M3/4A16 MSc Enhanced Coursework **Darryl Holm**
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[#1] $\mathbb{R}^3$ *bracket for the spherical pendulum*

(1a) *Identify the degrees of freedom for the spherical pendulum in* $T\mathbb{R}^3$ *and show that the constraint of constant length reduces the number of degrees of fredom from three to two by sending* $T\mathbb{R}^3 \to T\mathbb{S}^2$*.*

(1b) *Formulate the constrained Lagrangian in* $T\mathbb{R}^3$ *and show that its symmetry under rotations about the vertical axis implies a conservation law for angular momentum via Noether's theorem.*

(1c) *Derive the motion equations on* $T\mathbb{R}^3$ *from Hamilton's principle. Note that these equations preserve the defining conditions for*

$$T\mathbb{S}^2 : \{(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3 \,\big|\, \|\mathbf{x}\|^2 = 1 \text{ and } \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}\,.$$

That is, $T\mathbb{S}^2$ *is an invariant manifold of the equations in* $\mathbb{R}^3$*. Conclude that the constraints for remaining on* $T\mathbb{S}^2$ *may be regarded as dynamically preserved initial conditions for the spherical pendulum equations in* $T\mathbb{R}^3$*.*

(1d) *Legendre transform the Lagrangian defined on* $T\mathbb{R}^3$ *to find a constrained Hamiltonian (Routhian) with variables* $(\mathbf{x}, \mathbf{y}) \in T^*\mathbb{R}^3$ *whose dynamics preserves* $T\mathbb{S}^2$*.*

(1e) *There are six linear and quadratic variables in* $T^*\mathbb{R}^3/S^1$

$$\begin{aligned} \sigma_1 &= x_3 & \sigma_3 &= y_1^2 + y_2^2 + y_3^2 & \sigma_5 &= x_1y_1 + x_2y_2 \\ \sigma_2 &= y_3 & \sigma_4 &= x_1^2 + x_2^2 & \sigma_6 &= x_1y_2 - x_2y_1 \end{aligned}$$

However, these are not independent. They satisfy a cubic algebraic relation. Find this relation and write the TS^2 *constraints in terms of the* S^1 *invariants.*

(1f) *Write closed Poisson brackets among the six independent linear and quadratic* S^1*-invariant variables*

$$\sigma_k \in T^*\mathbb{R}^3/S^1, \quad k = 1, 2, \dots, 6.$$

(1g) *Show that the two quantities*

$$\sigma_3(1-\sigma_1^2) - \sigma_2^2 - \sigma_6^2 = 0 \quad \text{and} \quad \sigma_6$$

are Casimirs for the Poisson brackets on $T^*\mathbb{R}^3/S^1$*.*

(1h) *Use the orbit map* $T\mathbb{R}^3 \to \mathbb{R}^6$

$$\pi : \ (\mathbf{x}, \mathbf{y}) \to \{\sigma_j(\mathbf{x}, \mathbf{y}),\ j = 1, \dots, 6\} \tag{1}$$

to transform the energy Hamiltonian to S^1*-invariant variables.*

(1i) *Find the reduction* $T^*\mathbb{R}^3/S^1 \cap T\mathbb{S}^2 \to \mathbb{R}^3$*. Show that the motion follows the intersections of level surfaces of angular momentum and energy in* $\mathbb{R}^3$*. Compute the associated Nambu bracket in* $\mathbb{R}^3$ *and use it to characterise the types of motion available in the motion of this system.*

(1j) *Write the Hamiltonian, Poisson bracket and equations of motion in terms of the variables $\sigma_k \in T^*\mathbb{R}^3/S^1$, $k = 1, 2, 3$.*

(1k) *Interpret the solutions geometrically as intersections of Hamiltonian level sets (planes in $\mathbb{R}^3$) with a family of cup-shaped surfaces (Casimirs of the $\mathbb{R}^3$ bracket depending on angular momentum) whose limiting surface at zero angular momentum is a* pinched cup.

(1l) *Show that this geometrical interpretation implies that the two equilibria along the vertical axis at the North and South poles have the expected opposite stability. Explain why this was to be expected.*

(1m) *Show that the unstable vertical equilibrium at the North pole is connected to itself by homoclinic orbits.*

(1n) *Show that all other orbits are periodic.*

(1o) *Reduce the dynamics on a family of planes representing level sets of the Hamiltonian to single particle motion in a phase plane and compute the behaviour of its solutions. Identify its critical points and their stability.*

[#1] $\mathbb{R}^3$ *bracket for the spherical pendulum*

Answer Clip in from 2nd edition of GM1

(1a) The configuration space is S^2 and the Lagrangian (or Hamiltonian) is defined on the tangent bundle TS^2 of $S^2 \in \mathbb{R}^3$, which has dimension $\dim(TS^2) = 4$, not 6:

$$TS^2 = \big\{ (\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3 \simeq \mathbb{R}^6 \;\big|\; 1 - |\mathbf{x}|^2 = 0,\, \mathbf{x} \cdot \dot{\mathbf{x}} = 0 \big\}. \tag{2}$$

(1b) We begin with the Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}}) : T\mathbb{R}^3 \to \mathbb{R}$ given by

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \tfrac{1}{2}|\dot{\mathbf{x}}|^2 - g\hat{\mathbf{e}}_3 \cdot \mathbf{x} - \tfrac{1}{2}\mu(1 - |\mathbf{x}|^2), \tag{3}$$

in which the Lagrange multiplier μ constrains the motion to remain on the sphere S^2 by enforcing $(1 - |\mathbf{x}|^2) = 0$ when it is varied in Hamilton's principle.

