Expansions of fields by angular functions

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1. Angular functions

– Talk about some questions from the paper

> Boris Zilber, 'Non-commutative geometry and new stable structures', Newton Institute Preprint Series, NI05048-MAA, November 2005.

NOTATION: For the moment, fix:

- an algebraically closed field $(F; +, \cdot)$ of characteristic zero;
- multiplicatively independent $\alpha, \beta \in F$;
- a natural number N;
- a primitive N-th root of unity ϵ;
- $\Gamma = \langle \epsilon \rangle$

An *angular function* (with these data) is a function

$$\mathrm{ang} : F^* \to \Gamma$$

satisfying, for all $t \in F$:

$$\begin{aligned} \mathrm{ang}(\epsilon t) &= \mathrm{ang}(t) && (1) \\ \mathrm{ang}(\beta t) &= \mathrm{ang}(t) && (2) \\ \mathrm{ang}(\alpha t) &= \epsilon\mathrm{ang}(t) && (3) \end{aligned}$$

Zilber asks the following:

> QUESTION: Consider a structure $(F, +, \cdot, \mathrm{ang})$ which is existentially closed in the class of structures satisfying these equations. What is the model-theoretic status of this structure? Is it supersimple?

DEFINITIONS: Associated to any angular function ang there are two definable subgroups of the multiplicative group of the field:

- The group of *periods* of ang is

$$G = \{g \in F^* : \mathrm{ang}(gt) = \mathrm{ang}(t)\ \forall t \in F^*\};$$

- the group of *quasiperiods* of ang is

$$G^+ = \{h \in F^* : \exists \gamma \in \Gamma\ \forall t \in F^*\ \mathrm{ang}(ht) = \gamma \mathrm{ang}(t)\}.$$

Note that

- $\Gamma \leq G \leq G^+$;
- there is a definable homomorphism $\chi : G^+ \to \Gamma$ with kernel G (given by $\mathrm{ang}(ht) = \chi(h)\mathrm{ang}(t)$ for $h \in G^+$ and $t \in F^*$);
- the induced map $\bar{\chi} : G^+/G \to \Gamma$ is an isomorphism.

2. Zilber's examples

EXAMPLE 1: The following explains the terminology 'angular function.' Take:

- $F = \mathbb{C}$
- $\epsilon = \exp(2\pi i/N)$
- For $k = 0, \ldots, N-1$, let P_k be the sector of the complex plane consisting of non-zero complex numbers z with an argument $\arg z$ in the range $2\pi k/N \leq \arg z < 2\pi(k+1)/N$.

Define $\text{ang} : \mathbb{C}^* \to \Gamma$ by, for $t \in \mathbb{C}^*$:

$$\text{ang}(t) = \epsilon^k \Leftrightarrow t^N \in P_k.$$

NOTE:

- Group of periods: $G = \mathbb{R}^{>0}\Gamma$;
- Group of quasiperiods: $G^+ = \mathbb{R}^{>0}\langle\epsilon_1\rangle$, where $\epsilon_1 = \exp(2\pi i/N^2)$.
- $(\mathbb{C}, +, \cdot, \text{ang})$ has the strict order property (consider translates of the definable subset P_0).

EXAMPLE 2: An example of a non-classical Zariski curve (due to Hrushovski and Zilber) can be obtained from a suitable angular function.

Given $(F;+,\cdot,\alpha,\beta,\mathrm{ang})$ satisfying the above equations.
Define $U,V:F^*\to F^*$ by

$$\begin{aligned} U(t) &= \alpha t \\ V(t) &= \beta \mathrm{ang}(t) t. \end{aligned}$$

These are definable permutations of F^* and

$$VU(t)=\epsilon UV(t).$$

Let

- $(T;U,V)$ denote the set F with only the structure given by the definable permutations U,V;
- $p:T\to F$ be given by $p(t)=t^N$.

Then the structure $((T;U,V),(F;+,\cdot),p:T\to F)$ is a finite cover of $(F;+,\cdot)$ which is interpretable in $(F;+,\cdot,\mathrm{ang})$, but not in $(F;+,\cdot)$.

