Imperial College London

UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2011

M3/4A16

Geometric Mechanics I

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This paper is also taken for the relevant examination for the Associateship.

M3/4A16

Geometric Mechanics I

Date: Time:

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The Fish: A quadratically nonlinear oscillator

Consider the Hamiltonian dynamics on the symplectic manifold of a system with two degrees of freedom. In real phase space variables (x, y, θ, z), the symplectic form is

$$\omega = dx \wedge dy + d\theta \wedge dz$$

and Hamiltonian is

$$H = \tfrac{1}{2}y^2 + x\left(\tfrac{1}{3}x^2 - z\right) - \tfrac{2}{3}z^{3/2}$$

(a) Write the canonical Poisson bracket for this system.

(b) Write Hamilton's canonical equations for this system. Explain how to keep $z \geq 0$, so that H and θ remain real.

(c) At what values of x, y and H does the system have stationary points in the (x, y) plane?

(d) Propose a strategy for solving these equations. In what order should they be solved?

(e) Identify the constants of motion of this system and explain why they are conserved.

(f) Compute the associated Hamiltonian vector field X_H and show that it satisfies

$$X_H \,\lrcorner\, \omega = dH$$

(g) Write the Poisson bracket that expresses the Hamiltonian vector field X_H as a divergencefree vector field in $\mathbb{R}^3$ with coordinates $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Explain why this Poisson bracket satisfies the Jacobi identity.

(h) Identify the Casimir function for this $\mathbb{R}^3$ bracket. Show explicitly that it satisfies the definition of a Casimir function.

(i) Sketch a graph of the intersections of the level surfaces in $\mathbb{R}^3$ of the Hamiltonian and Casimir function. Show the directions of flow along these intersections. Identify the locations and types of any relative equilibria at the tangent points of these surfaces.

(j) Linearise around the relative equilibria on a level set of the Casimir (z), compute the eigenvalues to verify the locations and types of relative equilibria proposed in Part (i).

2. $\mathbb{R}^3$ bracket for the spherical pendulum.

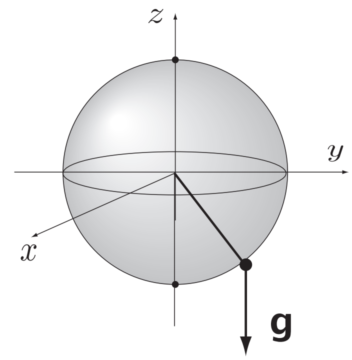

Figure 1: Spherical pendulum in $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. The mass of the pendulum bob is unity ($m = 1$).

(a) Derive the motion equation $\ddot{\mathbf{x}} = -g\hat{\mathbf{e}}_3 + \mu\mathbf{x}$ for the spherical pendulum from Hamilton's principle for the Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}}) : T\mathbb{R}^3 \to \mathbb{R}$ given by

$$L(\mathbf{x}, \dot{\mathbf{x}}; \mu) = \tfrac{1}{2}|\dot{\mathbf{x}}|^2 - g\hat{\mathbf{e}}_3 \cdot \mathbf{x} - \tfrac{1}{2}\mu(1 - |\mathbf{x}|^2),$$

in which g is the acceleration of gravity, $\hat{\mathbf{e}}_3$ is the vertical unit vector and the Lagrange multiplier μ constrains the motion to remain on the sphere S^2.

(b) Determine the Lagrange multiplier μ by requiring that these equations preserve the defining conditions for $T\mathbb{S}^2$,

$$T\mathbb{S}^2 : \left\{(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3 \,\Big|\, \|\mathbf{x}\|^2 = 1 \text{ and } \mathbf{x} \cdot \dot{\mathbf{x}} = 0\right\},$$

so that $T\mathbb{S}^2$ is an invariant manifold of the equations for a spherical pendulum in $\mathbb{R}^3$.

(c) Show via Noether's theorem that S^1 symmetry of this Lagrangian under rotations about the vertical axis implies conservation of the vertical component of angular momentum. Identify this quantity explicitly.

(d) Legendre transform the Lagrangian defined on $T\mathbb{R}^3$ to find its constrained Hamiltonian (Routhian) with variables $(\mathbf{x}, \mathbf{y}) \in T^*\mathbb{R}^3$ whose dynamics preserves $T\mathbb{S}^2$.

(e) A basis of six linear and quadratic forms for S^1-invariant polynomials in $T^*\mathbb{R}^3/S^1$ is

$$\begin{aligned} \sigma_1 &= x_3 & \sigma_3 &= y_1^2 + y_2^2 + y_3^2 & \sigma_5 &= x_1y_1 + x_2y_2 \\ \sigma_2 &= y_3 & \sigma_4 &= x_1^2 + x_2^2 & \sigma_6 &= x_1y_2 - x_2y_1 \end{aligned}$$

These S^1-invariant variables are not independent. They satisfy a cubic algebraic relation. Find this relation and write the TS^2 constraints in terms of the S^1 invariants.

(f) Write closed Poisson brackets among the six independent linear and quadratic S^1-invariant variables

$$\sigma_k \in T^*\mathbb{R}^3/S^1, \quad k = 1, 2, \ldots, 6.$$

(g) Show that the two quantities

$$\sigma_3(1-\sigma_1^2)-\sigma_2^2-\sigma_6^2=0 \quad \text{and} \quad \sigma_6$$

are Casimirs for the Poisson brackets on $T^*\mathbb{R}^3/S^1$ found in Part (f).

