1. Hamilton-Pontryagin metamorphosis
Consider the left-invariant action S for Hamilton's principle $\delta S = 0$ given by

$$S = \int L(\Omega, \omega, g)\, dt = \int l(\Omega) + \frac{1}{2\sigma^2}\big|\omega - \mathrm{Ad}_g\Omega\big|^2\, dt\,,$$

where $g \in G$ and $\omega = \dot{g}g^{-1} \in \mathfrak{g}$, for a matrix Lie group G and matrix Lie algebra $\mathfrak{g}$. Here $\sigma^2 \in \mathbb{R}$ is a positive constant and $|\cdot|$ is a Riemannian metric which defines a symmetric non-degenerate pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ between Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. (You may assume that $\mathfrak{g}^{**} \simeq \mathfrak{g}$.)

(1.a) Show that

$$(\mathrm{Ad}_g\Omega)' = \mathrm{Ad}_g\Omega' - \mathrm{ad}_{\mathrm{Ad}_g\Omega}\eta \quad \text{with} \quad \eta = g'g^{-1}$$

(1.b) Write ω' in terms of η, $\dot{\eta}$ and ad_ω using cross-derivatives of $\dot{g} = \omega g$ and $g' = \eta g$.

(1.c) Derive the Euler-Poincaré equation for $\partial l/\partial\Omega$ from $\delta S = 0$.
(You may ignore endpoint terms when integrating by parts.)

(1.d) Interpret this Euler-Poincaré equation as a conservation law.

1. Solution

(1.a) One computes

$$\begin{aligned}
(\mathrm{Ad}_g\Omega)' &= (g\Omega g^{-1})' \\
&= g'g^{-1}\mathrm{Ad}_g\Omega + \mathrm{Ad}_g\Omega' - (\mathrm{Ad}_g\Omega)g'g^{-1} \\
&= \mathrm{Ad}_g\Omega' - \mathrm{ad}_{\mathrm{Ad}_g\Omega}\eta \quad \text{with} \quad \eta = g'g^{-1}
\end{aligned}$$

(1.b) The cross-derivative identities for $\dot{g} = \omega g$ and $g' = \eta g$ yield

$$\dot{g}' = \omega' g + \omega g' = \dot{\eta} g + \eta \dot{g} \quad \Longrightarrow \quad \omega' = \dot{\eta} - \mathrm{ad}_\omega \eta\,.$$

(1.c) The variation of the action integral S is

$$\begin{aligned}
0 = \delta S &= \int \Big\langle \frac{\partial l}{\partial \Omega}\,,\, \Omega' \Big\rangle + \Big\langle \pi\,,\, \omega' - (\mathrm{Ad}_g\Omega)' \Big\rangle\, dt \\
&= \int \Big\langle \frac{\partial l}{\partial \Omega}\,,\, \Omega' \Big\rangle + \Big\langle \pi\,,\, \dot{\eta} - \mathrm{ad}_\omega\eta + \mathrm{ad}_{\mathrm{Ad}_g\Omega}\eta - \mathrm{Ad}_g\Omega' \Big\rangle\, dt \\
&= \int \Big\langle \frac{\partial l}{\partial \Omega} - \mathrm{Ad}^*_g\pi\,,\, \Omega' \Big\rangle - \Big\langle \dot{\pi} + \mathrm{ad}^*_\omega\pi - \mathrm{ad}^*_{\mathrm{Ad}_g\Omega}\pi\,,\, \eta \Big\rangle\, dt
\end{aligned}$$

where endpoint terms are being ignored and we have introduced the conjugate momentum for spatial angular velocity, π, given by

$$\pi := \frac{\partial L}{\partial \omega} = \frac{1}{\sigma^2}\big(\omega - \mathrm{Ad}_g\Omega\big)\,.$$

The pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is induced by the variational-derivative operation. Requiring independent variations to vanish yields,

$$\begin{aligned} \Pi := \frac{\partial l}{\partial \Omega} &= \mathrm{Ad}^*_g \pi\,, \\ \dot{\pi} + \mathrm{ad}^*_\omega \pi &= \mathrm{ad}^*_{\mathrm{Ad}_g\Omega}\pi\,. \end{aligned} \tag{1}$$

