M4A34 Assessed Homework 1 Spring Term 2009 Darryl Holm

Due at beginning of class one week from today

1. Heavy top motion: Approach A

The motion of a heavy top is determined from Euler's equations in vector form,

$$\mathbb{I}\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbb{I}\boldsymbol{\Omega} = -\,mg\boldsymbol{\chi} \times \boldsymbol{\Gamma}\,, \qquad (1)$$

$$\dot{\boldsymbol{\Gamma}} + \boldsymbol{\Omega} \times \boldsymbol{\Gamma} = 0\,, \qquad (2)$$

where $\boldsymbol{\Omega}$, $\boldsymbol{\Gamma}$, $\boldsymbol{\chi} \in \mathbb{R}^3$ are vectors in the rotating body frame, $\mathbb{I}$ is a real positive symmetric matrix and $mg\boldsymbol{\chi} = const.$

The angular velocity vector $\boldsymbol{\Omega}$ is related to the rotation rate by $\widehat{\boldsymbol{\Omega}} = \mathbf{R}^{-1}\dot{\mathbf{R}}$ and $\boldsymbol{\Gamma} = \mathbf{R}^{-1}\hat{\mathbf{z}}$ is the vertical axis as seen from the rotating body.

[a] Derive the Euler-Poincaré and Euler-Lagrange motion equations for a variational principle $\delta S = 0$, where

$$\begin{aligned} S &= \int_a^b L(\boldsymbol{\Omega}, \boldsymbol{\Gamma}, \dot{\boldsymbol{\Gamma}})\,\mathrm{d}t \\ &= \int_a^b \left(\frac{1}{2}\mathbb{I}\boldsymbol{\Omega}\cdot\boldsymbol{\Omega} - mg\boldsymbol{\chi}\cdot\boldsymbol{\Gamma} + \frac{1}{2\sigma^2}\big|\dot{\boldsymbol{\Gamma}} + \boldsymbol{\Omega}\times\boldsymbol{\Gamma}\big|^2 \right)\mathrm{d}t \qquad (3) \end{aligned}$$

with penalty σ^2, a positive constant.

[b] Legendre transform and write the resulting equations in canonical Hamiltonian form.

[c] Verify that these canonical equations recover the heavy top motion equation (1). What identity is used in the key step in this verification?

[d] Is the $\boldsymbol{\Gamma}$-equation also recovered from this variational principle?

Solution 1:

[a] Taking the variational derivative of the action integral S in (3) yields

$$\delta S = \int_a^b \Big(\mathbb{I}\boldsymbol{\Omega}\cdot\boldsymbol{\delta\Omega} - mg\boldsymbol{\chi}\cdot\boldsymbol{\delta\Gamma} + \boldsymbol{\Lambda}\cdot\Big(\boldsymbol{\delta}\dot{\boldsymbol{\Gamma}} + \boldsymbol{\delta\Omega}\times\boldsymbol{\Gamma} + \boldsymbol{\Omega}\times\boldsymbol{\delta\Gamma}\Big)\Big)\mathrm{d}t$$

where the momentum $\boldsymbol{\Lambda}$ canonically conjugate to $\boldsymbol{\Gamma}$ is found to be

$$\boldsymbol{\Lambda} := \frac{\partial L}{\partial \dot{\boldsymbol{\Gamma}}} = \frac{1}{\sigma^2}\big(\dot{\boldsymbol{\Gamma}} + \boldsymbol{\Omega} \times \boldsymbol{\Gamma}\big)\,.$$

Rearranging and integrating by parts in δS yields

$$\begin{aligned}\delta S &= \int_a^b \bigg(\Big(\mathbb{I}\boldsymbol{\Omega} + \boldsymbol{\Gamma} \times \boldsymbol{\Lambda}\Big) \cdot \boldsymbol{\delta\Omega} \\ &\qquad - \Big(\dot{\boldsymbol{\Lambda}} + \boldsymbol{\Omega} \times \boldsymbol{\Lambda} + mg\boldsymbol{\chi}\Big) \cdot \boldsymbol{\delta\Gamma}\bigg)\mathrm{d}t + \Big[\boldsymbol{\Lambda} \cdot \boldsymbol{\delta\Gamma}\Big]_a^b.\end{aligned}$$

Thus, stationarity $\delta S = 0$ implies

$$\begin{aligned}\boldsymbol{\delta\Omega} :&\qquad \mathbb{I}\boldsymbol{\Omega} + \boldsymbol{\Gamma} \times \boldsymbol{\Lambda} = 0\,,\\ \boldsymbol{\delta}\dot{\boldsymbol{\Gamma}} :&\qquad \sigma^2\boldsymbol{\Lambda} = \dot{\boldsymbol{\Gamma}} + \boldsymbol{\Omega} \times \boldsymbol{\Gamma}\,,\\ \boldsymbol{\delta\Gamma} :&\qquad \dot{\boldsymbol{\Lambda}} + \boldsymbol{\Omega} \times \boldsymbol{\Lambda} + mg\boldsymbol{\chi} = 0\,.\end{aligned}$$

Substituting the last two stationarity conditions into the time derivative of the first one easily leads to the motion equation (1) for the heavy top. However, the $\boldsymbol{\Gamma}$-equation (2) is only recovered in the limit $\sigma^2 \to 0$.

[b] Legendre transforming leads to the Hamiltonian,

$$\begin{aligned}H(\boldsymbol{\Gamma}, \boldsymbol{\Lambda}) &= \dot{\boldsymbol{\Gamma}} \cdot \boldsymbol{\Lambda} - L(\boldsymbol{\Omega}, \boldsymbol{\Gamma}, \dot{\boldsymbol{\Gamma}})\\ &= \frac{1}{2}(\boldsymbol{\Gamma} \times \boldsymbol{\Lambda}) \cdot \mathbb{I}^{-1}(\boldsymbol{\Gamma} \times \boldsymbol{\Lambda}) + \frac{\sigma^2}{2}\big|\boldsymbol{\Lambda}\big|^2 + mg\boldsymbol{\chi} \cdot \boldsymbol{\Gamma}\,.\end{aligned}$$