3. An answer to the question?

DEFINITION:

- The language L_0 contains:
  - the language of rings $+, -, \cdot, 0, 1$;
  - unary predicates $\mathbf{F}, \mathbf{\Gamma}, \mathcal{G}, \mathcal{G}^+$
  - a unary function symbol χ
- We have a fixed L_0-structure M consisting of:
  - an algebraically closed field $F = \mathbf{F}(M)$ of characteristic 0 ;
  - multiplicative subgroups

$$\Gamma = \mathbf{\Gamma}(M) \subseteq G = \mathcal{G}(M) \subseteq G^+ = \mathcal{G}^+(M) \subseteq F^*;$$

  - A surjective homomorphism $\chi : G^+ \to \Gamma$ with kernel G.
- T_0 is the L_0-theory of M.

  Assume (for our purposes, without loss) that T_0 is model complete.

Let $L_A = L_0 \cup \{A\}$ be the expansion of L_0 by an extra unary function symbol A. Define T_A to be the theory axiomatized by T_0 and axioms:

(i) $(A(0) = 0) \wedge (\forall t)(t \neq 0) \to \mathbf{\Gamma}(A(t))$;

(ii) $(\forall t)(\forall g)(\mathcal{G}(g) \to A(g \cdot t) = A(t))$;

(iii) $(\forall t)(\forall h)(\mathcal{G}^+(h) \wedge (t \neq 0) \to A(h \cdot t) = \chi(h) \cdot A(t))$

Zilber's Question can then be viewed as asking whether the class of existentially closed models of T_A is axiomatizable, and if so whether completions of its theory are supersimple.

Theorem 1 *(1) If T_0 eliminates the quantifier $\exists^\infty$ in the sorts $\mathbf{F}, \mathbf{F}/\mathcal{G}$ and $\mathbf{F}/\mathcal{G}^+$ then T_A has a model companion T_A^*.*

(2) If additionally T_0 is simple and Γ is finite then all completions of T_A^* are simple (and in the same simplicity class as T_0).

Discuss:

- why this follows easily from know results
- T_0 satisfying the hypotheses.

4. Proof of the Theorem

Suppose $M \models T_0$. Write $\mathcal{G}$ instead of $\mathcal{G}(M)$ etc.

4.1 From angular functions to sections

Consider the natural map $\nu : \mathbf{F}^*/\mathcal{G} \to \mathbf{F}^*/\mathcal{G}^+$, given by

$$\nu(x\mathcal{G}) = x\mathcal{G}^+.$$

A *section* of this is a map $s : \mathbf{F}^*/\mathcal{G}^+ \to \mathbf{F}^*/\mathcal{G}$ which satisfies:

$$(\forall y \in \mathbf{F}^*/\mathcal{G}^+)(s(y)\mathcal{G}^+ = y).$$

Given such an s we have that $s(t\mathcal{G}^+)^{-1}t\mathcal{G} \in \mathcal{G}^+/\mathcal{G}$, so

$$A(t) = \bar{\chi}(s(t\mathcal{G}^+)^{-1}t\mathcal{G}) \in \mathbf{\Gamma}.$$

This satisfies T_A. Let L_s be the expansion of L_0 obtained by adding a unary function symbol s between the indicated sorts, and T_s obtained from T_0 by adding the above axiom. Then:

Lemma 2 *There is a definable correspondence between the models of T_A and the models of T_s which preserves the property of existential closure. Thus T_A has a model companion if and only if T_s does.*

4.2. Skolem expansions

Suppose L is any first-order language and T any L-theory.

DEFINITION: Say that T *eliminates* $\exists^\infty$ (or is *algebraically bounded*) if for all L-formulas $\phi(x, \bar{y})$ there is a natural number N_ϕ with the property that for all models M of T and $\bar{a}$ in M, if $\phi[M, \bar{a}]$ has more than N_ϕ elements, then it is infinite.