S^1 symmetry and Noether's theorem The Lagrangian in (3) is invariant under S^1 rotations about the vertical axis, whose infinitesimal generator is $\delta\mathbf{x} = \hat{\mathbf{e}}_3 \times \mathbf{x}$. Consequently Noether's theorem, that each smooth symmetry of the Lagrangian in an action principle implies a conservation law for its Euler-Lagrange equations, in this case implies that the equations (5) conserve

$$J_3(\mathbf{x}, \dot{\mathbf{x}}) = \dot{\mathbf{x}} \cdot \delta\mathbf{x} = \mathbf{x} \times \dot{\mathbf{x}} \cdot \hat{\mathbf{e}}_3 \,, \tag{4}$$

which is the angular momentum about the vertical axis.

(1c) The Euler-Lagrange equation corresponding toLagrangian (3) is

$$\ddot{\mathbf{x}} = -g\hat{\mathbf{e}}_3 + \mu\mathbf{x} \,. \tag{5}$$

This equation preserves both of the TS^2 relations $1 - |\mathbf{x}|^2 = 0$ and $\mathbf{x} \cdot \dot{\mathbf{x}} = 0$, provided the Lagrange multiplier is given by

$$\mu = g\hat{\mathbf{e}}_3 \cdot \mathbf{x} - |\dot{\mathbf{x}}|^2 \,. \tag{6}$$

(1d)

The ***fibre derivative*** of the Lagrangian L in (3) is

$$\mathbf{y} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \dot{\mathbf{x}}\,. \tag{7}$$

The variable $\mathbf{y}$ will be the momentum canonically conjugate to the radial position $\mathbf{x}$, after the ***Legendre transform*** to the corresponding Hamiltonian,

$$H(\mathbf{x},\mathbf{y}) = \tfrac{1}{2}|\mathbf{y}|^2 + g\hat{\mathbf{e}}_3\cdot\mathbf{x} + \tfrac{1}{2}(g\hat{\mathbf{e}}_3\cdot\mathbf{x} - |\mathbf{y}|^2)(1-|\mathbf{x}|^2)\,, \tag{8}$$

whose ***canonical equations*** on $(1-|\mathbf{x}|^2)=0$, are

$$\dot{\mathbf{x}} = \mathbf{y} \quad\text{and}\quad \dot{\mathbf{y}} = -g\hat{\mathbf{e}}_3 + (g\hat{\mathbf{e}}_3\cdot\mathbf{x} - |\mathbf{y}|^2)\mathbf{x}\,. \tag{9}$$

(1e) The six S^1-invariants above satisfy the cubic algebraic relation

$$\sigma_5^2 + \sigma_6^2 = \sigma_4(\sigma_3 - \sigma_2^2)\,. \tag{10}$$

They also satisfy the positivity conditions

$$\sigma_4 \ge 0, \quad \sigma_3 \ge \sigma_2^2. \tag{11}$$

In these variables, the defining relations (2) for TS^2 become

$$\sigma_4 + \sigma_1^2 = 1 \quad\text{and}\quad \sigma_5 + \sigma_1\sigma_2 = 0\,. \tag{12}$$

As expected, being invariant under the S^1 rotations, TS^2 is also expressible in terms of S^1-invariants. The three relations in equations (10) – (12) will define the ***orbit manifold*** for the spherical pendulum in $\mathbb{R}^6$.

(1f)

$\{\cdot,\cdot\}$	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
σ_1	0	σ_4	$2\sigma_2$	0	$-\sigma_1\sigma_4$	0
σ_2		0	$-2\sigma_1\sigma_3$	$2\sigma_1\sigma_4$	$\sigma_2\sigma_4$	0
σ_3			0	$4\sigma_1\sigma_2$	$2(\sigma_2^2-\sigma_1^2\sigma_3)$	0
σ_4				0	$2\sigma_1^2\sigma_4$	0
σ_5					0	0
σ_6						0

(1g) The variable σ_6 is obviously a Casimir. It remains to check whether the S^1-invariant $\sigma_3(1-\sigma_1^2) - \sigma_2^2 - \sigma_6^2 = 0$ is one, too. This could be accomplished with the 3×3 bracket table:

$\{\cdot,\cdot\}$	σ_1	σ_2	σ_3
σ_1	0	$1-\sigma_1^2$	$2\sigma_2$
σ_2	$-1+\sigma_1^2$	0	$-2\sigma_1\sigma_3$
σ_3	$-2\sigma_2$	$2\sigma_1\sigma_3$	0

However, it's actually simpler to use the larger bracket table and check that the Casimir is $\sigma_3\sigma_4 - \sigma_2^2 - \sigma_6^2 = 0$. This works easily and it was guaranteed, because it is simply the cubic relation among the six S^1-invariants.

(1h) In S^1-invariant variables the energy Hamiltonian is

$$H = \tfrac{1}{2}\sigma_3 + g\sigma_1\,. \tag{13}$$

(1i) The motion may be expressed in Hamiltonian form by introducing the following bracket operation, defined for a function F of the S^1-invariant vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$ by

$$\{F, H\} = -\frac{\partial C}{\partial \boldsymbol{\sigma}} \cdot \frac{\partial F}{\partial \boldsymbol{\sigma}} \times \frac{\partial H}{\partial \boldsymbol{\sigma}} = -\,\epsilon_{ijk} \frac{\partial C}{\partial \sigma_i} \frac{\partial F}{\partial \sigma_j} \frac{\partial H}{\partial \sigma_k}\,, \tag{14}$$

where

$$C = \sigma_3(1 - \sigma_1^2) - \sigma_2^2 - \sigma_6^2 = 0$$

is the Casimir and one treats the other Casimir (σ_6) as a parameter.