This Hamiltonian is the kinetic plus potential energy for the heavy top. Hamilton's canonical equations of motion now are

$$\begin{aligned}\dot{\boldsymbol{\Gamma}} = \frac{\partial H}{\partial \boldsymbol{\Lambda}} &= -\,\boldsymbol{\Gamma} \times \mathbb{I}^{-1}(\boldsymbol{\Gamma} \times \boldsymbol{\Lambda}) + \sigma^2\boldsymbol{\Lambda} = -\,\boldsymbol{\Omega} \times \boldsymbol{\Gamma} + \sigma^2\boldsymbol{\Lambda}\,,\\ \dot{\boldsymbol{\Lambda}} = -\frac{\partial H}{\partial \boldsymbol{\Gamma}} &= -\,\boldsymbol{\Lambda} \times \mathbb{I}^{-1}(\boldsymbol{\Gamma} \times \boldsymbol{\Lambda}) - mg\boldsymbol{\chi} = -\,\boldsymbol{\Omega} \times \boldsymbol{\Lambda} - mg\boldsymbol{\chi}\,,\end{aligned}$$

where we have used $\boldsymbol{\Omega} = -\,\mathbb{I}^{-1}(\boldsymbol{\Gamma} \times \boldsymbol{\Lambda})$.

[c] Combining these canonical equations with the time derivative of $\mathbb{I}\boldsymbol{\Omega} = -\boldsymbol{\Gamma} \times \boldsymbol{\Lambda}$ yields the heavy top equation of motion (1), upon using the *Jacobi identity* for the triple vector product in $\mathbb{R}^3$.

The Jacobi identity is thus the key identity for deriving the equations of motion.

[d] The $\boldsymbol{\Gamma}$-equation is only recovered in the limit $\sigma \to 0$.

The limit $\sigma \to 0$ requires exact verification of the $\boldsymbol{\Gamma}$-equation, rather than merely a penalty for not satisfying it. The latter may be achieved by introducing it as a constraint, enforced by a Lagrange multiplier,

$$S = \int_a^b \left(\frac{1}{2} \mathbb{I}\boldsymbol{\Omega} \cdot \boldsymbol{\Omega} - mg\boldsymbol{\chi} \cdot \boldsymbol{\Gamma} + \boldsymbol{\Lambda} \cdot \left(\dot{\boldsymbol{\Gamma}} + \boldsymbol{\Omega} \times \boldsymbol{\Gamma} \right) \right) \mathrm{d}t$$

This is the action functional for the heavy top in the Clebsch approach that was discussed in class.

2. Heavy top motion: Approach B

The ***Kaluza-Klein Lagrangian*** for the heavy top is the map

$$L_{KK} : TQ_{KK} \simeq TSO(3) \times T\mathbb{R}^3 \mapsto \mathbb{R} \,,$$

defined by extending the tangent space $TSO(3)$ of the rigid body to include new auxiliary variables $\mathbf{q}, \dot{\mathbf{q}} \in T\mathbb{R}^3$ in the sum of squares

$$L_{KK}(\mathbf{R}, \dot{\mathbf{R}}, \mathbf{q}, \dot{\mathbf{q}}; \hat{\mathbf{z}}) = l_{KK}(\boldsymbol{\Omega}, \boldsymbol{\Gamma}, \dot{\mathbf{q}}) = \frac{1}{2} \langle \mathbb{I}\boldsymbol{\Omega} \,,\, \boldsymbol{\Omega} \rangle + \frac{1}{2} |\boldsymbol{\Gamma} + \dot{\mathbf{q}}|^2 \,. \tag{4}$$

The skew-symmetric matrix variable $\widehat{\boldsymbol{\Omega}} = \mathbf{R}^{-1}\dot{\mathbf{R}} \in \mathfrak{so}(3)$ and the unit vector $\boldsymbol{\Gamma} = \mathbf{R}^{-1}\hat{\mathbf{z}} \in S^2$ are defined as usual. The roles of the variables $(\mathbf{q}, \dot{\mathbf{q}}) \in T\mathbb{R}^3$ are to be determined. To begin this interpretation, note that $\mathbf{q} \in \mathbb{R}^3$ is an ignorable vector coordinate in the reduced Lagrangian l_{KK}; so its canonically conjugate momentum $\mathbf{p} \in \mathbb{R}^{3^*} \simeq \mathbb{R}^3$ is a constant of the motion.

[a] Derive the coupled Euler-Poincaré and Euler-Lagrange motion equations from the variational principle $\delta S = 0$, using the reduced Lagrangian l_{KK}, i.e., using the action

$$S = \int_a^b l_{KK}(\boldsymbol{\Omega}, \boldsymbol{\Gamma}, \dot{\mathbf{q}}) \, dt \,.$$

Explain how to obtain the $\boldsymbol{\Gamma}$-equation in this case.

[b] Perform the Legendre transformation of l_{KK} in (4) to obtain its momentum variables. Explain why the momentum $\mathbf{p}$ conjugate to the coordinate $\mathbf{q}$ is conserved, then identify it with the other constant vector in the heavy top formulation and thereby derive the correct motion equation for the heavy top.

[c] Continue the Legendre transformation of l_{KK} in (4) to obtain its Hamiltonian H_{KK} and the corresponding Hamiltonian equations.

[d] Evaluate H_{KK} on a level set of the constant $\mathbf{p}$ and rewrite the Euler-Poincaré equations in Lie-Poisson Hamiltonian form.

[e] Show that the Hamiltonian equations for $h = H_{KK}(\mathbf{\Pi}, \mathbf{\Gamma}, \mathbf{q}, \mathbf{p})$ split apart into a direct product of the heavy top equations in Lie-Poisson form, times a canonical Hamiltonian system. Write the Poisson bracket for the entire system.

Solution:

[a] According to its definition, the unit vector $\mathbf{\Gamma} = \mathbf{R}^{-1}\hat{\mathbf{z}} \in S^2$ satisfies the auxiliary equation,

$$\dot{\mathbf{\Gamma}} = -\mathbf{\Omega} \times \mathbf{\Gamma}\,,$$

upon taking the time derivative and using the hat map. So the $\mathbf{\Gamma}$-equation is part of the problem formulation in this case.

Variations in $\mathbf{\Omega}$ and $\mathbf{\Gamma}$ yields the Euler-Poincaré motion equation, while variations in $\mathbf{q}$ and $\dot{\mathbf{q}}$ yield the Euler-Lagrange equation.

