of generality concentrate on the σ case; the τ, υ cases being defined analogously.

By definition of ϕ_r, the tuple σ enumerates a substructure of M_f which is isomorphic to E^r. In the notation of Definition 1.10, denote its corresponding substructures A_i, B_i, C_i respectively as $A_i^\sigma, B_i^\sigma, C_i^\sigma$. Next, for $i, j \leq r$ let $Z(ij), Z'(ij), Z''(ij)$ satisfy

- $\mathrm{qftp}(A_i^\sigma B_j^\tau Z(ij)) = \mathrm{qftp}(D_1 D_2 D_{12})$
- $\mathrm{qftp}(B_i^\tau C_j^\upsilon Z'(ij)) = \mathrm{qftp}(D_2 D_3 D_{23})$
- $\mathrm{qftp}(C_i^\upsilon A_j^\sigma Z''(ij)) = \mathrm{qftp}(D_3 D_1 D_{31})$.

Finally let $|Z(ij)| = l_{AB}$, $|Z'(ij)| = l_{BC}$ and $|Z''(ij)| = l_{CA}$.

Claim: If r is large enough there exist $k, m, n \leq r$ such that $Z(km) \cap Z'(mn) = B_m^\tau$, $Z'(mn) \cap Z''(nk) = C_n^\upsilon$, and $Z''(nk) \cap Z(km) = A_k^\sigma$.

We will take $k = 1$. For $m \leq r$, let $C[m] = \{j : Z(1m) \cap Z'(mj) = B_m^\tau\}$. Since there are at most l_{AB} choices of j with $B_m^\tau \neq Z(1m) \cap Z'(mj)$ we have $|C[m]| \geq r - l_{AB}$. Now consider $C[1] \cap C[2] \ldots \cap C[2l_{CA}+1]$. Each term has size at least $r - l_{AB}$ and so this intersection excludes a maximum of $l_{AB}(2l_{CA} + 1)$ natural numbers $\leq r$; setting

$s = l_{AB}(2l_{CA} + 1) + 1$, then for $r \geq s$ this intersection will be non-empty.

Suppose $r \geq s$ and let n be an element in the above intersection. If $m \leq 2l_{CA} + 1$ then $Z(1m) \cap Z'(mn) = B_m^\tau$, by choice of n. The potential problem is that $Z''(n1)$ may intersect $Z(1m)$ by more than the desired A_1^σ or $Z'(mn)$ by more than the desired C_n^υ. There are at most l_{CA} choices of m such that $A_1^\sigma \neq Z(1m) \cap Z''(n1)$ and also at most l_{CA} choices of m such that $C_n^\upsilon \neq Z'(mn) \cap Z''(n1)$. However since there are $2l_{CA} + 1$ choices for m, there is at least one $m \leq 2l_{CA} + 1$ such that $A_1^\sigma = Z(1m) \cap Z''(n1)$ and $C_m^\upsilon = Z'(mn) \cap Z''(n1)$. This establishes the claim.

Now take k, m, n as in the claim. To ease the notation let $T_1 = A_k^\sigma, T_2 = B_m^\tau, T_3 = C_n^\upsilon$ and $T_{12} = Z(km), T_{23} = Z'(mn), T_{31} = Z''(nk)$. Put $T = T_{12} \cup T_{23} \cup T_{31}$. Since $T \subseteq M_f$ we have $T \in \mathcal{K}_f$. As $T_{ij} \cap T_{jk} = T_j$ and T_{ij} is isomorphic to D_{ij} from the ITD, we may construct a bijection $\xi : D \to T$ which preserves the L-relations R_i. Thus if $W \subseteq D$ then $d_0(W) \geq d_0(\xi W)$, and as $W \subseteq T \in \mathcal{K}_f$, this is $\geq f(|\xi(W)|)$. As ξ is a bijection, we obtain $d_0(W) \geq f(|W|)$ for all $W \subseteq D$. By definition of $\mathcal{K}_f$, we therefore have $D \in \mathcal{K}_f$: a contradiction. $\square$