DEFINITION: Suppose $\phi(x, \bar{y})$ is an L-formula, where $\bar{y}$ is an n-tuple of variables. Let L^+ be the expansion of L by a new n-ary function symbol σ. The L^+-theory T^+ is axiomatized by T together with:

$$(\forall \bar{y})((\exists x)\phi(x, \bar{y}) \to \phi(\sigma(\bar{y}), \bar{y})).$$

We refer to T^+ as a *Skolem expansion* of T.

Theorem 3 (P. Winkler, 1975) *Suppose T is a model-complete L-theory which eliminates $\exists^\infty$. Then any Skolem expansion T^+ of T has a model companion $(T^+)^*$.*

Proof of Theorem 1 (1) The theory T_s is the Skolem expansion of T_0 with respect to the formula $\phi(x, y)$:

$$(x \in \mathbf{F}/\mathcal{G}) \wedge (y \in \mathbf{F}/\mathcal{G}^+) \wedge (x\mathcal{G}^+ = y).$$

Apply Winkler's Theorem.

(2) If Γ is finite, then the map ν is $|\Gamma|$-to-1. So the Skolem expansion T_s is an *algebraic* Skolem expansion. Results of Nübling (*Arch. Math. Logic*, 2004) then give what we want.

5. Not the finite cover property

How to verify that $T_0 = Th(F; \Gamma, G, G^+, \chi, \ldots)$ eliminates $\exists^\infty$ in the various sorts?

THEOREM: (Shelah) *For a complete theory T, the following are equivalent:*

(1) T does not have the finite cover property;

(2) T is stable and T^{eq} eliminates $\exists^\infty$.

– The property of being algebraically bounded in all (real and imaginary) sorts is referred to as *weak nfcp*.

There are various technologies available for checking that a (stable) T has nfcp. For example:

- if T is non-multidimensional, then T has nfcp
- belles paires (Poizat)

6. A better answer?

Theorem 4 *Suppose F is an algebraically closed field of characteristic zero of infinite transcendence rank and $\alpha, \beta \in F$ are multiplicatively independent. Let N be a natural number, ϵ a primitive N-th root of 1 and $\Gamma = \langle \epsilon \rangle$; $G = \langle \alpha^N, \beta, \epsilon \rangle$; and $G^+ = \langle \alpha, \beta, \epsilon \rangle$. Define $\chi : G^+ \to \Gamma$ to have kernel G and $\chi(\alpha) = \epsilon$. Then:*

(i) $T_0 = Th(F; +, -, \cdot, 0, 1, \Gamma, G, G^+, \chi)$ is superstable of Lascar rank ω and has nfcp.

(ii) T_A has a model completion T_A^* and all completions of this are supersimple of SU-rank ω.

Proof: (i) All of the structure is definable in $(F, G, \alpha, \beta, \epsilon)$. It's well known that this is superstable of Lascar rank ω: see Pillay's paper 'Lang's conjecture and model theory.' The argument in Pillay's paper also gives non-multidimensionality, hence nfcp.
(ii) This follows from part (i) and Theorem 1. $\square$

QUESTION: Using an omitting-types theorem for e.c. models, one can show that there is a countable model of T_A^* in which the group of periods is precisely $G = \langle \alpha^N, \beta, \epsilon \rangle$. Is there a model in which the field is $\mathbb{C}$ and the group of periods is still G?

7. Green points and variations.

Zilber's paper: Do this where the group of periods G is (a variation of) Poizat's 'green points;' consider the case where Γ is infinite (but small).

7.1. Green points.

Bruno Poizat, 'L'Egalité au cube', JSL 66 (2001):

LANGUAGE L_G: $+, -, \cdot, 0, 1$ and a 1-ary predicate $\mathcal{G}$

CLASS $\mathcal{C}$: structures $(A, \mathcal{G}(A))$ where:

- A is an algebraically closed field of characteristic 0 and finite transcendence rank
- $G(A)$ is a torsion-free divisible subgroup of A^*
- $\delta(A_1) = 2\mathrm{trdeg}(A_1) - \mathrm{rk}_{\mathbb{Q}}(\mathcal{G}(A_1)) \geq 0$ for every algebraically closed subfield A_1 of A.