[b] The Legendre transformation for l_{KK} in (4) gives the momenta

$$\mathbf{\Pi} = \mathbb{I}\mathbf{\Omega} \qquad \text{and} \qquad \mathbf{p} = \mathbf{\Gamma} + \dot{\mathbf{q}}\,. \tag{5}$$

Since l_{KK} does not depend on $\mathbf{q}$, its Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial l_{KK}}{\partial \dot{\mathbf{q}}} = \frac{\partial l_{KK}}{\partial \mathbf{q}} = 0\,,$$

shows that $\mathbf{p} = \partial l_{KK}/\partial \dot{\mathbf{q}}$ is conserved. It's natural to identify the ***constant vector*** $\mathbf{p}$ in the body with the *other* constant vector in the problem,

$$\mathbf{p} = \mathbf{\Gamma} + \dot{\mathbf{q}} = -\,mg\,\boldsymbol{\chi}\,.$$

After this identification, the Euler-Poincaré motion equation for the Kaluza-Klein Lagrangian easily returns Euler's motion equation for the heavy top (1).

[c] The Hamiltonian H_{KK} associated to l_{KK} by the Legendre transformation (5) is

$$\begin{aligned} H_{KK}(\mathbf{\Pi},\mathbf{\Gamma},\mathbf{q},\mathbf{p}) &= \mathbf{\Pi}\cdot\mathbf{\Omega}+\mathbf{p}\cdot\dot{\mathbf{q}}-l_{KK}(\mathbf{\Omega},\mathbf{\Gamma},\mathbf{q},\dot{\mathbf{q}}) \\ &= \frac{1}{2}\mathbf{\Pi}\cdot\mathbb{I}^{-1}\mathbf{\Pi}-\mathbf{p}\cdot\mathbf{\Gamma}+\frac{1}{2}|\mathbf{p}|^2 \\ &= \frac{1}{2}\mathbf{\Pi}\cdot\mathbb{I}^{-1}\mathbf{\Pi}+\frac{1}{2}|\mathbf{p}-\mathbf{\Gamma}|^2-\frac{1}{2}|\mathbf{\Gamma}|^2\,. \end{aligned}$$

The Hamiltonian equations for the canonical variables are

$$\begin{aligned} \dot{\boldsymbol{q}} &= \frac{\partial H_{KK}}{\partial \boldsymbol{p}} = \mathbf{p}-\mathbf{\Gamma}\,, \\ \dot{\boldsymbol{p}} &= -\frac{\partial H_{KK}}{\partial \boldsymbol{q}} = 0\,. \end{aligned}$$

Therefore, as expected, $\boldsymbol{p}$ is a constant of the augmented motion and, consequently, the canonically conjugate $\boldsymbol{q}$-equation decouples from the rest.

[d] Recall that $\mathbf{\Gamma}$ is a unit vector. On the constant level set $|\mathbf{\Gamma}|^2=1$, the Kaluza-Klein Hamiltonian H_{KK} is a positive quadratic function, shifted by a constant. Moreover, on the constant level set $\mathbf{p}=-\,mg\,\boldsymbol{\chi}$, the Kaluza-Klein Hamiltonian H_{KK} is a function $h(\mathbf{\Pi},\mathbf{\Gamma})$ of only the variables $(\mathbf{\Pi},\mathbf{\Gamma})$ and is equal to the sum of the kinetic plus potential energy for the heavy top, plus a harmless constant,

$$h(\mathbf{\Pi},\mathbf{\Gamma}) := H_{KK}(\mathbf{\Pi},\mathbf{\Gamma},\mathbf{p})\Big|_{\mathbf{p}=-\,mg\,\boldsymbol{\chi}} = \frac{1}{2}\mathbf{\Pi}\cdot\mathbb{I}^{-1}\mathbf{\Pi}+mg\,\boldsymbol{\chi}\cdot\mathbf{\Gamma}+\frac{1}{2}\big|mg\,\boldsymbol{\chi}\big|^2\,,$$

in terms of whose derivatives one may rewrite the Euler-Poincaré equations in Lie-Poisson Hamiltonian form, as

$$\begin{bmatrix}\dot{\mathbf{\Pi}}\\ \dot{\mathbf{\Gamma}}\end{bmatrix} = \begin{bmatrix}\mathbf{\Pi}\times & \mathbf{\Gamma}\times\\ \mathbf{\Gamma}\times & 0\end{bmatrix}\begin{bmatrix}\mathbf{\Omega}\\ mg\boldsymbol{\chi}\end{bmatrix} = \begin{bmatrix}\{\mathbf{\Pi},h\}\\ \{\mathbf{\Gamma},h\}\end{bmatrix}. \tag{6}$$

As a result, the Hamiltonian equations for $h=H_{KK}(\mathbf{\Pi},\mathbf{\Gamma},\mathbf{q},\mathbf{p})$ split apart into a direct product of the heavy top equations in Lie-Poisson form, times a canonical Hamiltonian system consisting of a conserved momentum and the linear evolution of its conjugate coordinate.

$$\begin{bmatrix} \dot{\mathbf{\Pi}} \\ \dot{\mathbf{\Gamma}} \\ \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{\Pi}\times & \mathbf{\Gamma}\times & 0 & 0 \\ \mathbf{\Gamma}\times & 0 & 0 & 0 \\ 0 & 0 & 0 & Id \\ 0 & 0 & -Id & 0 \end{bmatrix} \begin{bmatrix} \mathbf{\Omega} \\ -\mathbf{p} \\ 0 \\ \mathbf{p}-\mathbf{\Gamma} \end{bmatrix}. \tag{7}$$

Equations (7) exactly recover the heavy top equations (1) and (2) upon evaluating

$$\mathbf{p} = -mg\boldsymbol{\chi}.$$

[e] Hence, we have the following.
The Lie-Poisson equations for the Kaluza-Klein Hamiltonian H_{KK} ***recover Euler's equations for the heavy top (1) and (2).***
The Lie-Poisson bracket may be written in matrix form as

$$\{\, f\,,\, h\,\} = \begin{bmatrix} \partial f/\partial\mathbf{\Pi} \\ \partial f/\partial\mathbf{\Gamma} \\ \partial f/\partial\mathbf{q} \\ \partial f/\partial\mathbf{p} \end{bmatrix}^T \begin{bmatrix} \mathbf{\Pi}\times & \mathbf{\Gamma}\times & 0 & 0 \\ \mathbf{\Gamma}\times & 0 & 0 & 0 \\ 0 & 0 & 0 & Id \\ 0 & 0 & -Id & 0 \end{bmatrix} \begin{bmatrix} \partial h/\partial\mathbf{\Pi} \\ \partial h/\partial\mathbf{\Gamma} \\ \partial h/\partial\mathbf{q} \\ \partial h/\partial\mathbf{p} \end{bmatrix}. \tag{8}$$

M4A34 Assessed Homework 2 Spring Term 2010 Darryl Holm

Due at beginning of class one week from today

1. Generalised rigid body

Review
Recall the following definitions for the left action of a Lie group G on the cotangent bundle T^*Q of a manifold Q:

- The diamond operation $\diamond : T^*Q \to \mathfrak{g}^*$ is defined by
$$\Big\langle p \diamond q \,,\, \xi \Big\rangle = \Big\langle p \,,\, -\pounds_\xi q \Big\rangle_V \,,$$
with pairings $\langle \cdot \,,\, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle \cdot \,,\, \cdot \rangle_V : TV^* \times TV \to \mathbb{R}$.