(1j) The individual components of the equations of motion may be obtained from Poisson bracket (14) with Hamiltonian (13) as

$$\dot{\sigma}_1 = -\sigma_2\,, \qquad \dot{\sigma}_2 = \sigma_1\sigma_3 + g(1 - \sigma_1^2)\,, \qquad \dot{\sigma}_3 = 2g\sigma_2\,.$$

(1k) The motion is along intersections of the Hamiltonian planes with the *pinched cup* given by $C = 0$.

(1l) The two equilibria lie along the vertical axis at the North and South poles, and they are the only ones. Moreover, they are expected to have opposite stability, because one corresponds to the pendulum down and the other to the pendulum up.

(1m,n,o) Substituting $\sigma_3 = 2(H - g\sigma_1)$ from equation (13) and setting the acceleration of gravity to be unity $g = 1$ yields

$$\ddot{\sigma}_1 = 3\sigma_1^2 - 2H\sigma_1 - 1\,, \tag{15}$$

which conserves the energy integral

$$\tfrac{1}{2}\dot{\sigma}_1^2 + V(\sigma_1) = E\,, \tag{16}$$

with cubic potential

$$V(\sigma_1) = -\,\sigma_1^3 + H\sigma_1^2 + \sigma_1\,. \tag{17}$$

Hereafter, the problem reduces to the σ_1 phase plane for the "fish" and everything else is elementary.

▲

[#2] The Hopf map

In coordinates $(a_1, a_2) \in \mathbb{C}^2$, the Hopf map $\mathbb{C}^2/S^1 \to S^3 \to S^2$ is obtained by transforming to the four quadratic S^1-invariant quantities

$$(a_1, a_2) \to Q_{jk} = a_j a_k^*\,, \quad \textit{with} \quad j, k = 1, 2\,.$$

Let the $\mathbb{C}^2$ coordinates be expressed as

$$a_j = q_j + ip_j$$

in terms of canonically conjugate variables satisfying the fundamental Poisson brackets

$$\{q_k, p_m\} = \delta_{km} \quad \textit{with} \quad k, m = 1, 2.$$

(2a) *Compute the Poisson brackets* $\{a_j, a_k^*\}$ *for* $j, k = 1, 2$.

(2b) *Is the transformation* $(q, p) \to (a, a^*)$ *canonical? Explain why, or why not.*

(2c) *Compute the Poisson brackets among the* Q_{jk}, *with* $j, k = 1, 2$.

(2d) *Make the linear change of variables,*

$$X_0 = Q_{11} + Q_{22}\,, \quad X_1 + iX_2 = Q_{12}\,, \quad X_3 = Q_{11} - Q_{22}\,.$$

Compute the Poisson brackets among the (X_0, X_1, X_2, X_3).

(2e) *Express the Poisson bracket* $\{F(\mathbf{X}), H(\mathbf{X})\}$ *in vector form among functions* F *and* H *of* $\mathbf{X} = (X_1, X_2, X_3)$

(2f) *Show that the quadratic invariants* (X_0, X_1, X_2, X_3) *themselves satisfy a quadratic relation. How is this relevant to the Hopf map?*

[#2] The Hopf map

Answer

(2a) The $\mathbb{C}^2$ coordinates $a_j = q_j + ip_j$ satisfy the Poisson bracket

$$\{a_j\,,\, a_k^*\} = -2i\,\delta_{jk}\,, \quad \text{for } j, k = 1, 2\,.$$

(2b) The transformation $(q, p) \mapsto (a, a^*)$ is indeed canonical. The constant $2i$ is inessential.

(2c) The quadratic S^1 invariants on $\mathbb{C}^2$ given by $Q_{jk} = a_j a_k^*$ satisfy the Poisson bracket relations,

$$\{Q_{jk}\,,\, Q_{lm}\} \;=\; 2i\,(\delta_{kl}Q_{jm} - \delta_{jm}Q_{kl})\,, \quad j, k, l, m = 1, 2\,.$$

Thus, they do close among themselves, but they do not satisfy canonical Poisson bracket relations.

(2d) The quadratic S^1 invariants (X_0, X_1, X_2, X_3) given by

$$X_0 = Q_{11} + Q_{22}\,, \quad X_1 + iX_2 = Q_{12}\,, \quad X_3 = Q_{11} - Q_{22}\,,$$

may be expressed in terms of the a_j, $j = 1, 2$ as

$$X_0^2 = |a_1|^2 + |a_2|^2\,, \quad X_1 + iX_2 = 2a_1 a_2^*\,, \quad X_3 = |a_1|^2 - |a_2|^2\,.$$

These satisfy the Poisson bracket relations,

$$\{X_0\,,\, X_k\} = 0\,, \quad \{X_j\,,\, X_k\} = -\,\epsilon_{jkl} X_l$$

(2e) The Poisson bracket $\{F(\mathbf{X}), H(\mathbf{X})\}$ is given in vector form as

$$\{F(\mathbf{X}), H(\mathbf{X})\} = -\,\mathbf{X}\cdot\frac{\partial F}{\partial \mathbf{X}}\times\frac{\partial H}{\partial \mathbf{X}}$$

(2f) The quadratic invariants (X_0, X_1, X_2, X_3) satisfy the quadratic relation

$$X_0^2 = X_1^2 + X_2^2 + X_3^2 \,.$$

This completes the Hopf map, because level sets of X_0 are spheres $S^2 \in S^3$.

