Imperial College London

UNIVERSITY OF LONDON

Course: M3/4A16
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Date: March 12, 2010

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2010

M3/4A16

Geometric Mechanics I

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This paper is also taken for the relevant examination for the Associateship.

M3/4A16

Geometric Mechanics I

Date: | | Time: | |

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The Planar Isotropic Simple Harmonic Oscillator (PISHO) satisfies Hamilton's canonical equations for $(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2$,

$$\dot{\mathbf{q}} = \{\mathbf{q},\, H\} = \frac{\partial H}{\partial \mathbf{p}} \quad \text{and} \quad \dot{\mathbf{p}} = \{\mathbf{p},\, H\} = -\frac{\partial H}{\partial \mathbf{q}}\,,$$

with Hamiltonian function $H : T^*\mathbb{R}^2 \to \mathbb{R}$ given by

$$H = \tfrac{1}{2}\big(|\mathbf{p}|^2 + |\mathbf{q}|^2\big).$$

(a) Show that the following transformations of coordinates are each canonical.

1. $T^*\mathbb{R}^2\ \{0\} \to T^*\mathbb{R}_+ \times T^*S^1$, given by $x + iy = re^{i\theta}\,, \quad p_x + ip_y = (p_r + ip_\theta/r)e^{i\theta}$.
2. $T^*\mathbb{R}^2 \to \mathbb{C}^2$, given by $a_k = q_k + ip_k\,, \quad a_k^* = q_k - ip_k$, with $k = 1, 2$.
3. $T^*\mathbb{R}^2\ \{0\} \to \mathbb{R}^2_+ \times T^2$, given by $I_k = \frac{1}{2}(q_k^2 + p_k^2)\,, \quad \phi_k = \tan^{-1}(p_k/q_k)$, with $k = 1, 2$.

(b) Write the Hamiltonian equations for PISHO using each of the canonical coordinate systems of the previous part.

(c) Use Stokes Law to compute the phase space area $\iint (dp_r \wedge dr + dp_\theta \wedge d\theta)$ of a periodic orbit in terms of the Hamiltonian, orbital period and the time average over the orbit of the squared radius given by the integral $\int_0^T r^2 dt = T\langle r^2 \rangle$.

2. This is more about Simple Harmonic Oscillators.

(a) Write the Poisson brackets among the variables for PISHO in the following two S^1-invariant cases

1. $(X_1, X_2, X_3) = (|\mathbf{q}|^2, |\mathbf{p}|^2, \mathbf{q}\cdot\mathbf{p})$ and $p_\theta^2 = |\mathbf{p}\times\mathbf{q}|^2$ with $(\mathbf{q},\mathbf{p}) \in \mathbb{R}^2 \times \mathbb{R}^2$.

2. $R = |a_1|^2 + |a_2|^2$, $Y_1 + iY_2 = 2a_1^* a_2$ and $Y_3 = |a_1|^2 - |a_2|^2$, with $a_k := q_k + ip_k \in \mathbb{C}^1$ for $k = 1, 2$.

(b) Write the Hamiltonian forms of the PISHO equations in terms of the Poisson brackets for the two sets of S^1-invariant quantities in the previous part.

(c) The Hamiltonian for the Planar Anisotropic Simple Harmonic Oscillator (PASHO) is,

$$\begin{aligned} H &= \frac{\omega_1}{2}(R + Y_3) + \frac{\omega_2}{2}(R - Y_3) \\ &= \frac{1}{2}(\omega_1 + \omega_2)R + \frac{1}{2}(\omega_1 - \omega_2)Y_3\,. \end{aligned}$$

1. Write Hamilton's equations for $\dot{Y}_1, \dot{Y}_2, \dot{Y}_3, \dot{R}$ in terms of the Poisson brackets for the S^1 invariants Y_1, Y_2, Y_3, R.
2. Write these equations equivalently in $\mathbb{R}^3$ vector form, with $\mathbf{Y} = (Y_1, Y_2, Y_3)^T$.
3. Describe this motion in terms of intersections in $\mathbb{R}^3$ of level sets of the Poincaré sphere and the PASHO Hamiltonian H above.