- The cotangent-lift momentum map for this action is given by
$$J = -\, p \diamond q : T^*Q \to \mathfrak{g}^*$$
for canonical variables $(q, p) \in T^*Q$ satisfying $\{q, p\} = Id$.

Let the Hamiltonian H_{grb} for a generalised rigid body (grb) be defined as the pairing of the cotangent-lift momentum map J with its dual $J^\sharp = K^{-1}J \in \mathfrak{g}$,
$$H_{grb} = \frac{1}{2}\Big\langle p \diamond q \,,\, (p \diamond q)^\sharp \Big\rangle = \frac{1}{2}\Big(p \diamond q \,,\, K^{-1}(p \diamond q) \Big) \,,$$
for an appropriate inner product $(\,\cdot\,,\,\cdot\,) : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ obtained, e.g., from the Killing form K on $\mathfrak{g}$ (which is symmetric and nondegenerate).

Problem statement

[a] Compute the canonical equations for the Hamiltonian H_{grb}.

[b] Use these equations to compute the evolution equation for $J = -\, p \diamond q$.

[c] Identify the resulting equation and give a plausible argument why this was to be expected, by writing out its associated Hamilton's principle and Euler-Poincaré equations.

[d] Write the dynamical equations for q, p and J on $\mathbb{R}^3$ and explain why the name generalised rigid body might be appropriate.

Solution:

[a] By rearranging the Hamiltonian, we find

$$H_{grb} \;=\; \frac{1}{2}\Big\langle p\,,\, -\pounds_{(p\diamond q)^\sharp}q \Big\rangle_V = \frac{1}{2}\Big\langle -\pounds^T_{(p\diamond q)^\sharp}p\,,\, q\Big\rangle_V .$$

Consequently, the canonical equations for this Hamiltonian are

$$\dot q \;=\; \frac{\delta H_{grb}}{\delta p} = -\pounds_{(p\diamond q)^\sharp}q\,, \tag{9}$$

$$\dot p \;=\; -\frac{\delta H_{grb}}{\delta q} = \pounds^T_{(p\diamond q)^\sharp}p\,. \tag{10}$$

[b] These equations allow us to compute the evolution equation for the left cotangent-lift momentum map $J = -p \diamond q$ as

$$\begin{aligned}
\Big\langle \dot J\,,\,\xi\Big\rangle &= \Big\langle -\dot p\diamond q - p\diamond \dot q\,,\,\xi\Big\rangle \\
&= \Big\langle \pounds^T_\gamma p\diamond q - p\diamond \pounds_\gamma q\,,\,\xi\Big\rangle, \quad \text{where} \quad \gamma = -(p\diamond q)^\sharp = J^\sharp \\
&= -\Big\langle \pounds^T_\gamma p\,,\,\pounds_\xi q\Big\rangle + \Big\langle p\,,\,\pounds_\xi\pounds_\gamma q\Big\rangle \\
&= \Big\langle p\,,\,-\pounds_{(\mathrm{ad}_\gamma\xi)}q\Big\rangle = \Big\langle p\diamond q\,,\,\mathrm{ad}_\gamma\xi\Big\rangle \\
&= \Big\langle \mathrm{ad}^*_\gamma(p\diamond q)\,,\,\xi\Big\rangle = \Big\langle \mathrm{ad}^*_{J^\sharp}J\,,\,\xi\Big\rangle, \quad \text{for any} \quad \xi\in\mathfrak{g}.
\end{aligned}$$

Thus, we find that the equation of motion for a generalised rigid body is the same as the Euler-Poincaré equation for a left-invariant Lagrangian, namely,

$$\dot J = \mathrm{ad}^*_{J^\sharp}J\,. \tag{11}$$

[c] Equation (11) also results from Hamilton's principle $\delta S = 0$ given by

$$S(\xi;p,q) = \int \Big(l(\xi) + \langle p\,,\,\dot q - \pounds_\xi q\rangle\Big)dt$$

for the Clebsch-constrained reduced Lagrangian defined in terms of variables $(\xi;p,q)\in\mathfrak{g}\times T^*Q$ when we identify $\delta l/\delta\xi = J$.

(For the right action of G on T^*Q we would have written the Clebsch constraint term as $\langle p\,,\,\dot q + \pounds_\xi q\rangle$.)

[d] On $\mathbb{R}^3$ the EP equation (11) for grb becomes

$$\dot{J} = -\,J^\sharp \times J\,,$$

which recovers the rigid body when J is the body angular momentum and $J^\sharp = K^{-1}J$ is the body angular velocity.

The corresponding canonical Hamiltonian equations (9) and (10) for $q, p \in \mathbb{R}^3$ are

$$\dot{q} = -\,J^\sharp \times q \quad \text{and} \quad \dot{p} = -\,J^\sharp \times p\,.$$

These equations describe rigid rotations of vectors $q, p \in \mathbb{R}^3$ at angular velocity $J^\sharp$.

2. Momentum map for cotangent lifts

Review

The formula determining the momentum map for the cotangent-lifted action of a Lie group G on a smooth manifold Q may be expressed in terms of the pairings $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle : T^*Q \times TQ \to \mathbb{R}$ as

$$\langle\, J\,,\,\xi\,\rangle = \langle\langle\, p\,,\,\pounds_\xi q\,\rangle\rangle\,,$$

where $(q,p) \in T^*_qQ$ and $\pounds_\xi q$ is the infinitesimal generator of the action of the Lie algebra element ξ on the coordinate q.