▲

[#3] Dynamics of vorticity gradient

(3a) *Write Euler's fluid equations in* $\mathbb{R}^3$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p \,, \quad \operatorname{div}\mathbf{u} = 0 \,, \tag{18}$$

in geometric form using the Lie derivative of the circulation 1-form.

(3b) *State and prove Kelvin's circulation theorem for Euler's fluid equations in geometric form.*

(3c) *The vorticity of the Euler fluid velocity* $\mathbf{u} = (u, v, w) \in \mathbb{R}^3$ *is given by* $\boldsymbol{\omega} := \operatorname{curl}\mathbf{u}$*. Write the Euler fluid equation for vorticity by taking the exterior derivative of the geometric form in part above.*

(3d) *For Euler fluid motion restricted to the* (x, y) *plane with normal unit vector* $\hat{\mathbf{z}}$*, the vorticity equation for* $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ *with scalar vorticity* $\omega(x, y, t)$ *simplifies to*

$$\partial_t \omega + \mathbf{u}\cdot\nabla\omega = 0 \quad \textit{with} \quad \mathbf{u} = (u, v, 0).$$

Using this scalar vorticity dynamics, compute the equation for

$$\left(\frac{\partial}{\partial t} + \pounds_u\right)(dz \wedge d\omega)$$

and express it as a 2D vector equation for the quantity

$$\mathcal{B} := \hat{\mathbf{z}} \times \nabla\omega \in \mathbb{R}^2.$$

(3e) *How is* $\mathcal{B}$ *related to* $\mathbf{u}$*? Compare the result of part for the dynamics of* $\mathcal{B}$ *with the dynamical equation for the vorticity vector in part and the defining equation for the flow velocity in terms of the stream function.*

(3f) *Find a conserved integral quantity associated with* $\mathcal{B}$ *and*$\mathbf{u}$*.*

[#3] *Dynamics of vorticity gradient*

Answer

(3a) Euler's fluid equations in $\mathbb{R}^3$ may be written in a nice geometric form as

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t} + \pounds_u\Big)(\mathbf{u}\cdot d\mathbf{x}) &= \big(\partial_t \mathbf{u} - \mathbf{u}\times \operatorname{curl}\mathbf{u} + \nabla|\mathbf{u}|^2\big)\cdot d\mathbf{x} \\
&= \big(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\tfrac{1}{2}|\mathbf{u}|^2\big)\cdot d\mathbf{x} \\
\text{by Euler's motion equation (18)} \quad &= -\,d\left(p - \tfrac{1}{2}|\mathbf{u}|^2\right). \qquad (19)
\end{aligned}
$$

(3b) Kelvin's circulation theorem states that the Euler equations (18) preserve the circulation integral $I(t)$ defined by

$$
I(t) = \oint_{c(\mathbf{u})} \mathbf{u}\cdot d\mathbf{x}\,, \qquad (20)
$$

where $c(\mathbf{u})$ is a closed circuit moving with the fluid at velocity $\mathbf{u}$.

The dynamical definition of Lie derivative yields the following for the time rate of change of the Kelvin circulation integral,

$$
\begin{aligned}
\frac{d}{dt}\oint_{c(\mathbf{u})} \mathbf{u}\cdot d\mathbf{x} &= \oint_{c(\mathbf{u})} \Big(\frac{\partial}{\partial t} + \pounds_u\Big)(\mathbf{u}\cdot d\mathbf{x}) \\
&= \oint_{c(\mathbf{u})} \Big(\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial x^j}u^j + u_j\frac{\partial u^j}{\partial \mathbf{x}}\Big)\cdot d\mathbf{x} \\
\text{by Euler's motion equation (18)} \quad &= -\oint_{c(\mathbf{u})} \nabla\Big(p - \frac{1}{2}|\mathbf{u}|^2\Big)\cdot d\mathbf{x} \\
&= -\oint_{c(\mathbf{u})} d\Big(p - \frac{1}{2}|\mathbf{u}|^2\Big) = 0\,.
\end{aligned}
$$

(3c) With $\boldsymbol{\omega} := \operatorname{curl}\mathbf{u}$ the exterior derivative of Euler's fluid equation and the Cartan formula for the Lie derivative yields

$$
\begin{aligned}
d\Big(\frac{\partial}{\partial t} + \pounds_u\Big)(\mathbf{u}\cdot d\mathbf{x}) &= \Big(\frac{\partial}{\partial t} + \pounds_u\Big)(\boldsymbol{\omega}\cdot d\mathbf{S}) \\
&= \big(\partial_t\boldsymbol{\omega} - \operatorname{curl}(\mathbf{u}\times\boldsymbol{\omega}) + \mathbf{u}\operatorname{div}\boldsymbol{\omega}\big)\cdot d\mathbf{S} \\
\text{by Euler's motion equation (18)} \quad &= -\,d^2\left(p - \tfrac{1}{2}|\mathbf{u}|^2\right) \\
&= 0\,.
\end{aligned}
$$

Thus, the vorticity satisfies the vector equation

$$
\partial_t\boldsymbol{\omega} - \operatorname{curl}(\mathbf{u}\times\boldsymbol{\omega}) = 0\,,
$$

since $\operatorname{div}\boldsymbol{\omega} = \operatorname{div}\operatorname{curl}\mathbf{u} = 0$.