(h) Use the orbit map $T\mathbb{R}^3 \to \mathbb{R}^6$

$$\pi:\ (\mathbf{x},\mathbf{y}) \to \{\sigma_j(\mathbf{x},\mathbf{y}),\ j=1,\dots,6\}$$

to transform the energy Hamiltonian to S^1-invariant variables.

(i) Write the equations of motion in terms of the variables $\sigma_k \in T^*\mathbb{R}^3/S^1$, $k=1,2,3$.

(j) Reduce the dynamics to single particle motion in a phase plane on level sets of the Hamiltonian in $T^*\mathbb{R}^3/S^1$.

3. Poisson brackets for 1:1 invariants

(a) Use the canonical Poisson brackets $\{q_i,\, p_j\} = \delta_{ij}$ to compute the Poisson brackets $\{Y_1, Y_2\}$, etc. among the three S^1-invariant quadratic phase space functions for a 1:1 resonance

$$Y_1 + iY_2 = 2a_1^* a_2 \quad \text{and} \quad Y_3 = |a_1|^2 - |a_2|^2 \,, \tag{1}$$

with $a_k := q_k + ip_k \in \mathbb{C}^1$ for $k = 1, 2$.

(b) Show that these Poisson brackets may be expressed as a closed system

$$\{Y_i, Y_j\} = c_{ij}^k Y_k, \qquad i, j, k = 1, 2, 3, \tag{2}$$

in terms of these invariants, by computing the coefficients c_{ij}^k.

(c) Write the Poisson brackets $\{Y_i, Y_j\}$ among these invariants as a 3×3 skew-symmetric table.

(d) Write the Poisson brackets for functions of these three invariants (Y_1, Y_2, Y_3) as a vector cross product of gradients of functions of $\mathbf{Y} \in \mathbb{R}^3$.

(e) Take the Poisson brackets of the three invariants (Y_1, Y_2, Y_3) with the function,

$$R = |a_1|^2 + |a_2|^2 \,. \tag{3}$$

Explain your answers geometrically in terms of vectors in $\mathbb{R}^3$.

(f) Write the results of applying the Poisson brackets in the form

$$\{\,\mathbf{a}\,,\, Y_k\} = c_k \mathbf{a} \quad k = 1, 2, 3\,,$$

for $\mathbf{a} = (a_1, a_2)^T \in \mathbb{C}^2$ and 2×2 matrices c_k, with $k = 1, 2, 3$. Identify the type of matrix that results (symmetric, skew symmetric, etc.) Write a 3×3 skew-symmetric table of their matrix commutation relations $[c_i,\, c_j]$, etc. Compare it with the table of Poisson brackets $\{Y_i, Y_j\}$ in Part (c).

(g) Show that the flows

$$\phi_k : \mathbf{z}(t) = e^{c_k t}\mathbf{z}(0) = \sum_{n=0}^{\infty} \frac{1}{n!}(c_k t)^n \mathbf{z}(0)$$

of the Hamiltonian vector fields $\{\,\cdot\,,\, Y_k\}$ arising from the three S^1 phase invariant quadratic phase space functions in (1) acting on the phase space vector $\mathbf{a} = (a_1, a_2)^T \in \mathbb{C}^2$ may be written as $SU(2)$ matrix transformations $\mathbf{a}(t) = U(t)\mathbf{a}(0)$, with $U^\dagger U = \mathrm{Id}$.

4. Lie derivative relations.

Recall that the pull-back ϕ_t^* of a smooth flow ϕ_t generated by a smooth vector field X defined on a smooth manifold M commutes with exterior derivative, wedge product and contraction. That is, for k-forms α, $\beta \in \Lambda^k(M)$, and $m \in M$, the pull-back ϕ_t^* satisfies

$$\begin{aligned} d(\phi_t^*\alpha) &= \phi_t^* d\alpha\,, \\ \phi_t^*(\alpha \wedge \beta) &= \phi_t^*\alpha \wedge \phi_t^*\beta\,, \\ \phi_t^*(X(m) \,\lrcorner\, \alpha) &= X(\phi_t(m)) \,\lrcorner\, \phi_t^*\alpha\,. \end{aligned}$$

Recall that the Lie derivative $\pounds_X\alpha$ of a k-form $\alpha \in \Lambda^k(M)$ by the vector field X tangent to the flow ϕ_t on M is defined as

$$\pounds_X\alpha = \frac{d}{dt}\bigg|_{t=0} (\phi_t^*\alpha) = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha)\,.$$

Verify the following Lie derivative relations:

(a) $\pounds_{fX}\alpha = f\pounds_X\alpha + df \wedge (X \,\lrcorner\, \alpha)$

(b) $\pounds_X d\alpha = d\big(\pounds_X\alpha\big)$

(c) $\pounds_X(X \,\lrcorner\, \alpha) = X \,\lrcorner\, \pounds_X\alpha$

(d) $\pounds_X(Y \,\lrcorner\, \alpha) = (\pounds_X Y) \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X\alpha)$

(e) $\pounds_X(\alpha \wedge \beta) = (\pounds_X\alpha) \wedge \beta + \alpha \wedge \pounds_X\beta$

(f) $[X\,,\,Y] \,\lrcorner\, \alpha = \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha)$

(g) Use Part (f) to verify $\pounds_{[X\,,\,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$

(h) Use Part (g) to verify the Jacobi identity for the Lie derivative.