Finite Covers

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Introduction

Connections between higher amalgamation properties and finite covers in the paper

[Hr]: Ehud Hrushovski, ‘Groupoids, imaginaries and internal covers’, ArXiv:math.LO/0603413v1, March 2006.

This talk: Outline a proof of

Hr, Proposition 3.11

Let T be a theory with a canonical 2-amalgamation (for example, T stable). There exists an expansion T^* of T to a language with additional sorts, such that:

(1) T is stably embedded in T^*, and the induced structure from T^* on the T-sorts is the structure of T. Each sort of T^* admits a 0-definable map to a sort of T, with finite fibres.

(2) T^* has existence and uniqueness for independent N-amalgamation over $\mathrm{acl}(\emptyset)$.

1. Independent amalgamation.

NOTATION: $N \in \mathbb{N}$; $\mathcal{P}(N)^- =$ set of proper subsets of $\{1, \ldots, N\}$. Think of this as a category with inclusion maps as morphisms.

Suppose T has QE and a canonical 2-amalgamation over algebraically closed sets. Let $\mathcal{C}$ be the category of algebraically closed substructures of models of T (and embeddings). Let $C \in \mathcal{C}$.

An (independent) N-amalgamation problem over C is a functor

$$A : \mathcal{P}(N)^- \to \mathcal{C}$$

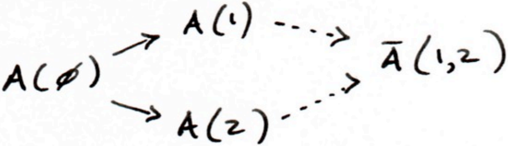
where $A(\emptyset) = C$ and for any $s \in \mathcal{P}(N)^-$ the set $\{A(i) : i \in s\}$ is independent over C, and $A(s) = \mathrm{acl}(A(i) : i \in s\}$.

A solution to this is an extension of A to a functor

$$\bar{A} : \mathcal{P}(N) \to \mathcal{C}$$

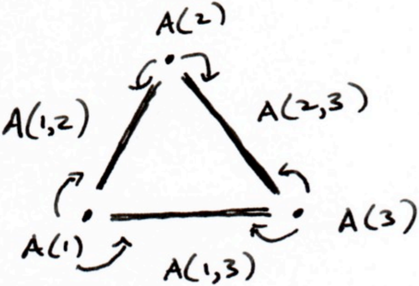
on the full power set, satisfying the same conditions (so including the case $s = \{1, \ldots, N\}$).

N = 2:



N = 3:

(ignore A(∅))



- Canonical 2-amalgamation means that each 2-amalgamation problem has a given solution and the resulting notion of independence is assumed to satisfy full transitivity and symmetry. The main result also requires definability.
- T has N-existence (for independent amalgamation over C) if every such amalgamation problem has a solution
- T has has N-uniqueness (for independent amalgamation over C) if every such amalgamation problem has at most one solution.

EXAMPLES:

(1) Stable theories have 2-existence and uniqueness over algebraically closed sets when independence is non-forking.

(2) A vector space of infinite dimension over a finite field has N-existence and uniqueness for all N.

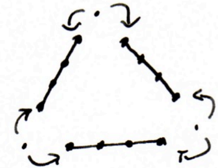
(3) The corresponding projective space does not have 3-uniqueness (if the field has at least 3 elements).

(4) Stable theories have N-existence and uniqueness over models.

Example for (3):

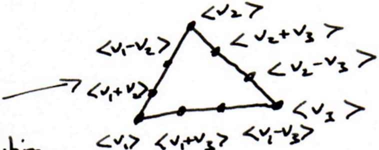
Field = GF(3) ; $V = V(\aleph_0, 3)$ v. space

$\mathbb{P}(V)$: projective space

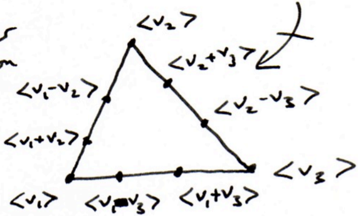


$A(i,j)$ = line in $\mathbb{P}(V)$

One solution



Another solution



Two reductions

Continue to assume T has a canonical notion of 2-amalgamation.