An alternative: By introducing the divergenceless vector field $\omega = \boldsymbol{\omega}\cdot\nabla$, one may also write Euler's vorticity equation by substituting $\omega \,\lrcorner\, d^3x = \boldsymbol{\omega}\cdot d\mathbf{S}$, as

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t} + \pounds_u\Big)(\boldsymbol{\omega}\cdot d\mathbf{S}) &= \Big(\frac{\partial}{\partial t} + \pounds_u\Big)\,(\omega \,\lrcorner\, d^3x) \\
&= (\partial_t \omega) \,\lrcorner\, d^3x + (\pounds_u \omega) \,\lrcorner\, d^3x + \omega \,\lrcorner\, (\pounds_u d^3x) \\
&= \big(\partial_t\omega + [u,\omega] + (\operatorname{div}\mathbf{u})\,\omega\big) \,\lrcorner\, d^3x \\
&= 0\,.
\end{aligned}
$$

Thus, the vorticity also satisfies the vector equation

$$
\partial_t\boldsymbol{\omega} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u} = 0\,,
$$

since $\operatorname{div}\mathbf{u} = 0$.

Setting the two final expressions for the vector vorticity equation equal to each other and substituting the equation for the Jacobi-Lie bracket yields the vector identity,

$$
-\operatorname{curl}(\mathbf{u}\times\boldsymbol{\omega}) + \mathbf{u}\,(\operatorname{div}\boldsymbol{\omega}) = \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\,(\operatorname{div}\mathbf{u})\,.
$$

This calculation supplies the geometric meaning of that famous vector identity:

$$
\pounds_u(\boldsymbol{\omega}\cdot d\mathbf{S}) = \pounds_u(\omega \,\lrcorner\, d^3x).
$$

These relations simplify for the Euler fluid theory, because both divergences vanish; that is, $(\operatorname{div}\boldsymbol{\omega}) = 0 = (\operatorname{div}\mathbf{u})$.

(3d) For Euler fluid motion restricted to the (x,y) plane with normal unit vector $\hat{\mathbf{z}}$, the vorticity equation for $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ with scalar vorticity $\omega(x,y,t)$ simplifies to

$$
\partial_t\omega + \mathbf{u}\cdot\nabla\omega = 0 \quad \text{with} \quad \mathbf{u} = (u,v,0).
$$

The 2D flow equation yields

$$
\begin{aligned}
0 = \big(\partial_t + \pounds_u\big)(dz\wedge d\omega) &= \big(\partial_t + \pounds_u\big)\big(\nabla z\times\nabla\omega\cdot d\mathbf{S}\big) \\
&= \Big(\partial_t\boldsymbol{\mathcal{B}} - \operatorname{curl}(\mathbf{u}\times\boldsymbol{\mathcal{B}}) + \mathbf{u}\operatorname{div}\boldsymbol{\mathcal{B}}\Big)\cdot d\mathbf{S}\,,
\end{aligned}
$$

with 2D vector $\boldsymbol{\mathcal{B}} := \hat{\mathbf{z}}\times\nabla\omega$, so that $\operatorname{div}\boldsymbol{\mathcal{B}} = 0$. Equivalently, the vector $\boldsymbol{\mathcal{B}}$ also satisfies

$$
\partial_t\boldsymbol{\mathcal{B}} + \mathbf{u}\cdot\nabla\boldsymbol{\mathcal{B}} - \boldsymbol{\mathcal{B}}\cdot\nabla\mathbf{u} + \boldsymbol{\mathcal{B}}(\operatorname{div}\mathbf{u}) = 0\,, \tag{21}
$$

with $\operatorname{div}\mathbf{u} = 0$. Substituting $\operatorname{div}\mathbf{u} = 0$ results in the following equation for the 2D divergenceless vector field $\mathcal{B} = \boldsymbol{\mathcal{B}}\cdot\nabla$,

$$
\partial_t\mathcal{B} + [u,\mathcal{B}] = 0 = \big(\partial_t + \pounds_u\big)\mathcal{B}\,,
$$

so $\mathcal{B}$ is an invariant vector field under the 2D Euler flow. Thus, $\mathcal{B}$ satisfies an equation in the *same* form in 2D as $\boldsymbol{\omega}$ satisfies in 3D.

(3e) The divergence-free 2D vector $\boldsymbol{\mathcal{B}}$ stands in relation to the scalar vorticity ω in

$$\boldsymbol{\mathcal{B}} := \hat{\mathbf{z}} \times \nabla\omega = (-\omega_y,\, \omega_x, 0),$$

as the divergence-free 2D Euler flow velocity $\mathbf{u}$ stands to the stream function ψ in the defining relation

$$\mathbf{u} := \hat{\mathbf{z}} \times \nabla\psi = (-\psi_y,\, \psi_x, 0)\,.$$

In fact, since $\omega = \hat{\mathbf{z}}\cdot\,\mathrm{curl}\,\mathbf{u} = \triangle\psi$, we may identify $\boldsymbol{\mathcal{B}} = -\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}$ in 3D.

