Imperial College London

UNIVERSITY OF LONDON

Course:	M4A34
Setter:	Holm
Checker:	Gibbons
Editor:	Turaev
External:	
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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2010

M4A34

Geometric Mechanics II

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This paper is also taken for the relevant examination for the Associateship.

M4A34

Geometric Mechanics II

Date: Time:

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Co-Adjoint motion

(a) If $U, O \in G$ and $\xi, \eta \in \mathfrak{g}$, define the maps $\mathrm{AD}_U O$, $\mathrm{Ad}_U \xi$, $\mathrm{ad}_\eta \xi$, $\mathrm{Ad}^*_{U^{-1}} \mu$, $\mathrm{ad}^*_\eta \mu$ for a matrix Lie group G and its matrix Lie algebra $\mathfrak{g}$. Recall that $\mathfrak{g}$ is the tangent space at the identity of G. In your definitions, be sure to tell the domain and range of each map. Use the trace pairing for matrices to compute explicit expressions.

(b) Prove the following.

Lemma 1 *For any fixed $\eta \in \mathfrak{g}$, in the matrix Lie algebra $\mathfrak{g}$ of a matrix Lie group G,*

$$\frac{d}{dt} \mathrm{Ad}_{g(t)^{-1}} \eta = -\, \mathrm{ad}_{\xi(t)} \left(\mathrm{Ad}_{g(t)^{-1}} \eta \right) , \tag{1}$$

where $\xi(t) = g(t)^{-1} \dot{g}(t) \in \mathfrak{g}$ and $g(t) \in G$.

(c) Use the Lemma in part [b] to prove the following.

Proposition 2 (Co-Adjoint motion equation)
Let $g(t)$ be a path in a matrix Lie group G and $\mu(t)$ be a path in its dual Lie algebra $\mathfrak{g}^*$. Then

$$\frac{d}{dt} \mathrm{Ad}^*_{g(t)^{-1}} \mu(t) = \mathrm{Ad}^*_{g(t)^{-1}} \left[\frac{d\mu}{dt} - \mathrm{ad}^*_{\xi(t)} \mu(t) \right] , \tag{2}$$

where $\xi(t) = g(t)^{-1} \dot{g}(t)$.

2. $SO(n)$ **rigid body motion**

(a) Compute the Euler-Poincaré equation for Hamilton's principle $\delta S = 0$ with $S = \int l(\Omega)\,dt$ for the Lagrangian

$$l = \frac{1}{2}\mathrm{tr}(\Omega^T \mathbb{A}\Omega)\,, \tag{3}$$

where $\Omega = O^{-1}\dot{O} \in so(n)$ is anti-symmetric, and the $n \times n$ matrix $\mathbb{A}$ is symmetric.

(b) Show that the solution of Euler-Poincaré equation in part (a) evolves by coadjoint motion,

$$M(t) =: \mathrm{Ad}^*_{O(t)}M(0) \quad \text{with} \quad M = \mathbb{A}\Omega + \Omega\mathbb{A}\,.$$

(c) If the $n \times n$ matrix $M(t)$ evolves by coadjoint motion, draw a conclusion about its eigenvalues. In particular, for the eigenvalue problem

$$M(t)\psi(t) = \lambda_t \psi(t)\,,$$

write an equation for λ_t and prove it, assuming that the eigenfunctions of $M(t)$ evolve according to

$$\psi(t) = O(t)^{-1}\psi(0)\,.$$

(d) Show that the Euler-Poincaré equation in part (a) may be rephrased as a system of two coupled **linear** equations. Namely, (i) the eigenvalue problem for $M(t)$ and (ii) an evolution equation for its eigenfunctions $\psi(t)$.

3. **Momentum maps**

Define appropriate pairings and determine the momentum maps explicitly for the following four actions.

(a) $\pounds_\xi q = \xi \times q$ for $\mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3$

(b) $\pounds_\xi q = \mathrm{ad}_\xi q$ for ad-action $\mathrm{ad}: \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ in a Lie algebra $\mathfrak{g}$

(c) AqA^{-1} for $A \in GL(3,R)$ acting on $q \in GL(3,R)$ by matrix conjugation

(d) $UQU^\dagger$ for a unitary matrix $U \in U(n)$ satisfying $U^\dagger = U^{-1}$ acting on Hermitian $Q \in H(n)$ satisfying $Q = Q^\dagger$.

4. **Generalised rigid body (grb)**

Let the Hamiltonian H_{grb} for a generalised rigid body (grb) be defined as the pairing of the cotangent-lift momentum map J with its dual $J^\sharp = K^{-1}J \in \mathfrak{g}$,

$$H_{grb} = \frac{1}{2}\Big\langle p \diamond q \,,\, (p \diamond q)^\sharp \Big\rangle = \frac{1}{2}\Big(p \diamond q \,,\, K^{-1}(p \diamond q) \Big)\,,$$

for an appropriate inner product $(\,\cdot\,,\,\cdot\,) : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$.

(a) Compute the canonical equations for the Hamiltonian H_{grb}.

(b) Use these equations to compute the evolution equation for $J = -\,p \diamond q$.

(c) Identify the resulting equation and give a plausible argument why this was to be expected, by writing out its associated Hamilton's principle and Euler-Poincaré equations for left and right actions.

(d) Write the dynamical equations for q, p and J on $\mathbb{R}^3$ and explain why the name 'generalised rigid body' might be appropriate.