Imperial College London

UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2012

M4A34

Geometric Mechanics II

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This paper is also taken for the relevant examination for the Associateship.

M4A34

Geometric Mechanics II

Date: | | Time: | |

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

4. The EPDiff(H^1) equation is obtained from the Euler-Poincaré reduction theorem for a right-invariant Lagrangian, when one defines this Lagrangian to be half the H^1 norm on the real line of the vector field of velocity $u = \dot{g}g^{-1}$, namely,

$$l(u) = \frac{1}{2}\|u\|^2_{H^1} = \frac{1}{2}\int_{-\infty}^{\infty} u^2 + u_x^2\,dx\,.$$

(Assume u and u_x vanishes as $|x| \to \infty$.)

(a) Derive the EPDiff(H^1) equation on the real line in terms of its velocity u and its momentum $m = \delta l/\delta u = u - u_{xx}$ in one spatial dimension.

(b) Use the Clebsch approach (hard constraint) to derive the peakon singular solution $m(x,t)$ of EPDiff(H^1) as a cotangent-lift momentum map in terms of canonically conjugate variables $q(t)$ and $p(t)$. Derive Hamilton's canonical equations for the conjugate variables $q(t)$ and $p(t)$.

1. Quaternionic rigid body dynamics

Problem statement: Formulate rigid body dynamics as an EP problem in quaternions.

(a) Show that the variation of the pure quaternion $\boldsymbol{\Omega} = 2\hat{\mathfrak{q}}^*\dot{\hat{\mathfrak{q}}}$ that expresses body angular velocity in terms of the pure unit quaternion $\hat{\mathfrak{q}} = [0, \hat{\mathbf{q}}]$ with $|\hat{\mathbf{q}}|^2 = 1$ satisfies the identity

$$\boldsymbol{\Omega}' - \dot{\boldsymbol{\Xi}} = \mathrm{Im}(\boldsymbol{\Omega}\,\boldsymbol{\Xi})\,, \tag{1}$$

where $\boldsymbol{\Xi} := 2\hat{\mathfrak{q}}^*\hat{\mathfrak{q}}'$ and $(\,\cdot\,)'$ denotes variation.

(b) Define the kinetic energy Lagrangian for the rigid body in terms of the quaternionic pairing, $\langle\, \mathfrak{p}\,,\, \mathfrak{q}\,\rangle = \mathrm{Re}(\mathfrak{p}\mathfrak{q}^*)$.

(c) State Hamilton's principle for the rigid body in terms of this Lagrangian.

(d) Use Hamilton's principle to derive the equations of motion for the rigid body in quaternionic form.

(e) Write the quaternionic formula (1) in its equivalent vector form,

2. **Adjoint and coadjoint actions of semidirect product $(S \circledS v)$ acting on $\mathbb{R}$**

The action of the scaling and translation group $(S \circledS v)$ on $\mathbb{R}$ may be represented by multiplying an *extended* vector $(r, 1)^T$ with $r \in \mathbb{R}$ by a 2×2 matrix, as

$$\begin{bmatrix} S & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ 1 \end{bmatrix} = \begin{bmatrix} S\,r + v \\ 1 \end{bmatrix}$$

for a scaling parameter $S \in \mathbb{R}$ and a translation $v \in \mathbb{R}$.

The group composition rule for $(S \circledS v)$ is

$$(\tilde{S}, \tilde{v})(S, v) = (\tilde{S}S, \tilde{S}v + \tilde{v})\,, \tag{2}$$

which can be represented by multiplication of 2×2 matrices, as

$$\begin{pmatrix} \tilde{S} & \tilde{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{S}S & \tilde{S}v + \tilde{v} \\ 0 & 1 \end{pmatrix}. \tag{3}$$

Problem statement

(a) Derive the AD, Ad and ad actions for $(S \circledS v)$. Use the notation $(S'(0), v'(0)) = (\sigma, \nu)$ for Lie algebra elements.

(b) Introduce a natural pairing in which to define the dual Lie algebra and derive its Ad^* and ad^* actions. Denote elements of the dual Lie algebra as (α, β).

(c) Compute its coadjoint motion equations as Euler-Poincaré equations.

(d) Legendre transform and find the corresponding canonical Poisson brackets.