In terms of $\Pi = \partial l/\partial\Omega$ the two stationarity relations in (1) imply

$$\begin{aligned} \frac{d}{dt}\Big\langle \Pi\,,\,\eta \Big\rangle &= \frac{d}{dt}\Big\langle \mathrm{Ad}^*_g\pi\,,\,\eta\Big\rangle \\ \text{taking } \tfrac{d}{dt}\mathrm{Ad}^*_g &= \Big\langle \mathrm{Ad}^*_g\big(\dot{\pi} + \mathrm{ad}^*_\omega\pi\big)\,,\,\eta\Big\rangle \\ \text{using } \pi\text{-eqn (1)} &= \Big\langle \mathrm{Ad}^*_g\big(\mathrm{ad}^*_{\mathrm{Ad}_g\Omega}\pi\big)\,,\,\eta\Big\rangle \\ \text{using Ad \& ad definitions} &= \Big\langle \pi\,,\,\mathrm{ad}_{\mathrm{Ad}_g\Omega}\big(\mathrm{Ad}_g\eta\big)\Big\rangle \\ \text{rearranging} &= \Big\langle \pi\,,\,\mathrm{Ad}_g\big(\mathrm{ad}_\Omega\eta\big)\Big\rangle \\ \text{taking duals} &= \Big\langle \mathrm{ad}^*_\Omega\mathrm{Ad}^*_g\pi\,,\,\eta\Big\rangle \\ \text{substituting the definition of } \Pi &= \Big\langle \mathrm{ad}^*_\Omega\,\Pi\,,\,\eta\Big\rangle \end{aligned}$$

This recovers the Euler-Poincaré equation,

$$\frac{d}{dt}\frac{\partial l}{\partial\Omega} = \mathrm{ad}^*_\Omega\,\frac{\partial l}{\partial\Omega}\,,$$

for coadjoint motion on the dual of the left-invariant Lie-algebra of G.

(1.d) The definition of Ad^* gives

$$\frac{d}{dt}\frac{\partial l}{\partial\Omega} = \mathrm{ad}^*_\Omega\,\frac{\partial l}{\partial\Omega}\,, \quad \text{is equivalent to} \quad \frac{d}{dt}\left(\mathrm{Ad}^*_{g^{-1}}\frac{\partial l}{\partial\Omega}\right) = 0\,,$$

which is a conservation law (for π).

2. Momentum map for unitary transformations
Consider the matrix Lie group $\mathcal{Q}$ of $n \times n$ Hermitian matrices, so that $Q^\dagger = Q$ for $Q \in \mathcal{Q}$. The Poisson (symplectic) manifold is $T^*\mathcal{Q}$, whose elements are pairs (Q, P) of Hermitian matrices. The corresponding Poisson bracket is

$$\{F, H\} = \mathrm{tr}\left(\frac{\partial F}{\partial Q}\frac{\partial H}{\partial P} - \frac{\partial H}{\partial Q}\frac{\partial F}{\partial P}\right).$$

Let G be the group $U(n)$ of $n \times n$ unitary matrices: G acts on $T^*\mathcal{Q}$ through

$$(Q, P) \mapsto (UQU^\dagger, UPU^\dagger)\,, \quad UU^\dagger = Id$$

(2.a) What is the linearization of this group action?

(2.b) What is its momentum map?

(2.c) Is this momentum map equivariant? Explain why, or why not.

(2.d) Is the momentum map conserved by the Hamiltonian $H = \frac{1}{2}\mathrm{tr}\,P^2$? Prove it.

2. Solution

(2.a.i) The linearization of this group action with $U = \exp(t\xi)$, with skew-Hermitian $\xi^\dagger = -\xi$ yields the vector field

$$X_\xi = \big([Q, \xi], [P, \xi]\big)$$

(2.a.ii) This is the Hamiltonian vector field for

$$H_\xi = \mathrm{tr}\,([Q, P]\xi)$$

thus yielding the momentum map $J(Q, P) = [Q, P]$.

(2.a.iii) Being defined by a cotangent lift, this momentum map is equivariant.

(2.b) For $H = \frac{1}{2}\mathrm{tr}\,P^2$,

$$\Big\{[Q, P], H\Big\} = \mathrm{tr}\left(\frac{\partial[Q, P]}{\partial Q}\frac{\partial H}{\partial P}\right) = \mathrm{tr}\Big(P^2 - P^2\Big) = 0$$

so the momentum map $J(Q, P) = [Q, P]$ is conserved by this Hamiltonian.