3. Newton's equation for the **reduced Kepler problem** for planetary motion is

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = 0\,, \tag{1}$$

in which μ is a constant and $r = |\mathbf{r}|$ with $\mathbf{r} \in \mathbb{R}^3$.

(a) Show that Newton's equation (1) conserves the quantities,

$$\begin{aligned} E &= \frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{\mu}{r} \quad \text{(energy)}\,, \\ \mathbf{L} &= \mathbf{r} \times \dot{\mathbf{r}} \quad \text{(specific angular momentum)}\,. \end{aligned}$$

Since, $\mathbf{r} \cdot \mathbf{L} = 0$, the planetary motion in $\mathbb{R}^3$ takes place in a plane to which vector $\mathbf{L}$ is perpendicular. This is the orbital plane. Constancy of magnitude $L = |\mathbf{L}|$ means the orbit sweeps out equal areas in equal times. This is **Kepler's Second Law**.

(b) The unit vectors for polar coordinates in the orbital plane are $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. Show that these vectors satisfy

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\,\hat{\boldsymbol{\theta}} \quad \text{and} \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\,\dot{\theta}\,\hat{\mathbf{r}}\,, \quad \text{where} \quad \dot{\theta} = \frac{L}{r^2}\,.$$

Show that Newton's equation (1) also conserves the following two vector quantities,

$$\begin{aligned} \mathbf{K} &= \dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}} \quad \text{(Hamilton's vector)}, \\ \mathbf{J} &= \dot{\mathbf{r}} \times \mathbf{L} - \mu\mathbf{r}/r \quad \text{(Laplace-Runge-Lenz vector, or LRL vector)}\,, \end{aligned}$$

which both lie in the orbital plane, since $\mathbf{J} \cdot \mathbf{L} = 0 = \mathbf{K} \cdot \mathbf{L}$.

(c) From their definitions, show that these conserved quantities are related by

$$L^2 + \frac{J^2}{(-2E)} = \frac{\mu^2}{(-2E)} \quad \text{and} \quad \mathbf{J} \cdot \mathbf{K} \times \mathbf{L} = K^2L^2 = J^2\,, \tag{2}$$

where $J^2 := |\mathbf{J}|^2$, etc. and $-2E > 0$ for bounded orbits.

(d) Choose the conserved LRL vector $\mathbf{J}$ in the orbital plane to point along the reference line for the measurement of the polar angle θ, say from the center of the orbit (Sun) to the perihelion (point of nearest approach), so that

$$\mathbf{r} \cdot \mathbf{J} = rJ\cos\theta = \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{L} - \mu\,\mathbf{r}/r)\,.$$

Use this relation to write the Kepler orbit $r(\theta)$ in plane polar coordinates. Show that the orbit $r(\theta)$ is a conic section. This is **Kepler's First Law**.

4. The real-valued Maxwell-Bloch system for $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ is given by

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = x_1 x_3\,, \quad \dot{x}_3 = -x_1 x_2\,.$$

(a) Write this system in three-dimensional vector $\mathbb{R}^3$-bracket notation as

$$\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2\,,$$

where H_1 and H_2 are two conserved functions. Show that the level sets of one of these (let it be H_1) are **circular cylinders** oriented along the x_1-direction and that the level sets of the other (let it be H_2) are **parabolic cylinders** oriented along the x_2-direction.

(b) Restrict the equations and their $\mathbb{R}^3$ Poisson bracket to a level set of H_2. Show that the Poisson bracket on the parabolic cylinder $H_2 = const$ is symplectic.

(c) Derive the equation of motion on a level set of H_2 and express them in the form of Newton's Law. Do they reduce to something familiar?

(d) Identify steady solutions and determine which are unstable (saddle points) and which are stable (centers).