Problem statement:

Define appropriate pairings and determine the momentum maps explicitly for the following actions, *then compute their symmetric double-bracket canonical equations.*

[a] $\pounds_\xi q = \xi \times q$ for $\mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3$

[b] $\pounds_\xi q = \mathrm{ad}_\xi q$ for ad-action $\mathrm{ad} :\, \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ in a Lie algebra $\mathfrak{g}$

[c] AqA^{-1} for $A \in GL(3,R)$ acting on $q \in GL(3,R)$ by matrix conjugation

[d] Aq for left action of $A \in SO(3)$ on $q \in SO(3)$

[e] AqA^T for $A \in GL(3,R)$ acting on $q \in Sym(3)$, that is $q = q^T$.

[f] $UQU^\dagger$ for a unitary matrix $U \in U(n)$ satisfying $U^\dagger = U^{-1}$ acting on Hermitian $Q \in H(n)$ satisfying $Q = Q^\dagger$.

[g] $\pounds_\xi \phi = \{\xi, \phi\} = (\partial_p \xi)(\partial_q \phi) - (\partial_q \xi)(\partial_p \phi)$ for the canonical Poisson bracket $\{\,\cdot\,,\,\cdot\,\} : \mathcal{F}^* \times \mathcal{F} \to \mathcal{F}$.

Solution:

[a] $\pounds_\xi q = \xi \times q$ for $\mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ under *spatial* rotations

The definition of cotangent-lift momentum map yields

$$\langle J\,,\,\xi\,\rangle \quad = \quad \langle p\,,\,\xi \times q \rangle = p \cdot \xi \times q = -\,p \times q \cdot \xi$$

Hence, one has $J = -\,p \times q = q \times p \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ for this action.

The Hamiltonian for geodesic motion on $SO(3)$ with respect to the metric given by the pairing $\mathbb{I} : \mathfrak{so}(3) \to \mathfrak{so}(3)^*$ is

$$H = \frac{1}{2} J \cdot \mathbb{I}^{-1} J = \frac{1}{2} (q \times p) \cdot \mathbb{I}^{-1} (q \times p),$$

with canonical Poisson bracket $\{\,q\,,\,p\,\} = 1$, which yields canonical equations in symmetric double-bracket form,

$$\begin{aligned} \dot{q} &= \{q, H\} = \frac{\delta H}{\delta p} = -\,q \times \mathbb{I}^{-1}(q \times p), \\ \dot{p} &= \{p, H\} = -\,\frac{\delta H}{\delta q} = -\,p \times \mathbb{I}^{-1}(q \times p). \end{aligned}$$

By the Jacobi identity for the cross product, these canonical equations yield

$$\dot{J} = J \times \mathbb{I}^{-1} J$$

in the form

$$\frac{d}{dt}(q \times p) = (q \times p) \times \Big(\mathbb{I}^{-1}(q \times p) \Big)$$

These equations provide the symmetric phase-space formulation of *spatial* rotations of a rigid body.

[b] $\pounds_\xi q = \mathrm{ad}_\xi q$ for $\mathrm{ad}: \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ in a Lie algebra $\mathfrak{g}$.

The definition of cotangent-lift momentum map yields

$$\langle J \,,\, \xi \rangle \;\;=\;\; \langle p \,,\, \mathrm{ad}_\xi q \rangle = -\left\langle \mathrm{ad}^*_q p \,,\, \xi \right\rangle$$

Hence, one has

$$J = -\,\mathrm{ad}^*_q p \in \mathfrak{g}^*$$

for this action.

The Hamiltonian for geodesic motion on the Lie group G with respect to the metric defined as the pairing of the cotangent-lift momentum map J with its dual $J^\sharp = K^{-1} J \in \mathfrak{g}$ is given by,

$$H_G = \frac{1}{2}\Big\langle \mathrm{ad}^*_q p \,,\, (\mathrm{ad}^*_q p)^\sharp \Big\rangle = \frac{1}{2}\Big(\mathrm{ad}^*_q p \,,\, K^{-1}(\mathrm{ad}^*_q p) \Big) \,,$$

for an appropriate inner product $(\,\cdot\,,\,\cdot\,): \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ obtained, e.g., from the Killing form K on $\mathfrak{g}$ (which is symmetric and nondegenerate).

The canonical Poisson bracket $\{\, q \,,\, p \,\} = 1$ yields canonical equations for the Hamiltonian H_G in symmetric double-bracket form,

$$\dot{q} \;\;=\;\; \{q, H\} = \frac{\delta H}{\delta p} = \mathrm{ad}^*_{J^\sharp}\, q, \tag{12}$$

$$\dot{p} \;\;=\;\; \{p, H\} = -\frac{\delta H}{\delta q} = \mathrm{ad}^*_{J^\sharp}\, p. \tag{13}$$

After a short calculation using the Jacobi identity for the ad-operation in $\mathfrak{g}$, these canonical equations yield

$$\frac{d}{dt}\Big\langle J \,,\, \xi \Big\rangle = \Big\langle \dot{J} \,,\, \xi \Big\rangle = \Big\langle \mathrm{ad}^*_{J^\sharp}\, J \,,\, \xi \Big\rangle \tag{14}$$

in the form

$$\frac{d}{dt}(\mathrm{ad}^*_q p) = \mathrm{ad}^*_{K^{-1}(\mathrm{ad}^*_q p)} \mathrm{ad}^*_q p \,. \tag{15}$$

These equations provide the symmetric phase-space formulation of the motions of a generalised rigid body.

[c] AqA^{-1} for $A \in GL(3, R)$ acting on $q \in GL(3, R)$ by matrix conjugation.

In this case, one computes the ad-action for $GL(3,R)$ conjugation as

$$T_e(AqA^{-1}) = \xi q - q\xi = [\xi, q]\,,$$

for $\xi = A'(0) \in gl(3,R)$ acting on $q \in GL(3,R)$ by matrix Lie bracket $[\cdot\,,\,\cdot]$.

