Cotangent-lift momentum maps

Background

Suppose a Lie group G acts on a manifold Q from the *left*, as

$$G \times Q \to Q : q_s = U_s q_0 \quad \text{for} \quad q \in Q, \quad U_{s=0} = Id \quad \text{and} \quad U \in G.$$

The tangent lift of this action is given by

$$q'_s\big|_{s=0} = \big[U'_s U_s^{-1} q_s\big]_{s=0} =: \pounds_\xi q =: \xi_Q(q),$$

with $\xi \in \mathfrak{g}$, the Lie algebra corresponding to the tangent space of G at the identity U_0 and $\pounds_\xi q$ is the Lie derivative of $q \in Q$ with respect to $\xi \in \mathfrak{g}$. Sometimes these relations are encoded by writing $U_s = \exp(s\xi)$. The quantity $\xi_Q(q)$ is called the infinitesimal generator of the Lie group action.

Consider Hamilton's principle defined on $\mathfrak{g} \times TQ$ by the action integral

$$S(\xi; p, q) = \int \Big(l(\xi, q) + \Big\langle p \,,\, \dot{q} - \pounds_\xi q \Big\rangle_Q \Big) dt. \tag{1}$$

The action integral $S(\xi; p, q)$ contains a Lagrangian l and a constraint enforced by pairing

$$\langle \,\cdot\,,\,\cdot\,\rangle_Q : T^*Q \times TQ \to \mathbb{R}.$$

In terms of this pairing, the tangent lift of the action of G on Q is enforced as a *Clebsch constraint*, in which the momentum $p \in T^*Q$ canonically conjugate to $q \in Q$ is used as a Lagrange multiplier.

Notation: In preparation for taking the variations in Hamilton's principle, $\delta S = 0$, we shall define some notation.

- The diamond-operation $(\,\diamond\,)$ is defined as

$$\Big\langle p \,,\, -\pounds_{\delta\xi} q \Big\rangle_Q =: \Big\langle p \diamond q \,,\, \delta\xi \Big\rangle_{\mathfrak{g}} \,,$$

  for the *two* pairings $\langle \cdot\,,\,\cdot\rangle_Q : T^*Q \times TQ \to \mathbb{R}$ and $\langle \cdot\,,\,\cdot\rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$.

- The dual (or transpose) $\pounds_\xi^T$ of the Lie derivative $\pounds_\xi$ with respect to the pairing on $T^*Q \times TQ$ is defined as

$$\Big\langle p \,,\, -\pounds_\xi \delta q \Big\rangle_Q =: \Big\langle -\pounds_\xi^T p \,,\, \delta q \Big\rangle_Q$$

In this notation, the variation of the action integral in (1) may be expressed as

$$\begin{aligned} \delta S(\xi; p, q) \;\; &= \int \bigg(\Big\langle \frac{\partial l}{\partial \xi} + p \diamond q \,,\, \delta\xi \Big\rangle_{\mathfrak{g}} + \Big\langle \delta p \,,\, \dot{q} - \pounds_\xi q \Big\rangle_Q \\ &+ \;\; \Big\langle \frac{\partial l}{\partial q} - \dot{p} - \pounds_\xi^T p \,,\, \delta q \Big\rangle_Q \bigg) dt \end{aligned}$$

Momentum maps

Proposition 1. *The quantity defined by the pairing*

$$J^\eta := \Big\langle -\, p \diamond q \,,\, \eta \Big\rangle_{\mathfrak{g}} =: \Big\langle J \,,\, \eta \Big\rangle_{\mathfrak{g}}$$

is the Hamiltonian *for the action of the Lie algebra* $\mathfrak{g}$ *on* T^*Q.

Proof. One computes the Hamiltonian vector field for J^η with fixed η as

$$(\dot q,\, \dot p) = \left(\frac{\partial J^\eta}{\partial p},\, -\frac{\partial J^\eta}{\partial q} \right) = \Big(\pounds_\eta q,\, -\pounds_\eta^T p \Big),$$

from the variations

$$\begin{aligned}
\delta J^\eta &= \Big\langle -\, \delta p \diamond q \,,\, \eta \Big\rangle_{\mathfrak{g}} + \Big\langle -\, p \diamond \delta q \,,\, \eta \Big\rangle_{\mathfrak{g}} \\
&= \Big\langle \delta p \,,\, \pounds_\eta q \Big\rangle_Q + \Big\langle p \,,\, \pounds_\eta \delta q \Big\rangle_Q \\
&= \Big\langle \delta p \,,\, \pounds_\eta q \Big\rangle_Q + \Big\langle \pounds_\eta^T p \,,\, \delta q \Big\rangle_Q .
\end{aligned}$$

Remark 2. *The relation* $\partial l/\partial \xi = -\, p \diamond q$ *defines the map* $J : T^*Q \to \mathfrak{g}^*$*. This is the **cotangent-lift momentum map** for left action of Lie group* G *on manifold* Q *discussed in the lectures.*

The ***evolution equation*** for the cotangent-lift momentum map $J = -\, p \diamond q$ may be computed as

$$\begin{aligned}
\Big\langle \dot J \,,\, \eta \Big\rangle &= \Big\langle -\, \dot p \diamond q - p \diamond \dot q \,,\, \eta \Big\rangle \\
&= \Big\langle \pounds_\xi^T p \diamond q - p \diamond \pounds_\xi q - \frac{\partial l}{\partial q} \diamond q \,,\, \eta \Big\rangle \\
&= \Big\langle \pounds_\xi^T p \,,\, -\pounds_\eta q \Big\rangle + \Big\langle p \,,\, \pounds_\eta \pounds_\xi q \Big\rangle - \Big\langle \frac{\partial l}{\partial q} \diamond q \,,\, \eta \Big\rangle \\
&= \Big\langle p \,,\, -\pounds_{(\mathrm{ad}_\xi \eta)} q \Big\rangle - \Big\langle \frac{\partial l}{\partial q} \diamond q \,,\, \eta \Big\rangle \\
&= \Big\langle p \diamond q \,,\, \mathrm{ad}_\xi \eta \Big\rangle - \Big\langle \frac{\partial l}{\partial q} \diamond q \,,\, \eta \Big\rangle \\
&= \Big\langle \mathrm{ad}_\xi^* (p \diamond q) - \frac{\partial l}{\partial q} \diamond q \,,\, \eta \Big\rangle \\
&= \Big\langle -\, \mathrm{ad}_\xi^* J - \frac{\partial l}{\partial q} \diamond q \,,\, \eta \Big\rangle \qquad \text{for any} \quad \eta \in \mathfrak{g} .
\end{aligned}$$

This produces the ***Euler-Poincaré equation with advected quantities*** (q) acted on from the *left* by the group G and moving with *right-invariant* velocity ξ. Namely, for a given $l(\xi, q)$,

$$\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathrm{ad}_\xi^* \frac{\partial l}{\partial \xi} + \frac{\partial l}{\partial q} \diamond q = 0 \qquad \text{and} \qquad \frac{dq}{dt} = \pounds_\xi q \,.$$

Of course, when $\partial l/\partial q = 0$, the system reduces to the usual ***Euler-Poincaré equation***. The cotangent-lift momentum map conveys the *Hamiltonian meaning* of the Euler-Poincaré equation.