For the stream function $\psi(x,y)$ one may compute this relation by alternating actions on ψdz with exterior derivative (d) and Hodge dual ($*$) in $\mathbb{R}^3$,

$$\begin{aligned}
d(\psi dz) &= \psi_x dx\wedge dz + \psi_y dy \wedge dz \\
*\,d(\psi dz) &= -\psi_x dy + \psi_y dx = -\mathbf{u}\cdot d\mathbf{x} \\
d*d(\psi dz) &= -\psi_{xx} dx\wedge dy + \psi_{yy} dy\wedge dx \\
&= -\triangle\psi\, dx\wedge dy \\
&= -\omega\, dx\wedge dy = -\boldsymbol{\omega}\cdot d\mathbf{S} \qquad (22)\\
*\,d*d(\psi dz) &= -\omega\, dz \\
d*d*d(\psi dz) &= -d(\omega\, dz) \\
&= \omega_x dz\wedge dx - \omega_y dy\wedge dz = \boldsymbol{\mathcal{B}}\cdot d\mathbf{S} \\
*\,d*d*d(\psi dz) &= \omega_x dy - \omega_y dx = \boldsymbol{\mathcal{B}}\cdot d\mathbf{x} \quad \text{and so forth} \ldots
\end{aligned}$$

We recognize that these equations imply a series of these higher curl quantities, with $\psi, \triangle\psi, \triangle^2\psi, \ldots$.

(3f) To find a conserved integral quantity associated with $\mathcal{B}$ and$\mathbf{u}$, we note that Euler's equations

$$\big(\partial_t + \pounds_u\big)(\mathbf{u}\cdot d\mathbf{x}) = -\,d\left(p - \tfrac{1}{2}|\mathbf{u}|^2\right)$$

along with $(\partial_t + \pounds_u)(\boldsymbol{\mathcal{B}}\cdot d\mathbf{S}) = 0$ and $\mathrm{div}\boldsymbol{\mathcal{B}} = 0$ also imply that

$$\big(\partial_t + \pounds_u\big)\big(\boldsymbol{\mathcal{B}}\cdot d\mathbf{S}\wedge\mathbf{u}\cdot d\mathbf{x}\big) \;=\; -\boldsymbol{\mathcal{B}}\cdot d\mathbf{S}\wedge d\left(p - \tfrac{1}{2}|\mathbf{u}|^2\right),$$

so that

$$\begin{aligned}
\big(\partial_t + \pounds_u\big)\big(\mathbf{u}\cdot\boldsymbol{\mathcal{B}}\, d^3x\big) &= \Big(\partial_t(\mathbf{u}\cdot\boldsymbol{\mathcal{B}}) + \,\mathrm{div}\big((\mathbf{u}\cdot\boldsymbol{\mathcal{B}})\mathbf{u}\big)\Big)\, d^3x \\
&= -\,\mathrm{div}\Big(\big(p - \tfrac{1}{2}|\mathbf{u}|^2\big)\boldsymbol{\mathcal{B}}\Big)\, d^3x\,.
\end{aligned}$$

Hence, we have the conservation law for the integral,

$$\frac{d}{dt}\int dz \int_{\mathcal{D}} \mathbf{u}\cdot\boldsymbol{\mathcal{B}}\, dxdy = -\int dz \oint_{\partial\mathcal{D}} \Big((\mathbf{u}\cdot\boldsymbol{\mathcal{B}})\mathbf{u} + \big(p - \tfrac{1}{2}|\mathbf{u}|^2\big)\boldsymbol{\mathcal{B}}\Big)\cdot\hat{\mathbf{n}}\, ds = 0\,,$$

in which the right-hand side vanishes for Neumann boundary conditions; that is,

$$\mathbf{u}\cdot\hat{\mathbf{n}} = 0 = \boldsymbol{\mathcal{B}}\cdot\hat{\mathbf{n}} = \nabla\omega\cdot\hat{\mathbf{n}}\times\hat{\mathbf{z}} \quad \text{on the boundary.}$$

Therefore, the integral of the cross product of the 2D vectors $\nabla\omega$ and $\mathbf{u}$, namely

$$\int dz \int_{\mathcal{D}} \mathbf{u}\cdot\boldsymbol{\mathcal{B}}\, dxdy = \int dz \int_{\mathcal{D}} \hat{\mathbf{z}}\cdot\nabla\omega\times\mathbf{u}\, dxdy,$$

is conserved by the Euler equations for planar incompressible fluid flow, provided the velocity $\mathbf{u}$ is tangential to the boundary and the gradient $\nabla\omega$ is normal to the boundary. From the formulas in (22) above we compute the relation,

$$\begin{aligned}\mathbf{u}\cdot d\mathbf{x}\wedge\boldsymbol{\mathcal{B}}\cdot d\mathbf{S} &= (-\psi_y dx + \psi_x dy)\wedge(\omega_x dz\wedge dx - \omega_y dy\wedge dz)\\ &= \big(\mathrm{div}(\omega\nabla\psi) - \omega^2\big)dxdydz.\end{aligned}$$

The vertical integral may be ignored, since the integrand is independent of z. This relation shows that $\int_{\mathcal{D}}\mathbf{u}\cdot\boldsymbol{\mathcal{B}}\,dxdy = -\int_{\mathcal{D}}\omega^2\,dxdy$ is actually minus the well-known enstrophy, up to a boundary term. In vector notation, this is

$$\int_{\mathcal{D}}\mathbf{u}\cdot\boldsymbol{\mathcal{B}}\,dxdy = -\int_{\mathcal{D}}\mathrm{curl}^{-1}(\omega\hat{\mathbf{z}})\cdot\mathrm{curl}\,(\omega\hat{\mathbf{z}})\,dxdy = -\int_{\mathcal{D}}\omega^2\,dxdy,$$

again up to a boundary term. Lastly, we confirm the expected preservation of the enstrophy,

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{D}}\omega^2\,dxdy &= 2\int_{\mathcal{D}}\omega\partial_t\omega\,dxdy = -2\int_{\mathcal{D}}\omega\,\mathbf{u}\cdot\nabla\omega\,dxdy\\ &= -\int_{\mathcal{D}}\mathrm{div}(\omega^2\mathbf{u})\,dxdy = -\oint_{\partial\mathcal{D}}\omega^2\,\mathbf{u}\cdot\hat{\mathbf{n}}\,ds = 0\,,\end{aligned}$$

on using $\mathrm{div}\mathbf{u}=0$ and Neumann boundary conditions.