Consequently, one finds the momentum map by using the matrix trace-pairing, $\langle\, A\,,\, B\,\rangle = \mathrm{tr}(A^TB)$,

$$\langle\, p\,,\, \mathrm{ad}_\xi q\,\rangle = \mathrm{tr}(p^T[\xi,q]) = \mathrm{tr}\Big((pq^T - q^Tp)^T\xi\Big)$$

Thus,

$$J = pq^T - q^Tp = [p, q^T] = -\,\mathrm{ad}_{q^T}p = -\,\mathrm{ad}^*_q p\,,$$

where $[\,\cdot\,,\,\cdot\,]$ is the matrix commutator, so that $J^T = [q, p^T]$ exchanges $q \leftrightarrow p$ in J. (Page 175 of the book)

Then for a symmetric matrix K^{-1}, we have

$$H = \frac{1}{2}\Big\langle J\,,\, J^\sharp\Big\rangle = \frac{1}{2}\mathrm{tr}(J^TK^{-1}J)\,,$$

whose canonical Hamiltonian equations are, cf. (12) and (13)

$$\begin{aligned}
\dot q &= \frac{\delta H}{\delta p} = -\big[q^T, K^{-1}J\big] =: \mathrm{ad}^*_{J^\sharp}\, q,\\
\dot p &= -\frac{\delta H}{\delta q} = -\,\big[p^T, K^{-1}J\big] = -\,\big[(K^{-1}J)^T, p\big] =: \mathrm{ad}^*_{J^\sharp}\, p,
\end{aligned}$$

in which the last step uses a relation for the transpose of the matrix commutator $[\,A^T\,,\,B\,] = -[\,A\,,\,B^T\,]^T$ and antisymmetry $J^T = -J$. Hence, cf. (15)

$$\dot J = \mathrm{ad}^*_{J^\sharp}\, J = -\,\Big[(K^{-1}J)^T, J\Big] \qquad (16)$$

[d] Aq for left action of $A \in SO(3)$ on $q \in SO(3)$

Compute $T_e(Aq) = \xi q$ for $\xi = A'(0) \in so(3)$ acting on $q \in SO(3)$ by left matrix multiplication. For the matrix pairing $\langle\, A\,,\, B\,\rangle =$

trace(A^TB), one finds the following expression for the momentum map,

$$\mathrm{trace}(p^T\xi q) = \mathrm{trace}((pq^T)^T\xi) \quad \Rightarrow \quad J = \frac{1}{2}(pq^T - qp^T) = -J^T\,,$$

upon using antisymmetry of the matrix $\xi \in so(3)$.

Then for a symmetric matrix K^{-1}, we have

$$H = \frac{1}{2}\Big\langle J\,,\, J^\sharp\Big\rangle = \frac{1}{2}\mathrm{tr}(J^TK^{-1}J) = -\,\frac{1}{2}\mathrm{tr}(JK^{-1}J)\,,$$

whose canonical Hamiltonian equations are, cf. (12) and (13)

$$\begin{aligned}\dot q &= \frac{\delta H}{\delta p} = -\,q^T(K^{-1}J + JK^{-1})\\ \dot p &= -\,\frac{\delta H}{\delta q} = -\,p^T(K^{-1}J + JK^{-1})\end{aligned}$$

Compute equation for $(\dot q, \dot p, \dot J)$

[e] AqA^T for $A \in GL(3,R)$ acting on $q \in Sym(3)$; that is, $q = q^T$.

Compute

$$T_e(AqA^T) = \xi q + q\xi^T = \xi q + (\xi q)^T\,,$$

for $\xi = A'(0) \in gl(3,R)$ acting on $q \in Sym(3)$. For the matrix pairing $\langle\, A\,,\, B\,\rangle = \mathrm{tr}(A^TB)$, one finds

$$\mathrm{tr}(p^T(\xi q + q\xi^T)) = \mathrm{tr}((qp^T + q^Tp)\xi) \Rightarrow J = pq^T + p^Tq = J^T\,,$$

Compute equation for $(\dot q, \dot p, \dot J)$

[f] $UQU^\dagger$ for a unitary matrix $U \in U(n)$ satisfying $U^\dagger = U^{-1}$ acting on Hermitian $Q \in H(n)$ satisfying $Q = Q^\dagger$.

The canonically conjugate elements of $T^*\mathcal{Q}$ are pairs (Q,P) of Hermitian matrices. The corresponding Poisson bracket is

$$\{F,H\} = \mathrm{tr}\bigg(\frac{\partial F}{\partial Q}\frac{\partial H}{\partial P} - \frac{\partial H}{\partial Q}\frac{\partial F}{\partial P}\bigg).$$

The action of the group $U(n)$ on $T^*\mathcal{Q}$ is given by

$$(Q,P) \mapsto (UQU^\dagger, UPU^\dagger)\,, \quad UU^\dagger = Id\,.$$

The linearisation of this group action with $U = \exp(t\xi)$, with skew-Hermitian $\xi^\dagger = -\,\xi$ yields the vector field with (Q,P) components,

$$X_\xi = \Big([\xi, Q], [\xi, P]\Big)\,.$$

This is the Hamiltonian vector field for

$$J_\xi = \operatorname{tr}\big([Q,P]\xi\big)\,,$$

thus yielding the *momentum map* $J(Q,P) = [Q,P]$.

Compute equations of motion for for $(\dot{Q}, \dot{P}, \dot{J})$ for the Hamiltonian

$$H(Q,P) = \frac{1}{2}\operatorname{tr}\Big([Q,P]^T K^{-1}[Q,P]\Big) = \frac{1}{2}\operatorname{tr}\big(J^T J^\sharp\big)$$

with $J^\sharp := K^{-1}J$. Hence, we find

$$\begin{aligned}
\dot{Q} &= \frac{\partial H}{\partial P} = -\,[J^\sharp, Q]\,,\\
\dot{P} &= -\,\frac{\partial H}{\partial Q} = -\,[J^\sharp, P]\,,\\
\dot{J} &= \frac{d}{dt}[Q,P] = -\,\big[J^\sharp, [Q,P]\big]\\
&= -\,[J^\sharp, J] = -\left[\frac{\partial H}{\partial J}, J\right].
\end{aligned}$$

As a result, we have for any smooth function $F(J)$,

$$\frac{d}{dt}F(J) = -\operatorname{tr}\!\left(J\left[\frac{\partial F}{\partial J}, \frac{\partial H}{\partial J}\right]\right) =: \{F,\,H\}$$

[g] $\pounds_\xi \phi = \{\xi, \phi\} = (\partial_p \xi)(\partial_q \phi) - (\partial_q \xi)(\partial_p \phi)$ for canonical Poisson bracket $\{\,\cdot\,,\,\cdot\,\} : \mathcal{F}^* \times \mathcal{F} \to \mathcal{F}$.