EMBEDDINGS: $A \leq B$ ($\in \mathcal{C}$) means $\delta(A) \leq \delta(B_1)$ for all algebraically closed $A \subseteq B_1 \subseteq B$.

$(\mathcal{C}, \leq)$ has the amalgamation property and we can construct a universal-homogeneous structure $(F, \mathcal{G}(F))$ for the class. Poizat shows how to axiomatize $T_G = Th(F, \mathcal{G}(F))$ (using results of Ax/ Zilber on intersecting algebraic varieties with tori). T_G is ω-stable of Morley rank $\omega.2$ and the subgroup $\mathcal{G}$ has Morley rank ω. Using similar methods and belles paires one has:

Proposition 5 *T_G has nfcp.*

7.2. Variations:

VARIATION 1:

$\tilde{T}_G$: Same as Poizat's T_G, but $\mathcal{G} \equiv \mathbb{Z}^2$. This is superstable of U-rank $\omega.2$ and has nfcp.

For suitable $\alpha, \beta \in \mathbb{C}^\times$ and $h \in \mathbb{R}$ let

$$G_0 = \exp(\frac{2\pi i}{hN}\mathbb{Z} + \frac{\alpha}{h}\mathbb{Z} + \beta\mathbb{R}).$$

THEOREM: (Zilber) *Assuming Schanuel's conjecture,* $\tilde{T}_G = Th(\mathbb{C}, G_0)$.

As for Theorem 4, (and assuming SC) one then has:

Theorem 6 *With the above notation, let* $\Gamma = \langle \epsilon \rangle$; $G = G_0.\Gamma$; *and* $G^+ = \exp((2\pi i/hN)\mathbb{Z} + (\alpha/hN)\mathbb{Z} + \beta\mathbb{R}).\Gamma$ *and* $\chi(a) = \epsilon$. *Then:*

- *(i)* $T_0 = Th(\mathbb{C}; +, -, \cdot, 0, 1, \Gamma, G, G^+, \chi)$ *is superstable of Lascar rank* $\omega.2$ *and has nfcp.*
- *(ii)* T_A *has a model companion* T_A^* *and all completions of this are supersimple of* SU*-rank* $\omega.2$.

QUESTION: Zilber gives an explicit construction of an angular function ang_N with the above data. Is it a model of T_A^*?

VARIATION 2: Infinite Γ.

LANGUAGE: $+, -, \cdot, 0, 1$; unary predicates $\mathcal{G}, \Gamma, \Gamma^+$ and unary functions χ, χ^{-1}.

CLASS: algebraically closed fields A of characteristic zero; multiplicative subgroups $\mathcal{G}(A), \Gamma(A), \Gamma^+(A)$ with the properties that

- $\Gamma(A) \subseteq \mathcal{G}(A)$ and $\mathcal{G}(A) \cap \Gamma^+(A) = 1$;
- the groups $\mathcal{G}(A)$ and $\Gamma(A)$ are elementarily equivalent to $\mathbb{Z}$;
- $\chi : \Gamma^+(A) \to \Gamma(A)$ and $\chi^{-1} : \Gamma(A) \to \Gamma^+(A)$ are mutually inverse group isomorphisms;
- the predimension inequality $\delta \geq 0$ holds, where

$$\delta(A) = 2.\mathrm{trdeg}(A) - \mathrm{rk}_{\mathbb{Q}}(\mathcal{G}(A)) - 3.\mathrm{rk}_{\mathbb{Q}}(\Gamma(A)).$$

THEORY $\hat{T}_G$: like Poizat's T_G.

Define $\mathcal{G}^+ = \mathcal{G}.\Gamma^+$ and extend $\chi : \mathcal{G}^+ \to \Gamma$.

Obtain L_0-theory T_0 which is superstable of Lascar rank $\omega.2$ with nfcp. So part (i) of Theorem 1 applies and we have a model companion T_A^* of T_A. As Γ is infinite, this is NOT simple.

QUESTION: Zilber has natural candidates for models of $\hat{T}_G$ and T_A^*. Are they in fact models?