▲

[#4] The C. Neumann problem [1859]

(4a) *Derive the equations of motion*

$$\ddot{\mathbf{x}} = -\mathsf{A}\mathbf{x} + (\mathsf{A}\mathbf{x}\cdot\mathbf{x} - \|\dot{\mathbf{x}}\|^2)\mathbf{x}$$

of a particle of unit mass moving on the sphere S^{n-1} under the influence of a quadratic potential

$$V(\mathbf{x}) = \tfrac{1}{2}\mathsf{A}\mathbf{x}\cdot\mathbf{x} = \tfrac{1}{2}a_1x_1^2 + \tfrac{1}{2}a_2x_2^2 + \cdots + \tfrac{1}{2}a_nx_n^2\,,$$

for $\mathbf{x}\in\mathbb{R}^n$, where $\mathsf{A} = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ is a fixed $n\times n$ diagonal matrix. Here $V(\mathbf{x})$ is a harmonic oscillator with spring constants that are taken to be fully anisotropic, with $a_1 < a_2 < \cdots < a_n$.

Hint: These are the Euler-Lagrange equations obtained when a Lagrange multiplier μ is used to restrict the motion to a sphere by adding a term,

$$\mathcal{L}(\mathbf{x},\dot{\mathbf{x}}) = \tfrac{1}{2}\|\dot{\mathbf{x}}\|^2 - \tfrac{1}{2}\mathsf{A}\mathbf{x}\cdot\mathbf{x} - \mu(1 - \|\mathbf{x}\|^2)\,, \tag{23}$$

on the tangent bundle

$$TS^{n-1} = \{(\mathbf{x},\dot{\mathbf{x}})\in\mathbb{R}^n\times\mathbb{R}^n|\ \|\mathbf{x}\|^2 = 1,\ \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}.$$

(4b) *Form the matrices*

$$\mathsf{Q} = (x^i x^j) \quad \textit{and} \quad \mathsf{L} = (x^i \dot{x}^j - x^j \dot{x}^i)\,,$$

and show that the Euler-Lagrange equations for the Lagrangian in (23) are equivalent to

$$\dot{\mathsf{Q}} = [\mathsf{L}, \mathsf{Q}] \quad \textit{and} \quad \dot{\mathsf{L}} = [\mathsf{Q}, \mathsf{A}]\,.$$

Show further that for a constant parameter λ these Euler-Lagrange equations imply

$$\frac{d}{dt}(-\mathsf{Q} + \mathsf{L}\lambda + \mathsf{A}\lambda^2) = [-\mathsf{Q} + \mathsf{L}\lambda + \mathsf{A}\lambda^2, -\mathsf{L} - \mathsf{A}\lambda]\,.$$

Explain why this formula is important from the viewpoint of conservation laws.

(4c) *Verify that the energy*

$$E(\mathsf{Q}, \mathsf{L}) = -\frac{1}{4}\mathrm{trace}(\mathsf{L}^2) + \frac{1}{2}\mathrm{trace}(\mathsf{AQ})$$

is conserved for this system.

(4d) *Prove that the following $(n-1)$ quantities for $j = 1, 2, \ldots, n-1$ are also conserved*

$$\Phi_j = \dot{x}_j^2 + \frac{1}{2}\sum_{i \neq j} \frac{(x^i \dot{x}^j - x^j \dot{x}^i)^2}{a_j - a_i}\,,$$

where $(\mathbf{x}, \dot{\mathbf{x}}) = (x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) \in TS^{n-1}$ and the a_j are the eigenvalues of the diagonal matrix A.

[#4] The C. Neumann problem [1859]

Answer All of these are straightforward verifications.

(4a)

(4b)

(4c)

(4d)

▲

[#5] Three-wave equations

The three-wave equations of motion take the symmetric form

$$i\dot{A} = B^*C\,, \quad i\dot{B} = CA^*\,, \quad i\dot{C} = AB\,, \quad \textit{for} \quad (A, B, C) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} \simeq \mathbb{C}^3\,. \tag{24}$$

(5a) *Write these equations as a Hamiltonian system. How many degrees of freedom does it have?*

(5b) *Find two additional constants of motion for it, besides the Hamiltonian.*

(5c) *Use the Poisson bracket to identify the symmetries of the Hamiltonian associated with the two additional constants of motion, by computing their Hamiltonian vector fields and integrating their characteristic equations.*

(5d) *Set:*

$$A = |A| \exp(i\xi)\,, \quad B = |B| \exp(i\eta)\,, \quad C = Z \exp(i(\xi + \eta))\,.$$

Determine whether this transformation is canonical.