The definition of cotangent-lift momentum map yields

$$\langle J\,,\,\xi\rangle \;=\; \langle \pi\,,\,\pounds_\xi\phi\rangle = \int \pi\{\xi,\phi\}dqdp = \int \xi\{\phi,\pi\}dqdp$$

Hence, one has $J = \{\pi,\phi\}$ for this action.

Writing $f = \{\pi,\phi\}$ in the Hamiltonian for geodesic Vlasov motion

$$H = \frac{1}{2}\int f\,K * f dqdp = \frac{1}{2}\int \{\pi,\phi\}\,K * \{\pi,\phi\}dqdp$$

with canonical Poisson bracket $\{\,\cdot\,,\,\cdot\,\} : \mathcal{F}^* \times \mathcal{F} \to \mathcal{F}$ yields canonical equations in symmetric double-bracket form,

$$\begin{aligned}
\dot{\phi} &= \frac{\delta H}{\delta \pi} = \Big\{\phi,\,K * \{\pi,\phi\}\Big\},\\
\dot{\pi} &= -\frac{\delta H}{\delta \phi} = \Big\{\pi,\,K * \{\pi,\phi\}\Big\}.
\end{aligned}$$

By the general theory, these canonical equations recover

$$\partial_t f = \Big\{f, \frac{\delta H}{\delta f}\Big\}$$

in the form

$$\partial_t\{\pi,\phi\} = \Big\{\{\pi,\phi\}, K * \{\pi,\phi\}\Big\}$$

These equations would appear in the symmetric phase-space formulation of optimal control of geodesic Vlasov motion.

3. Nambu representation of Euler's rigid-body equations

Review of the Nambu bracket

A Hamiltonian vector field X_H defined on phase-plane coordinates $(p,q) = (x_1,x_2) \in \mathbb{R}^2$ satisfies

$$X_H \,\lrcorner\, d^2x = -\,dH$$

where $H : \mathbb{R}^2 \to \mathbb{R}$. The corresponding Poisson bracket may be be given a 3D vector representation in Cartesian coordinates $(x_1, x_2, x_3) \in \mathbb{R}^3$ by writing it as

$$X_H = \{\,\cdot\,, H\}_x = \frac{\partial H}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial H}{\partial x_1}\frac{\partial}{\partial x_2} = \nabla x_3 \times \nabla H \cdot \nabla\,,$$

in which $\nabla x_3 = (0, 0, 1)$ is the unit vector normal to the $\mathbb{R}^2$ plane.
Consequently, a phase-plane function $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$\dot{F} = X_H F = \{F\,, H\}_x = \nabla F \cdot \nabla x_3 \times \nabla H =: \{F\,, x_3\,, H\}_x\,,$$

in which the last expression defines the ***Nambu bracket*** for the ordered triple (F, x_3, H).
The level set of x_3 is not special, so a Nambu bracket may be defined for the ordered triple (F, S, H) with any smooth function $S : \mathbb{R}^3 \to \mathbb{R}$ as

$$\{F\,, S\,, H\}_x\, dx_1 \wedge dx_2 \wedge dx_3 := \nabla F \cdot \nabla S \times \nabla H\, dx_1 \wedge dx_2 \wedge dx_3 = dF \wedge dS \wedge dH\,.$$

A Nambu vector field $X_{SH} = \{\,\cdot\,, S\,, H\}_x$ on $\mathbb{R}^3$ satisfies

$$X_{SH} \,\lrcorner\, d^3x = -\, dH \wedge dS$$

for functions $S, H : \mathbb{R}^3 \to \mathbb{R}$. Thus, the Nambu bracket $\{F\,, S\,, H\}$ reduces to the canonical Poisson bracket at any nonsingular point on a level surface of S. (This holds for $S = x_3 = constant$, of course, and is guaranteed by Darboux's theorem for any regular surface $S = constant$.)
Euler's equations for rigid-body motion in principal-axis body angular momentum coordinates $\mathbf{x} \in \mathbb{R}^3$ consist of

$$\dot{\mathbf{x}} = \frac{1}{4}\nabla|\mathbf{x}|^2 \times \nabla(\mathbf{x} \cdot \mathbb{I}^{-1}\mathbf{x})\,,$$

with diagonal momentum of inertia tensor $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$.
These are expressed in Nambu form by identifying $S = \frac{1}{2}|\mathbf{x}|^2$ and $H = \frac{1}{2}\mathbf{x} \cdot \mathbb{I}^{-1}\mathbf{x}$. Consequently, the motion in body coordinates $\mathbf{x}$ occurs along intersections of level sets of the conserved angular momentum S and the kinetic energy H.

Problem statement:

[a] Verify from its definition that the Nambu bracket is trilinear, has even (odd) parity under corresponding permutations of its three entries, satisfies the Jacobi identity, and is invariant under the volume-preserving transformations of $\mathbb{R}^3$.

[b] Show that volume-preserving transformations are the analogs for the Nambu bracket of canonical transformations of the symplectic Poisson bracket by specialising the last property in Part [a] to $S = x_3$.

[c] Show that Euler's rigid-body equations transform into

$$\dot{\mathbf{X}} = \frac{1}{4}\nabla(X_1^2 - X_2^2) \times \nabla(X_1^2 - X_3^2)$$

upon defining

$$X_i := x_i/\gamma_i \quad \text{(no sum), where} \quad \gamma_i = -1/I_j + 1/I_k\,,$$

with cyclic permutation in $i, j, k = 1, 2, 3$. In performing this calculation, check that their definitions imply that the three γ's satisfy the relation

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 0$$

and impose $\gamma_1\gamma_2\gamma_3 = 1$ on their product so that volume is preserved. Hint: The easy way to do this is to use the wedge product and its properties.

[d] Discuss the geometric meaning of this representation of rigid-body motion and sketch its solution trajectories for $\mathbf{X} \in \mathbb{R}^3$. Explicitly write the transformed Euler equations in components of $\mathbf{X} = (X_1, X_2, X_3)$ and check that the divergence of the resulting vector field $\dot{\mathbf{X}}$ vanishes.

[e] Reduce the motion to a level surface of $S = \frac{1}{2}(X_1^2 - X_2^2)$. Write the equations of motion explicitly and show that they are canonically Hamiltonian.

