1 M345 PA16 Assessed Coursework #1 Spring Term 2020

Exercise 1.1 Compute the Euler-Lagrange equations from Hamilton's principle for the following two simple mechanical systems: $L(q,\dot q) = T(\dot q) - V(q) = KE - PE$.

1. Charged particle in a magnetic field, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$:
   $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 + \frac{e}{c}\dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}) \quad\Longrightarrow\quad \ddot{\mathbf{x}} = \frac{e}{mc}\dot{\mathbf{x}}\times\mathbf{B}$ with $\mathbf{B} = \operatorname{curl}\mathbf{A}$

2. Spherical pendulum: $L(\mathbf{x},\dot{\mathbf{x}}) = \frac12|\dot{\mathbf{x}}|^2 - g\,\hat{\mathbf{e}}_3\cdot\mathbf{x}$ with $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^3$, constrained to $TS^2 = \{\mathbf{x},\dot{\mathbf{x}} \in T\mathbb{R}^3 \,|\, 1 - |\mathbf{x}|^2 = 0 \;\&\; \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}$.

   One may transform motion in $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^3$ to motion on TS^2, by noticing that rotations in $SO(3)$ leave the sphere invariant. Thus, motion on the unit sphere may be expressed as motion on $SO(3)$ starting from an initial point $\mathbf{x}_0$ with $|\mathbf{x}_0|^2 = 1$. Namely,

   $$\mathbf{x}(t) = O(t)\mathbf{x}_0\,,\quad \dot{\mathbf{x}}(t) = \dot O(t)\mathbf{x}_0 \quad\text{for}\quad (O,\,\dot O) \in TSO(3)\,,$$

   where $\mathbf{x}_0 = \mathbf{x}(0)$ is the initial position of the pendulum bob and $O^T = O^{-1}$.

   (a) Notice that the Lagrangian for the spherical pendulum is *not* invariant under the full $SO(3)$. This is because the presence of gravity breaks the symmetry of the Lagrangian from $SO(3)$ to the $SO(2)$ subgroup of rotations around the vertical direction, $\hat{\mathbf{e}}_3$. What to do? We will need to keep track of the evolution of $\boldsymbol{\Gamma}(t) := O^{-1}(t)\hat{\mathbf{e}}_3$, which is the vertical direction, as seen from the direction of the pendulum bob.
   Compute the equation of motion for $\boldsymbol{\Gamma}(t)$.

   (b) Transform the Lagrangian for the spherical pendulum from $T\mathbb{R}^3$ to $T_eSO(3) \simeq \mathfrak{so}(3)$.

   $$\begin{aligned} L(\mathbf{x},\dot{\mathbf{x}}) &= \frac12|\dot{\mathbf{x}}|^2 - g\,\hat{\mathbf{e}}_3\cdot\mathbf{x} = \frac12|\dot O(t)\mathbf{x}_0|^2 - g\,O^T(t)\hat{\mathbf{e}}_3\cdot\mathbf{x}_0 =: L(O(t),\dot O(t);\hat{\mathbf{e}}_3) \\ &= \frac12|O^{-1}\dot O(t)\mathbf{x}_0|^2 - g\,(O^{-1}(t)\hat{\mathbf{e}}_3)\cdot\mathbf{x}_0 =: L(O^{-1}\dot O(t);O^{-1}\hat{\mathbf{e}}_3) \end{aligned} \tag{1}$$

   Setting $\mathbf{x}(t) = O(t)\mathbf{x}_0$ avoids the need for the constraint $|\mathbf{x}|^2 = 1$, since rotations preserve length. However, the presence of gravity requires an equation for $\boldsymbol{\Gamma}(t) = O^{-1}\hat{\mathbf{e}}_3$.