(e) Choose the Hamiltonian $H = \frac{1}{2}\alpha^2 + \frac{1}{2}(\log \beta)^2$ and solve its coadjoint motion equations.

3. Momentum maps for cotangent lifts

Recall that the formula determining the momentum map for the cotangent-lifted action of a Lie group G on a smooth manifold Q may be expressed in terms of the pairings $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle : T^*Q \times TQ \to \mathbb{R}$ as

$$\langle\, J(q,p)\,,\,\xi\,\rangle = \langle\langle\, p\,,\,\pounds_\xi q\,\rangle\rangle\,,$$

where $(q,p) \in T^*_qQ$ and $\pounds_\xi q$ is the infinitesimal generator of the action of the Lie algebra element ξ on the coordinate q.

Problem statement:

(a) Define appropriate pairings and determine the momentum maps explicitly for the following,

(i) $\pounds_\xi q = \xi \times q$ for $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

(ii) $\pounds_\xi q = \mathrm{ad}_\xi q$ for ad-action $\mathrm{ad} :\, \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ in a Lie algebra $\mathfrak{g}$

(iii) $UQU^\dagger$ for a unitary matrix $U \in U(n)$ satisfying $U^\dagger = U^{-1}$ acting on Hermitian $Q \in H(n)$ satisfying $Q = Q^\dagger$.

(b) For case (a.ii), compute the *canonical* equations in phase space for the Hamiltonian corresponding to the norm of the momentum map,

$$H(q,p) = \frac{1}{2}\langle\, J(q,p)\,,\, J^\sharp(q,p)\,\rangle := \frac{1}{2}\Big(J(q,p)\,,\, K^{-1}J(q,p)\Big) = \frac{1}{2}\|J\|_K^2\,,$$

with $J^\sharp(q,p) = K^{-1}J(q,p)$ for a constant positive symmetric matrix $K^{-1} : \mathfrak{g}^* \to \mathfrak{g}$ and a nondegenerate pairing $(\,\cdot\,,\,\cdot\,) : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$.

(c) Use the canonical equations for case (a.ii) to derive the Euler-Poincaré equation for the momentum map J.

(d) For case (a.i), compute the *gradient flow* equations for the norm of the momentum map in phase space and show that the norm is non-increasing.

4. **Lie derivative relations.**

Recall that the pull-back ϕ_t^* of a smooth flow ϕ_t generated by a smooth vector field X defined on a smooth manifold M commutes with the exterior derivative d, wedge product $\wedge$ and contraction $\lrcorner$.

That is, for k-forms α, $\beta \in \Lambda^k(M)$, and $m \in M$, the pull-back ϕ_t^* satisfies

$$\begin{aligned} d(\phi_t^*\alpha) &= \phi_t^* d\alpha\,, \\ \phi_t^*(\alpha \wedge \beta) &= \phi_t^*\alpha \wedge \phi_t^*\beta\,, \\ \phi_t^*(X \,\lrcorner\, \alpha) &= \phi_t^* X \,\lrcorner\, \phi_t^*\alpha\,. \end{aligned}$$

Recall that the Lie derivative $\pounds_X\alpha$ of a k-form $\alpha \in \Lambda^k(M)$ by the vector field X tangent to the flow ϕ_t on M is defined either dynamically or geometrically as

$$\pounds_X\alpha \;=\; \frac{d}{dt}\bigg|_{t=0} (\phi_t^*\alpha) = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha),$$

in which the last is Cartan's geometric formula for the Lie derivative.

Problem statement:

Verify the following formulas

(a) $X \,\lrcorner\, (Y \,\lrcorner\, \alpha) = -\,Y \,\lrcorner\, (X \,\lrcorner\, \alpha)$.

(b) $[X\,,\,Y] \,\lrcorner\, \alpha = \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha)$.

(c) Use (b) to verify $\pounds_{[X\,,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$.

(d) Use (c) to verify the Jacobi identity.

(e) For a top form α and divergenceless vector fields X and Y, show that

$$[X\,,\,Y] \,\lrcorner\, \alpha = d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big)\,. \tag{4}$$

(f) Write the equivalent of equation (4) as a formula in vector calculus.