Alternatively, one may simply observe that the map

$$(Q, P) \mapsto (UQU^\dagger, UPU^\dagger)\,, \quad UU^\dagger = Id$$

preserves $\mathrm{tr}(P^2)$, since it takes

$$\mathrm{tr}(P^2) \mapsto \mathrm{tr}(UPU^\dagger UPU^\dagger) = \mathrm{tr}(P^2)$$

3. $GL(n,\mathbb{R})$**-invariant motions**
Consider the Lagrangian

$$L=\frac{1}{2}\operatorname{tr}\Big(\dot{S}S^{-1}\dot{S}S^{-1}\Big)+\frac{1}{2}\,\dot{\mathbf{q}}\cdot S^{-1}\dot{\mathbf{q}}\,,$$

where S is an $n\times n$ symmetric matrix and $\mathbf{q}\in\mathbb{R}^n$ is an $n-$component column vector.

(3.a) Legendre transform to construct the corresponding Hamiltonian and canonical equations.

(3.b) Show that the Lagrangian and Hamiltonian are invariant under the group action

$$\mathbf{q}\to G\mathbf{q}\quad\text{and}\quad S\to GSG^T$$

for any constant invertible $n\times n$ matrix, G.

(3.c) Compute the infinitesimal generator for this group action and construct its corresponding momentum map. Is this momentum map equivariant? Prove it.

(3.d) Verify directly that this momentum map is a conserved $n\times n$ matrix quantity by using the equations of motion.

3. Solution

(3.a) Legendre transform as

$$P=\frac{\partial L}{\partial\dot{S}}=S^{-1}\dot{S}S^{-1}\quad\text{and}\quad\mathbf{p}=\frac{\partial L}{\partial\dot{\mathbf{q}}}=S^{-1}\dot{\mathbf{q}}$$

Thus, the Hamiltonian $H(Q,P)$ and its canonical equations are:

$$H(\mathbf{q},\mathbf{p},S,P)=\frac{1}{2}\operatorname{tr}\big(PS\cdot PS\big)+\frac{1}{2}\mathbf{p}\cdot S\mathbf{p}\,,$$

$$\dot{S}=\frac{\partial H}{\partial P}=SPS\,,\quad\dot{P}=-\,\frac{\partial H}{\partial S}=-\Big(PSP+\frac{1}{2}\mathbf{p}\otimes\mathbf{p}\Big),$$

$$\dot{\mathbf{q}}=\frac{\partial H}{\partial\mathbf{p}}=S\mathbf{p}\,,\quad\dot{\mathbf{p}}=\frac{\partial H}{\partial\mathbf{q}}=0\,.$$

(3.b) Under the group action $\mathbf{q}\to G\mathbf{q}$ and $S\to GSG^T$ for any constant invertible $n\times n$ matrix, G, one finds $\dot{S}S^{-1}\to G\dot{S}S^{-1}G^{-1}$ and $\dot{\mathbf{q}}\cdot S^{-1}\dot{\mathbf{q}}\to\dot{\mathbf{q}}\cdot S^{-1}\dot{\mathbf{q}}$. Hence, $L\to L$. Likewise, $P\to G^{-T}PG^{-1}$ so $PS\to G^{-T}PSG^T$ and $\mathbf{p}\to G^{-T}\mathbf{p}$ so that $S\mathbf{p}\to GS\mathbf{p}$. Hence, $H\to H$, as well; so both L and H for the system are invariant.

(3.c) The infinitesimal actions for $G(\epsilon) = Id + \epsilon A + O(\epsilon^2)$, where $A \in gl(n)$ are

$$X_A \mathbf{q} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} G(\epsilon)\mathbf{q} = A\mathbf{q}$$

and

$$X_A S = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \Big(G(\epsilon) S G(\epsilon)^T \Big) = AS + SA^T$$

The defining relation for the corresponding momentum map yields

$$\begin{aligned} \langle J, A \rangle = \langle (Q,P), X_A \rangle &= \operatorname{tr}\left(P X_A S\right) + \mathbf{p}\cdot X_A \mathbf{q} \\ &= \operatorname{tr}\big(P(AS + SA^T)\big) + \mathbf{p}\cdot A\mathbf{q} \end{aligned}$$

Hence, $\langle J, A \rangle := \operatorname{tr}\big(JA^T\big) = \operatorname{tr}\big((2SP + \mathbf{q}\otimes\mathbf{p})A\big)$, so

$$J = (2PS + \mathbf{p}\otimes\mathbf{q})$$

This momentum map is a cotangent lift, so it is equivariant.