Solution:

[a] For example, the last property in **[c]** follows for the smooth transformations $(x_1, x_2, x_3) \to (X_1, X_2, X_3) \in \mathbb{R}^3$ as

$$\begin{aligned} &\{F\,, S\,, H\}_x\, dx_1 \wedge dx_2 \wedge dx_3 = dF \wedge dS \wedge dH \\ &= \{F\,, S\,, H\}_X\, dX_1 \wedge dX_2 \wedge dX_3\,, \end{aligned}$$

so that

$$\{F\,, S\,, H\}_x = \{F\,, S\,, H\}_X\,,$$

when $dX_1 \wedge dX_2 \wedge dX_3 = dx_1 \wedge dx_2 \wedge dx_3$.

[b] The specialisation to $S = x_3$ in the last of the properties in Part **[c]** immediately recovers the area-preserving (symplectic) transformations of the plane.

[c] Direct calculation.

[d] The solution trajectories for rigid-body motion in $\mathbf{X} \in \mathbb{R}^3$ follow along intersections of two families of hyperbolic cylinders whose axes of translation symmetry are orthogoanal to each other. The Euler equations in components of $\mathbf{X} = (X_1, X_2, X_3)$ are $\dot{X}_1 = X_2 X_3$ and cyclic permutations.

[e] Direct calculation.

4. Trilinear bracket representation of the heavy top equations

A certain trilinear bracket for any three smooth functions $F, S, H : \mathbb{R}^6 \to \mathbb{R}$ defines a dynamical system for $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^6$ as

$$\begin{aligned} \dot{F} &= \{F\,, S\,, H\} \\ &:= \frac{\partial F}{\partial \mathbf{y}} \cdot \frac{\partial S}{\partial \mathbf{x}} \times \frac{\partial H}{\partial \mathbf{x}} + \frac{\partial S}{\partial \mathbf{y}} \cdot \frac{\partial H}{\partial \mathbf{x}} \times \frac{\partial F}{\partial \mathbf{x}} + \frac{\partial H}{\partial \mathbf{y}} \cdot \frac{\partial F}{\partial \mathbf{x}} \times \frac{\partial S}{\partial \mathbf{x}} \,. \end{aligned} \tag{17}$$

Let the dynamics of a given system be determined from $\dot{\mathbf{x}} = \{\mathbf{x}\,, S\,, H\}$ and $\dot{\mathbf{y}} = \{\mathbf{y}\,, S\,, H\}$.

Problem statement:

[a] Write the dynamical equations of the variables $\mathbf{x}(t)$ and $\mathbf{y}(t)$ generated by the trilinear bracket (17) for arbitrary smooth functions S and H.

[b] Show that this dynamics conserves the functions S and H, so that the motion in $\mathbb{R}^6$ takes place on the intersections of level sets of S and H. What is the dimensionality of these intersections? Is the dimensionality an even integer?

[c] Assume that $S = \mathbf{x} \cdot \mathbf{y}$ and $H = \frac{1}{2}\mathbf{x} \cdot \mathbb{I}^{-1}\mathbf{x} + \mathbf{a} \cdot \mathbf{y}$ for constant vector $\mathbf{a} \in \mathbb{R}^3$ and constant symmetric positive invertible 3×3 matrix $\mathbb{I}$. Reduce the problem to an equivalent rigid body problem when initial conditions are chosen such that $S = 0$.

[d] Explain what the reduced motion on $S = 0$ in part [c] means physically.

[e] From its definition that the bracket (17) is trilinear. Show that it has even (odd) parity under corresponding permutations of its three entries and satisfies the Jacobi identity.

[f] What is the interpretation of the trilinear bracket (17) when $S = \mathbf{x} \cdot \mathbf{y}$?

Solution:

[a] The dynamical equations of the variables $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are

$$\begin{aligned} \dot{\mathbf{x}} &= \{\mathbf{x}\,, S\,, H\} = \frac{\partial S}{\partial \mathbf{y}} \times \frac{\partial H}{\partial \mathbf{x}} - \frac{\partial H}{\partial \mathbf{y}} \times \frac{\partial S}{\partial \mathbf{x}}\,, \\ \dot{\mathbf{y}} &= \{\mathbf{y}\,, S\,, H\} = \frac{\partial S}{\partial \mathbf{x}} \times \frac{\partial H}{\partial \mathbf{x}}\,. \end{aligned}$$

[b] When $F = S$ or $F = H$, the trilinear bracket (17) vanishes by antisymmetry of cross product of vectors in $\mathbb{R}^3$. The dimensionality of the intersections of level sets of these quantities is 5+5-6=4, which is even and thus is a candidate for a symplectic Hamiltonian formulation.

[c] The preserved condition $S = \mathbf{x} \cdot \mathbf{y} = 0$ implies that the relation $\mathbf{y} = \mathbf{b} \times \mathbf{x}$ is preserved for a constant vector $\mathbf{b} \in \mathbb{R}^3$. Hence, $H = \frac{1}{2}\mathbf{x} \cdot \mathbb{I}^{-1}\mathbf{x} + \mathbf{a} \cdot \mathbf{y}$ becomes

$$H = \frac{1}{2}\mathbf{x} \cdot \mathbb{I}^{-1}\mathbf{x} + \mathbf{x} \cdot (\mathbf{a} \times \mathbf{b}) \,.$$

Substitution of this reduced Hamiltonian into the trilinear bracket (17) results in

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{x} \times (\mathbb{I}^{-1}\mathbf{x} + \mathbf{a} \times \mathbf{b}) \,, \\ \dot{\mathbf{y}} &= \mathbf{b} \times \dot{\mathbf{x}} \,. \end{aligned}$$

[d] The 1st equation (for $\dot{\mathbf{x}}$) may be identified as eccentrically rotating rigid body motion. The second equation (for $\dot{\mathbf{y}}$) decouples; so it may be solved separately.

When $\mathbf{b} = 0$, the heavy top is supported at its center of mass and its motion reduces to ordinary rigid body motion.

[e] The various properties of the trilinear bracket (17) may be verified directly.

[f] When $S = \mathbf{x} \cdot \mathbf{y}$, the trilinear bracket (17) reduces to the Lie-Poisson bracket on $se(3)^*$. Its various properties follow in that case by being linearly dual to the defining properties of a Lie algebra. The quantity $S = \mathbf{x} \cdot \mathbf{y}$ is the Casimir for the Lie-Poisson bracket on $se(3)^*$.