   (c) For $O^{-1}\dot O(t) := \widehat\Omega(t) = \boldsymbol{\Omega}(t)\times$ (by the hat map $\widehat\Omega_{ij} = -\epsilon_{ijk}\Omega_k$) and defining $\boldsymbol{\Gamma}(t) := O^{-1}(t)\hat{\mathbf{e}}_3$), show that

   $$\begin{aligned} L(Id, O^{-1}\dot O(t);O^{-1}\mathbf{x}_0) &= \frac12|\widehat\Omega(t)\mathbf{x}_0|^2 - g\,(O^{-1}(t)\hat{\mathbf{e}}_3)\cdot\mathbf{x}_0 \\ &= \frac12|\boldsymbol{\Omega}(t)\times\mathbf{x}_0|^2 - g\,\boldsymbol{\Gamma}(t)\cdot\mathbf{x}_0 =: \ell(\boldsymbol{\Omega}(t),\boldsymbol{\Gamma}(t))\,. \end{aligned} \tag{2}$$

   (d) From its definition $\boldsymbol{\Gamma} := O^{-1}(t)\hat{\mathbf{e}}_3$ show that $\boldsymbol{\Gamma}$ satisfies

   $$\dot{\boldsymbol{\Gamma}} := -\,\widehat\Omega\boldsymbol{\Gamma} = -\,\boldsymbol{\Omega}\times\boldsymbol{\Gamma} \quad\text{and}\quad \delta\boldsymbol{\Gamma} = -\,\widehat\Xi\,\boldsymbol{\Gamma} = -\,\boldsymbol{\Xi}\times\boldsymbol{\Gamma}\,,$$

   with $\widehat\Xi = O^{-1}\delta O = \boldsymbol{\Xi}\times$.

   (e) From their definitions $\widehat\Omega := O^{-1}\dot O = \boldsymbol{\Omega}\times$ and $\widehat\Xi = O^{-1}\delta O = \boldsymbol{\Xi}\times$ show that $\widehat\Omega$ and $\widehat\Xi$ satisfy

   $$\delta\widehat\Omega = \frac{d\widehat\Xi}{dt} + \big[\widehat\Omega\,,\,\widehat\Xi\big]\,,\quad \delta\boldsymbol{\Omega} = \frac{d\boldsymbol{\Xi}}{dt} + \boldsymbol{\Omega}\times\boldsymbol{\Xi}$$

   (f) Define

   $$\boldsymbol{\Pi}(t) := \frac{\partial\ell}{\partial\boldsymbol{\Omega}} = \mathbf{x}_0\times\big(\boldsymbol{\Omega}(t)\times\mathbf{x}_0\big)\,.$$

and use Hamilton's principle with the reduced Lagrangian $0 = \delta S_{red} = \delta \int_a^b \ell(\boldsymbol{\Omega}(t), \boldsymbol{\Gamma}(t))\, dt$ to derive the following motion equation for the spherical pendulum

$$\dot{\boldsymbol{\Pi}} + \boldsymbol{\Omega} \times \boldsymbol{\Pi} = g\, \boldsymbol{\Gamma} \times \mathbf{x}_0 \,, \quad \text{with auxiliary equation} \quad \dot{\boldsymbol{\Gamma}} = -\, \boldsymbol{\Omega} \times \boldsymbol{\Gamma} \,.$$

Note that these equations can be written in matrix operator form as

$$\frac{d}{dt} \begin{bmatrix} \boldsymbol{\Pi} \\ \boldsymbol{\Gamma} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Pi}\times & \boldsymbol{\Gamma}\times \\ \boldsymbol{\Gamma}\times & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega} \\ g\mathbf{x}_0 \end{bmatrix} .$$

We will see this pattern again when we write these equations in Hamiltonian form as

$$\frac{d}{dt} \begin{bmatrix} \boldsymbol{\Pi} \\ \boldsymbol{\Gamma} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Pi}\times & \boldsymbol{\Gamma}\times \\ \boldsymbol{\Gamma}\times & 0 \end{bmatrix} \begin{bmatrix} \partial h / \partial \boldsymbol{\Pi} \\ \partial h / \partial \boldsymbol{\Gamma} \end{bmatrix} .$$

Exercise 1.2

(a) Compute the Legendre transforms

$$p := \frac{\partial L}{\partial \dot{q}} \,, \quad H(q,p) = \langle p,\, \dot{q} \rangle - L(q, \dot{q})$$
$$\boldsymbol{\Pi} := \frac{\partial \ell}{\partial \boldsymbol{\Omega}} \,, \quad h(\boldsymbol{\Pi},\, \boldsymbol{\Gamma}) = \langle \boldsymbol{\Pi},\, \boldsymbol{\Omega} \rangle - \ell(\boldsymbol{\Omega}, \boldsymbol{\Xi})$$

for the two simple mechanical systems in Exercise 1.1.