(3.d) Conservation of the momentum map is verified directly by:

$$\dot{J} = (2\dot{P}S + 2P\dot{S} + \mathbf{p}\otimes\dot{\mathbf{q}}) = 0$$

4. Euler-Poincaré equation EPDiff in one dimension
The EPDiff(H^1) equation is obtained from the Euler-Poincaré reduction theorem for a right-invariant Lagrangian, when one defines this Lagrangian to be half the H^1 norm on the real line of the vector field of velocity $u = \dot{g}g^{-1}$, namely,

$$l(u) = \frac{1}{2}\|u\|^2_{H^1} = \frac{1}{2}\int_{-\infty}^{\infty} u^2 + u_x^2\,dx\,.$$

(Assume u vanishes as $|x| \to \infty$.)

(4.a) Derive the EPDiff(H^1) equation on the real line in terms of its velocity u and its momentum $m = \delta l/\delta u = u - u_{xx}$ in one spatial dimension.

(4.b) Use the Clebsch approach (hard constraint) to derive the peakon singular solution $m(x,t)$ of EPDiff(H^1) as a momentum map in terms of canonically conjugate variables $q_i(t)$ and $p_i(t)$, with $i = 1, 2, \ldots, N$.

4. Solution

(4.a) The EPDiff(H^1) equation is written on the real line in terms of its velocity u and its momentum $m = \delta l/\delta u$ in one spatial dimension as

$$m_t + um_x + 2mu_x = 0\,, \quad \text{where} \quad m = u - u_{xx}$$

where subscripts denote partial derivatives in x and t. This equation is derived from the variational principle with $l(u) = \frac{1}{2}\|u\|^2_{H^1}$ as follows.

$$
\begin{aligned}
0 = \delta S &= \delta \int l(u) dt = \frac{1}{2}\,\delta \iint u^2 + u_x^2 \, dx\, dt \\
&= \iint (u - u_{xx})\,\delta u \, dx\, dt =: \iint m\,\delta u\, dx\, dt \\
&= \iint m\,(\xi_t - \mathrm{ad}_u \xi)\, dx\, dt \\
&= \iint m\,(\xi_t + u\xi_x - \xi u_x)\, dx\, dt \\
&= -\iint (m_t + (um)_x + m u_x)\, \xi \, dx\, dt \\
&= -\iint (m_t + \mathrm{ad}^*_u m)\, \xi\, dx\, dt\,,
\end{aligned}
$$

where $u = \dot{g}g^{-1}$ implies $\delta u = \xi_t - \mathrm{ad}_u \xi$ with $\xi = \delta g g^{-1}$.

(4.b) The constrained Clebsch action integral is

$$
S(u,p,q) = \int l(u)\, dt + \int p(t)\big(\dot{q}(t) - u(q(t),t)\big) dt
$$

whose variation in u is gotten by inserting a delta function, so that

$$
\begin{aligned}
0 = \delta S &= \int \left(\frac{\delta l}{\delta u} - p(t)\delta(x - q(t)) \right) \delta u \, dx\, dt \\
&\quad - \int \left(\dot{p}(t) + \frac{\partial u}{\partial q}\, p(t) \right) \delta q - \delta p \Big(\dot{q}(t) - u(q(t),t) \Big) dt\,.
\end{aligned}
$$

The singular momentum solution $m(x,t)$ of EPDiff(H^1) is written as $m(x,t) = \delta l/\delta u = p(t)\delta(x - q(t))$ with canonical equations for (q,p),

$$
\dot{q}(t) = u(q(t),t) = \frac{\partial h}{\partial p}\,, \qquad \dot{p}(t) = -\frac{\partial u}{\partial q}\, p(t) = -\frac{\partial h}{\partial q}\,,
$$

with Hamiltonian $h(p,q) = \frac{1}{2}p^2 G(q)$ and $u(q(t),t) = p(t)G(q(t))$.