Imperial College London

UNIVERSITY OF LONDON

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Setter:	Holm
Checker:	Gibbons
Editor:	Chen
External:	
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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2009

M4A34

GEOMETRICAL MECHANICS, Part 2

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May-June 2009

This paper is also taken for the relevant examination for the Associateship.

M4A34

GEOMETRICAL MECHANICS, Part 2

Date: Time:

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Consider the following action S for Hamilton's principle $\delta S = 0$ given by

$$S = \int L(\Omega, \omega, g)\, dt = \int l(\Omega) + \frac{1}{2\sigma^2}\big|\omega - \mathsf{Ad}_g\Omega\big|^2\, dt\,,$$

where $g \in G$ and $\omega = \dot{g}g^{-1}(t) \in \mathfrak{g}$, for a matrix Lie group G and its right-invariant matrix Lie algebra $\mathfrak{g}$. Here $\sigma^2 \in \mathbb{R}$ is a positive constant and $|\cdot|$ is a Riemannian metric which defines a symmetric non-degenerate pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ between the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. (The variables ω and $\mathsf{Ad}_g\Omega$ are both elements of the Lie algebra $\mathfrak{g}$.)

(a) Denote variations as, e.g., $\delta g = g'$ and show that

$$(\mathsf{Ad}_g\Omega)' = \mathsf{Ad}_g\Omega' - \mathsf{ad}_{\mathsf{Ad}_g\Omega}\eta \quad \text{with} \quad \eta = g'g^{-1} \in \mathfrak{g}\,.$$

(b) Express $\delta\omega = \omega'$ in terms of η, $\dot{\eta}$ and ad_ω using cross-derivatives of $\dot{g} = \omega g$ and $g' = \eta g$.

(c) Use the relations from Parts a and b to derive the Euler-Poincaré equation for $\partial l/\partial\Omega$ from Hamilton's principle, $\delta S = 0$.
(You may ignore endpoint terms when integrating by parts.)

(d) Interpret this Euler-Poincaré equation as a conservation law.

2. (a) Consider the matrix Lie group $\mathcal{Q}$ of $n \times n$ Hermitian matrices, so that $Q^\dagger = Q$ for $Q \in \mathcal{Q}$. The Poisson (symplectic) manifold is $T^*\mathcal{Q}$, whose elements are pairs (Q, P) of Hermitian matrices. The corresponding Poisson bracket is

$$\{F, H\} = \mathrm{tr}\left(\frac{\partial F}{\partial Q}\frac{\partial H}{\partial P} - \frac{\partial H}{\partial Q}\frac{\partial F}{\partial P}\right).$$

Let G be the group $U(n)$ of $n \times n$ unitary matrices: G acts on $T^*\mathcal{Q}$ through

$$(Q, P) \mapsto (UQU^\dagger, UPU^\dagger)\,, \quad UU^\dagger = Id$$

(i) What is the linearization of this group action?

(ii) What is its momentum map?

(iii) Is this momentum map equivariant? Explain why, or why not.

(b) Is the momentum map in part (a) conserved by the Hamiltonian $H = \frac{1}{2}\mathrm{tr}\, P^2$? Prove it.

3. Consider the Lagrangian

$$L = \frac{1}{2}\operatorname{tr}\Big(\dot{S}S^{-1}\dot{S}S^{-1}\Big) + \frac{1}{2}\,\dot{\mathbf{q}}\cdot S^{-1}\dot{\mathbf{q}}\,,$$

where S is an $n\times n$ symmetric matrix and $\mathbf{q}\in\mathbb{R}^n$ is an $n-$component column vector.

(a) Legendre transform to construct the corresponding Hamiltonian and canonical equations.

(b) Show that the Lagrangian and Hamiltonian are invariant under the group action

$$\mathbf{q}\to G\mathbf{q} \quad \text{and} \quad S\to GSG^T$$

for any constant invertible $n\times n$ matrix, G.

(c) Compute the infinitesimal generator for this group action and construct its corresponding momentum map. Is this momentum map equivariant? Prove it.

(d) Verify directly that this momentum map is a conserved $n\times n$ matrix quantity by using the equations of motion.

4. The EPDiff(H^1) equation is obtained from the Euler-Poincaré reduction theorem for a right-invariant Lagrangian, when one defines this Lagrangian to be half the H^1 norm on the real line of the vector field of velocity $u=\dot{g}g^{-1}$, namely,

$$l(u) = \frac{1}{2}\|u\|^2_{H^1} = \frac{1}{2}\int_{-\infty}^{\infty} u^2 + u_x^2\,dx\,.$$

(Assume u and u_x vanishes as $|x|\to\infty$.)

(a) Derive the EPDiff(H^1) equation on the real line in terms of its velocity u and its momentum $m=\delta l/\delta u = u - u_{xx}$ in one spatial dimension.

(b) Use the Clebsch approach (hard constraint) to derive the peakon singular solution $m(x,t)$ of EPDiff(H^1) as a cotangent-lift momentum map in terms of canonically conjugate variables $q(t)$ and $p(t)$. Derive Hamilton's canonical equations for the conjugate variables $q(t)$ and $p(t)$.