Proof of Theorem 0.1: We already noted in the preamble to the Theorem that (2) $\Rightarrow$ (1). The argument in this subsection shows that if $\mathcal{K}_f$ is not closed under ITD's, then M_f has SOP_3. As simplicity always implies $NSOP_3$ we therefore have (1) $\Rightarrow$ (3) $\Rightarrow$ (2). $\square$

2. $Th(M_0)$ IS BAD

To keep the ideas clear, we shall work with a particular example. So in this section we assume that the language L has (apart from equality) a single ternary relation R. The class $\overline{\mathcal{K}}$ consists of L-structures in which this is symmetric and only realised by distinct triples of elements. If $A \in \mathcal{K}$, then $d_0(A) = |A| - |R[A]|$, where by $R[A]$ we mean the set of 3-subsets of A picked out by R (rather than the set of 3-tuples).

We shall be working with the class $\mathcal{K}_0 = \{A \in \mathcal{K} : \emptyset \leq A\}$ and the notion of embedding $A \leq B$, meaning $d_0(A) < d_0(B')$ for all $A \subset B' \subseteq B$, as before. Note that there is a different notion of embedding introduced in [6]: $A \leq^* B$ meaning $d_0(A) \leq d_0(B')$ for all $A \subset B' \subseteq B$. Both $(\mathcal{K}_0, \leq)$ and $(\mathcal{K}_0, \leq^*)$ are amalgamation classes, however, their respective generic structures M_0 and M_0^* behave very differently. It is well-known that M_0^* is ω-stable (of Morley rank ω), but as we shall see, the model theory of M_0 is bad.

Note that if $a, b, c \in A \in \overline{\mathcal{K}}$ and $A \models R(a, b, c)$, then $c \in \mathrm{cl}_A(a, b)$. The idea is to encode graphs into the closures of pairs of elements. A similar (but more difficult) type of encoding is used in Section 3 of [12].

Define predicates V, E as follows:

$$V(x; y, z) \leftrightarrow R(x, y, z)$$

and

$$E(x_1, x_2; y, z) \leftrightarrow V(x_1; y, z) \wedge V(x_2; y, z) \wedge (\exists w)R(x_1, x_2, w).$$

If $a, b \in A \in \overline{\mathcal{K}}_0$ then $\Gamma(a, b, A)$ is the graph with vertex set $V[A; a, b]$ and edges $E[A; a, b]$. Note that the vertex set here is in $\mathrm{cl}_A(a, b)$ and any edge is witnessed in $\mathrm{cl}_A(a, b)$. Thus if $A \leq B$ then $\Gamma(a, b, A) = \Gamma(a, b, B)$.

Now, if Γ is any graph, there is $A_\Gamma \in \overline{\mathcal{K}}_0$ and $a_\Gamma, b_\Gamma \in A_\Gamma$ with $\Gamma(a_\Gamma, b_\Gamma, A_\Gamma)$ isomorphic to Γ. Indeed, suppose Γ has vertex set S and edge set $U \subseteq [S]^2$. Let A_Γ be the disjoint union of $\{a_\Gamma, b_\Gamma\}$, S and U with the relation $R[A_\Gamma]$ consisting of $\{a_\Gamma, b_\Gamma, s\}$ for all $s \in S$, and $\{s_1, s_2, u\}$ for all $u = \{s_1, s_2\} \in U$. It is easy to check that $A_\Gamma \in \overline{\mathcal{K}}_0$ and all points in A_Γ are in the closure of a_Γ, b_Γ.

Given any first-order sentence σ in the language of graphs (with binary relation S) we construct an L-formula $\theta_\sigma(y, z)$ by replacing all atomic subformulas $S(x_1, x_2)$ in σ by $E(x_1, x_2; y, z)$ and replacing any quantifier $\forall x$ by $\forall x \in V(x; y, z)$ (and likewise $\exists x$ by $\exists x \in V(x; y, z)$).