Lemma (cf. Hr, 3.1)

Suppose that T has N-uniqueness over $\mathrm{acl}(\emptyset)$ for all $N \geq 2$. Then T has N-existence over $\mathrm{acl}(\emptyset)$ for all $N \geq 2$.

Lemma (Hr, Prop 3.5)

T has N-uniqueness over $\mathrm{acl}(\emptyset)$ iff the following condition holds for all independent $a_1, \ldots, a_N$:

***(*)** if $c \in \mathrm{acl}(a_1, \ldots, a_{N-1})$ is in the definable closure of $\bigcup_{i=1}^{N-1} \mathrm{acl}(a_1 \ldots \widehat{a}_i \ldots a_{N-1} a_N)$, then it is in the definable closure of $\bigcup_{i=1}^{N-1} \mathrm{acl}(a_1 \ldots \widehat{a}_i \ldots a_{N-1})$.*

2. Finite covers and definable types

DEFINITIONS: Work with multi-sorted structures. Suppose $L \subseteq L^*$ are languages; T is an L-theory and $T^* \supseteq T$ an L^*-theory.

- T is embedded in T^* if the the induced structure on the T-sorts from the 0-definable sets of T^* is the 0-definable structure of T.
- T is stably embedded if the same is true without the 0
- T is fully embedded if it is embedded and stably embedded.
- T^* is an algebraic cover of T if T is fully embedded and each sort of T^* admits a 0-definable finite-to-one map to a sort of T.
- An algebraic cover T^* is a finite cover of T if it is in the definable closure of the T-sorts and a single T^*-sort.

REMARKS: If T is fully embedded in T^* and $M^* \models T^*$ is saturated, then any automorphism of the T-part of M^* extends to an automorphism of M^*. For a finite cover, stable embeddedness follows automatically from embeddedness.

Finite covers from definable types

Suppose T is a complete L-theory. Work in a large model M^* of T. Say that a type $p(x) \in S(\emptyset)$ is definable if

for each L-formula $\phi(x, y)$ there is an L-formula $\psi_\phi^p(y)$ with the property that for every D

$$p|D = \{\phi(x, d) : \phi(x, y) \text{ an } L\text{-formula }, d \in D \text{ and } \models \psi_\phi^p(d)\}$$

is a complete type over D, and $p|\emptyset = p$.

Let p be definable, as above. Suppose θ is an L-formula such that if $\models \theta(a', b', c')$, then c' is algebraic over $a'b'$. Let $M \models T$.

Let $a^* \models p|M$ and

$$C = \{(b', c^*) : c^* \in M^*, b' \in M \text{ and } M^* \models \theta(a^*, b', c^*)\}.$$

We make the disjoint union $M \cup C$ into a structure $M^+ = C(M, a^*)$ by giving it the induced structure from (M^*, a^*).

Lemma

Suppose $M \preceq \tilde{M}$ are ω-saturated.

1. *If $a^* \models p|\tilde{M}$ then $C(M, a^*) \preceq C(\tilde{M}, a^*)$.*
2. *If d, e are tuples in M^+ then:*

$$\mathrm{tp}^{M^+}(d) = \mathrm{tp}^{M^+}(e) \Leftrightarrow \mathrm{tp}^{M^*}(d/a^*) = \mathrm{tp}^{M^*}(e/a^*).$$

3. *M is fully embedded in M^+.*

Denote $Th(M^+)$ by $T_{p,\theta}$.

Note that M^+ is a finite cover of M. We call it a *definable* finite cover.

3. Splitting of finite covers

An algebraic cover T' of T splits over T if there is an expansion of T' to an algebraic cover T'' of T which is interdefinable with T. For a sufficiently saturated model M' of T', this implies that there is an expansion M'' of M' with

$$\mathrm{Aut}(M') = \mathrm{Aut}(M'/M) \rtimes \mathrm{Aut}(M'').$$

Lemma (Free Amalgamation)

Suppose $M_1 \supseteq M_0$ and $M \supseteq M_0$ are algebraic covers. Let $M' = M_1 \coprod_{M_0} M$ be the disjoint union of M_1 and M over M_0. Then M' is an algebraic cover of M and if M_1 is 0-interpretable in M over M_0, then M' splits over M.