(b) Compute the Hamiltonian equations for each system and show equivalence with their corresponding Euler-Lagrange equations.

1. Charged particle in a magnetic field, $(\mathbf{x}, \mathbf{p}) \in T^*\mathbb{R}^2$: $H = \frac{1}{2m}|\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x})|^2$ $\mathbf{p} := \partial L / \partial \dot{\mathbf{x}} = m\dot{\mathbf{x}} + \frac{e}{c}\mathbf{A}(\mathbf{x}) \in T^*M$

2. Spherical pendulum, $(O,\, \dot{O}) \in TSO(3)$, $\widehat{\Omega} = O^{-1}\dot{O} \in T(SO(3) \simeq \mathfrak{so}(3)$, $\boldsymbol{\Pi} = \partial \ell / \partial \boldsymbol{\Omega} \in T^*(SO(3) \simeq \mathfrak{so}(3)^* \simeq \mathbb{R}^3$, $H = \frac{1}{2}\boldsymbol{\Pi} \cdot I^{-1}\boldsymbol{\Pi} + g\, \boldsymbol{\Gamma} \cdot \mathbf{x}_0$ with $\boldsymbol{\Pi} := \frac{\partial \ell}{\partial \boldsymbol{\Omega}} = I\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}(t) = O^{-1}\hat{\mathbf{e}}_3$.

   This is done most easily from the symmetry reduced Hamilton's principle with Frobenius pairing of matrices $\big\langle A\,,\, B \big\rangle = \frac{1}{2}\mathrm{tr}(A^T B)$,

$$0 = \delta S_{red} = \delta \int_a^b \big\langle \widehat{\Pi}\,,\, O^{-1}\dot{O} \big\rangle - h(\widehat{\Pi}, O^{-1}(t)\hat{\mathbf{e}}_3)\, dt \,.$$

   One first shows that

$$\delta(O^{-1}\dot{O}) = \frac{d}{dt}\widehat{\Xi} + [\widehat{\Omega}, \widehat{\Xi}] \quad \text{and} \quad \delta(O^{-1}(t)\hat{\mathbf{e}}_3)_i = -\,\widehat{\Xi}_{ij}\Gamma^j = -\,(\boldsymbol{\Xi} \times \boldsymbol{\Gamma})_i$$

   Then, after using the hat map to prove $\big\langle \widehat{\Pi}\,,\, [\widehat{\Omega}, \widehat{\Xi}] \big\rangle = \boldsymbol{\Pi} \cdot (\boldsymbol{\Omega} \times \boldsymbol{\Xi})$, one finds

$$\delta \boldsymbol{\Omega} = \frac{d\boldsymbol{\Xi}}{dt} + \boldsymbol{\Omega} \times \boldsymbol{\Xi}$$

$$0 = \delta S_{red} = \int_a^b \left(\delta\boldsymbol{\Pi} \cdot \Big(\boldsymbol{\Omega} - \frac{\partial h}{\partial \boldsymbol{\Pi}} \Big) + \boldsymbol{\Pi} \cdot \Big(\frac{d\boldsymbol{\Xi}}{dt} + \boldsymbol{\Omega} \times \boldsymbol{\Xi} \Big) + \frac{\partial h}{\partial \boldsymbol{\Gamma}} \cdot (\boldsymbol{\Xi} \times \boldsymbol{\Gamma}) \right) dt \,.$$

   Show by integrating by parts and rearranging to factor out $\boldsymbol{\Xi}$ that this version of Hamilton's principle recovers the last matrix equation appearing in the previous exercise.

   Compute the corresponding Lie–Poisson bracket by contracting that matrix equation with the covector of derivatives $(\partial f / \partial \boldsymbol{\Pi}\,,\, \partial h / \partial \boldsymbol{\Gamma})$ to compute $df(\boldsymbol{\Pi},\, \boldsymbol{\Gamma})/dt = \{f, h\}(\boldsymbol{\Pi},\, \boldsymbol{\Gamma})$.