Lemma 2.1. *For any $M \in \overline{\mathcal{K}}_0$ and $a, b \in M$ we have:*

$$\Gamma(a, b, M) \models \sigma \Leftrightarrow M \models \theta_\sigma(a, b).$$

Proof. This is essentially a triviality: cf. Theorem 5.3.2 in [4]. □

Now let M_0 be the generic structure for the class $(\mathcal{K}_0, \leq)$, as in the introduction.

Theorem 2.2. *Suppose σ is a sentence in the language of graphs. Then there is a finite model of σ iff $M_0 \models (\exists y, z)\theta_\sigma(y, z)$.*

Proof. If there is a finite model Γ of σ then we can find $A \leq M_0$ with $A_\Gamma \cong A$. Then by the lemma, $M_0 \models \theta_\sigma(a_\Gamma, b_\Gamma)$, as required.

Conversely suppose $a, b \in M_0$ and $M_0 \models \theta_\sigma(a, b)$. Then $\Gamma(a, b, M_0)$ is a graph which is a model of σ. It is finite, as it is contained in $\mathrm{cl}_{M_0}(a, b)$. $\square$

Corollary 2.3. *$Th(M_0)$ is undecidable.*

Proof. The construction of θ_σ from σ is obviously recursive. On the other hand, the theory of all finite graphs is undecidable (by Trakhtenbrot's Theorem, cf. [4]). So the same is true of $Th(M_0)$, by the above. $\square$

Theorem 2.4. *Suppose σ is a sentence in the language of graphs which has arbitrarily large finite models. Then some infinite model of σ is interpretable in a model of $Th(M_0)$.*

Proof. The formulas $\theta_\sigma(a, b) \wedge$ '$|V(x; a, b)| \geq n$' (for $n \in \mathbb{N}$) are consistent with $Th(M_0)$ by assumption. So by compactness there is a model M of $Th(M_0)$ and $a, b \in M$ such that $\Gamma(a, b, M)$ is an infinite model of σ (by Lemma 2.1). $\square$

Corollary 2.5. *$Th(M_0)$ has the strict order property.*

Proof. We can construct a family of finite graphs in which arbitrarily large finite linear orderings are uniformly interpretable. There is a sentence in the language of graphs which implies that the interpreted stucture is a linear ordering (again, this is by Theorem 5.3.2 of [4]). Thus, arguing by compactness as in the previous proof, there is a model M of $Th(M_0)$ and $a, b \in M$ such that the interpreted structure in $\Gamma(a, b, M)$ is an infinite linear ordering. But $\Gamma(a, b, M)$ is itself interpreted in M. $\square$

The reader will have noticed that the proofs used only the local finiteness of closure and $\leq$-universality of M_0 (i.e. every $A \in \mathcal{K}_0$ is isomorphic to some self-sufficient substructure of M_0).

The undecidability result means that $Th(M_0)$ is not recursively axiomatisable. In particular, the *semigeneric theory* T_{sgen} given in ([10], Definition 3.27) following [1], does not axiomatize $Th(M_0)$ (for the notation there, we take T_0 as the universal theory describing $\overline{\mathcal{K}}_0$). We have $T_{sgen} \subseteq Th(M_0)$ (essentially, because of the full form of the amalgamation property), so we conclude that T_{sgen} is not complete. In fact, it is useful to see this in a different way. It is fairly easy to show that if $A \in \overline{\mathcal{K}}_0$, then there is a model M of T_{sgen} which has A as a self-sufficient substructure. Let σ be some formula in the language of graphs which has only infinite models, let Γ be such a model and $A = A_\Gamma$. Then $M \models (\exists y, z)\theta_\sigma(y, z)$, but of course $M_0 \not\models (\exists y, z)\theta_\sigma(y, z)$.

The undecidability and SOP rested on transferring properties of finite structures to M_0. So one possible positive property left for $Th(M_0)$ is the following:

Question 2.6. Does $Th(M_0)$ have the finite model property?

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