(5e) *Express the three-wave problem entirely in terms of the variable $Z = |Z|e^{i\zeta}$, reduce the motion to a single equation for $|Z|$ then reconstruct the full solution as,*

$$A = |A| \exp(i\xi)\,, \quad B = |B| \exp(i\eta)\,, \quad C = |Z| \exp\big(i(\xi + \eta + \zeta)\big)\,.$$

That is, reduce the motion to a single equation for $|Z|$ then write the various differential equations for $|A|$, ξ, $|B|$, η and η.

[#5] Three-wave equations

Answer

(5a) The three-wave interaction equations (24) may be written in canonical form with Hamiltonian $H = \Re(ABC^*)$ and Poisson brackets

$$\{A, A^*\} = \{B, B^*\} = \{C, C^*\} = -2i\,.$$

There are 3 complex-canonical degrees of freedom.

(5b) The three-wave equations conserve the following three quantities:

$$\begin{aligned} H &= \frac{1}{2}(ABC^* + A^*B^*C) = \Re(ABC^*)\,, & (25)\\ J &= |A|^2 - |B|^2\,, & (26)\\ N &= |A|^2 + |B|^2 + 2|C|^2\,. & (27) \end{aligned}$$

(5c) The Hamiltonian vector field $X_H = \{\,\cdot\,, H\}$ generates the motion, while $X_J = \{\,\cdot\,, J\}$ and $X_N = \{\,\cdot\,, N\}$ generate S^1 symmetries $S^1 \times \mathbb{C}^3 \mapsto \mathbb{C}^3$ of the Hamiltonian H. The S^1 symmetries associated to J and N are the following:

$$J: \begin{pmatrix} A \\ B \\ C \end{pmatrix} \to \begin{pmatrix} e^{-2i\phi}A \\ e^{2i\phi}B \\ C \end{pmatrix} \qquad N: \begin{pmatrix} A \\ B \\ C \end{pmatrix} \to \begin{pmatrix} e^{-2i\psi}A \\ e^{-2i\psi}B \\ e^{-4i\psi}C \end{pmatrix}$$

The constant of motion J represents the angular momentum about the vertical in the new variables, while N is the new conserved quantity arising from phase-averaging in the Lagrangian L to obtain $\langle L \rangle$.
The following positive-definite combinations of N and J are physically significant:

$$N_+ \equiv \frac{1}{2}(N + J) = |A|^2 + |C|^2\,, \qquad N_- \equiv \frac{1}{2}(N - J) = |B|^2 + |C|^2\,.$$

These combinations are known as the ***Manley-Rowe invariants*** in the extensive literature about three-wave interactions. The quantities H, N_+ and N_- provide three independent constants of the motion.

(5d) The transformation

$$\begin{aligned} A &= |A| \exp(i\xi)\,, \\ B &= |B| \exp(i\eta)\,, \\ C &= Z \exp(i(\xi+\eta))\,. \end{aligned} \tag{28}$$

is canonical, since it preserves the symplectic form

$$dA \wedge dA^* + dB \wedge dB^* + dC \wedge dC^* = dZ \wedge dZ^*\,.$$

In these variables, the Hamiltonian is a function of only Z and Z^*

$$H = \frac{1}{2}(Z + Z^*) \cdot \sqrt{N_+ - |Z|^2} \cdot \sqrt{N_- - |Z|^2}\,.$$

The Poisson bracket is $\{Z, Z^*\} = -2i$ and the canonical equations reduce to

$$i\dot{Z} = i\{Z, H\} = 2\frac{\partial H}{\partial Z^*}\,.$$

This provides the dynamics of both the amplitude and phase of $Z = |Z|e^{i\zeta}$.

(5e) The amplitude $|Z| = |C|$ is obtained in closed form in terms of Jacobi elliptic functions as the solution of

$$\left(\frac{d\mathcal{Q}}{d\tau}\right)^2 = \left[\mathcal{Q}^3 - 2\mathcal{Q}^2 + (1 - \mathcal{J}^2)\mathcal{Q} + 2\mathcal{E}\right], \tag{29}$$

where $\mathcal{Q}$, $\mathcal{J}$, $\mathcal{E}$ and τ are normalised by the constant of motion N as

$$\mathcal{Q} = \frac{2|Z|^2}{N}\,, \quad \mathcal{J} = \frac{J}{N}\,, \quad \mathcal{E} = -\frac{4H^2}{N^3}\,, \quad \tau = \sqrt{2N}t\,. \tag{30}$$

Once $|Z|$ is known, $|A|$ and $|B|$ follow immediately from the Manley-Rowe relations,

$$|A| = \sqrt{N_+ - |Z|^2}\,, \qquad |B| = \sqrt{N_- - |Z|^2}\,.$$

The phases ξ and η may now be determined. Using the three-wave equations (24) together with (28), one finds

$$\dot{\xi} = -\frac{H}{|A|^2}\,, \qquad \dot{\eta} = -\frac{H}{|B|^2}\,, \tag{31}$$

so that ξ and η can be integrated by quadratures once $|A|(t)$ and $|B|(t)$ are known. Finally, the phase ζ of Z is determined unambiguously by

$$\frac{d|Z|^2}{dt} = -2H \tan\zeta \qquad \text{and} \qquad H = |A||B||Z| \cos\zeta\,. \tag{32}$$

Hence, we can now reconstruct the full solution as,

$$A = |A| \exp(i\xi)\,, \quad B = |B| \exp(i\eta)\,, \quad C = |Z| \exp\big(i(\xi + \eta + \zeta)\big)\,.$$

▲