Proof: There is an injective map $f : M_1 \to M$ which is the identity on M_0 and which sends 0-definable sets to 0-definable sets. If we expand M' by f to obtain M'', then $\mathrm{Aut}(M') = \mathrm{Aut}(M'/M) \rtimes \mathrm{Aut}(M'')$.

4. Splitting and N-uniqueness

Lemma (Splitting Lemma)

Suppose $M \subseteq M'$ is a split algebraic cover, $X_1, \ldots, X_r \subseteq M$ and $\mathrm{acl}^M(X_i) = X_i$ for $i = 1, \ldots, r$. Then

$$\mathrm{Aut}(\bigcup_i \mathrm{acl}^{M'}(X_i)/\bigcup_i X_i) = \mathrm{Aut}(\bigcup_i \mathrm{acl}^{M'}(X_i)/M).$$

Proof: Do this with $i = 1$. Write $\mathrm{Aut}(M') = \mathrm{Aut}(M'/M) \rtimes \mathrm{Aut}(M'')$. Restriction to the T-sorts gives a (topological) group isomorphism $\mathrm{Aut}(M'') \to \mathrm{Aut}(M)$. In the lemma the inclusion $\supseteq$ is clear. Suppose the other direction does not hold. Then there is $c \in \mathrm{acl}^{M'}(X_1)$ which is fixed by $\mathrm{Aut}(M'/M)$ but not by $\mathrm{Aut}(M'/X_1)$. Thus, $\mathrm{Aut}(M''/X_1, c)$ is a proper open subgroup of finite index in $\mathrm{Aut}(M''/X_1)$. Restricting to the T-sorts gives a proper open subgroup of finite index in $\mathrm{Aut}(M/X_1)$, contradicting algebraic closure of X_1 in M. □

Corollary (Main Lemma)

Let $M \models T$ be ω-saturated, p a complete type definable over $\emptyset$ and $a^* \models p|M$. Suppose all $T_{p,\theta}$ split over T. Suppose $b_0, b_1, \ldots, b_r \in M$ and $c \in \mathrm{acl}(a^* b_0)$, $e_i \in \mathrm{acl}(a^* b_i)$ are such that $c \in \mathrm{dcl}(a^* e_1 \ldots e_r M)$. Then

$$c \in \mathrm{dcl}(a^* e_1 \ldots e_r B_0 B_1 \ldots B_r),$$

where $B_i = \mathrm{acl}(b_i)$.

Corollary

Suppose $p_1, \ldots, p_N$ are types definable over $\emptyset$ and $a_1, \ldots, a_N$ are such that $a_i \models p_i | \{a_1, \ldots, \hat{a}_i, \ldots a_N\}$. Let

$$c \in \mathrm{acl}(a_1 \ldots a_{N-1}) \cap \mathrm{dcl}(\bigcup_{i=1}^{N-1} \mathrm{acl}(a_1 \ldots \widehat{a}_i \ldots a_N)).$$

Suppose further that for all $i < N$ each $T_{p_i,\theta}$ splits over T. Then

$$c \in \mathrm{dcl}(\bigcup_{i=1}^{N-1} \mathrm{acl}(a_1 \ldots \widehat{a}_i \ldots a_{N-1})).$$

Using this and the characterization of N-uniqueness we get:

Theorem (Theorem A)

Suppose T has a definable canonical 2-amalgamation and all definable finite covers of T split over T. Then T has N-uniqueness for independent amalgamation, for all $N \geq 2$.

5. Guaranteeing the splitting

GIVEN: T_0 with a definable canonical 2-amalgamation.

WANT: Algebraic cover $T \supseteq T_0$ with a definable canonical 2-amalgamation extending that of T_0 such that all definable finite covers of T split over T.

MAIN POINTS:

- A definable finite cover of T_0 inherits a definable canonical 2-amalgamation from T
- Taking a sequence of definable finite covers we obtain T with the property that any definable finite cover of T_0 and a finite set of sorts of T is interpretable in T.
- As in the free amalgamation lemma, any definable finite cover of T splits over T.