1 M3-4-5A16 Assessed Homework Set # 1 Autumn Term 2018

Exercise 1.1 Compute the Euler-Lagrange equations from Hamilton's principle for any two of the first five and any three of the second five of the following simple mechanical systems:

$$L(q,\dot q) = T(\dot q) - V(q) = KE - PE.$$

1. Planar isotropic oscillator, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$:
$L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - \frac{k}{2}|\mathbf{x}|^2 \quad\Longrightarrow\quad \ddot{\mathbf{x}} = -\,\omega^2\mathbf{x} \quad \text{with} \quad \omega^2 = k/m$

2. Planar anisotropic oscillator, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$:
$L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - \frac{k_1}{2}x_1^2 - \frac{k_2}{2}x_2^2 \quad\Longrightarrow\quad \ddot{x}_i = -\,\omega_i^2 x_i \quad \text{with} \quad \omega_i^2 = k_i/m \quad i = 1,2$

3. Planar pendulum, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$, constrained to $TS^1 = \{\mathbf{x},\dot{\mathbf{x}} \in T\mathbb{R}^2 |\, 1 - |\mathbf{x}|^2 = 0 \;\&\; \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}$:
$L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_2\cdot\mathbf{x} - \frac{\mu}{2}(1-|\mathbf{x}|^2) \quad\Longrightarrow\quad m\ddot{\mathbf{x}} = -\,mg\hat{\mathbf{e}}_2(\mathrm{Id} - \mathbf{x}\otimes\mathbf{x}) - \mathrm{m}|\dot{\mathbf{x}}|^2\mathbf{x}$, (gravity & centripetal force)

4. Planar pendulum motion lifted to a curve in $SO(2)$: $\mathbf{x}(t) = O(\theta(t))\mathbf{x}_0 \in \mathbb{R}^2\,,\quad O(\theta(t)) \in SO(2)\,,\quad |\mathbf{x_0}|^2 = R^2$, where $\mathbf{x}(0) = \mathbf{x}_0$. $\dot{\mathbf{x}}(t) = \dot{O}O^{-1}(t)\mathbf{x} = \dot\theta(t)\,\hat{\mathbf{e}}_3\times\mathbf{x}$ for $(\theta,\dot\theta) \in TSO(2)\,,$
$L = \frac{m}{2}R^2\dot\theta^2 - mgR(1-\cos\theta) \quad\Longrightarrow\quad \ddot\theta = -\omega^2\sin\theta \quad \text{with} \quad \omega^2 = g/R$

5. Charged particle in a magnetic field, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$:
$L = \frac{m}{2}|\dot{\mathbf{x}}|^2 + \frac{e}{c}\dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}) \quad\Longrightarrow\quad \ddot{\mathbf{x}} = \frac{e}{mc}\dot{\mathbf{x}}\times\mathbf{B} \quad \text{with} \quad \mathbf{B} = \mathrm{curl}\,\mathbf{A}$

6. Kepler problem in Cartesian coordinates, $(\mathbf{r},\dot{\mathbf{r}}) \in T\mathbb{R}^3$:
$L(\mathbf{r},\dot{\mathbf{r}}) = \frac{1}{2}|\dot{\mathbf{r}}|^2 - V(r)$ with $V(r) = -\,\mu/r$ and $r := |\mathbf{r}| = \sqrt{\mathbf{r}\cdot\mathbf{r}}$. $\quad\Longrightarrow\quad \ddot{\mathbf{r}} + \frac{\mu\mathbf{r}}{r^3} = 0.$

7. Kepler problem in polar coordinates, $(r,\dot r,\theta,\dot\theta) \in T\mathbb{R}_+\times TS^1$: $|\dot{\mathbf{r}}|^2 = \dot r^2 + r^2\dot\theta^2$
$L = \frac{1}{2}\left(\dot r^2 + r^2\dot\theta^2\right) + \frac{\mu}{r} \quad\Longrightarrow\quad \ddot r = -\frac{\mu}{r^2} + \frac{J^2}{r^3} \quad \text{with} \quad J = r^2\dot\theta = const$

8. Spherical pendulum (a), $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^3$, on $TS^2 = \{(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^3 : |\mathbf{x}| = 1 \;\&\; \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}$:
$L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} + \frac{1}{2}\mu(1-|\mathbf{x}|^2)$

9. Spherical pendulum (b), set $\mathbf{x}(t) = O(t)\mathbf{x}_0\,,\quad \dot{\mathbf{x}}(t) = \dot{O}(t)\mathbf{x}_0$ for $(O,\dot O) \in TSO(3)\,,$ where $\mathbf{x}_0 = \mathbf{x}(0)$ is the initial position of the particle and $O^T = O^{-1}$

$$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} = \frac{m}{2}|\dot{O}(t)\mathbf{x}_0|^2 - mg\,O^T(t)\hat{\mathbf{e}}_3\cdot\mathbf{x}_0\,.$$

Setting $\mathbf{x}(t) = O(t)\mathbf{x}_0$ avoids the need for the constraint $|\mathbf{x}|^2 = 1$, since rotations preserve length. $\Longrightarrow$

$$\dot{\boldsymbol{\Pi}} + \boldsymbol{\Omega}\times\boldsymbol{\Pi} = -\,g\,\boldsymbol{\Gamma}\times\mathbf{x}_0 \quad \text{with} \quad \boldsymbol{\Pi} := \mathbf{x}_0\times(\boldsymbol{\Omega}\times\mathbf{x}_0) = \boldsymbol{\Omega}|\mathbf{x}_0|^2 - \mathbf{x}_0(\mathbf{x}_0\cdot\boldsymbol{\Omega})\,.$$

Set $g = 0$ to get free motion on the sphere. Finally, from its definition, $\boldsymbol{\Gamma} := O^{-1}(t)\hat{\mathbf{e}}_3$ satisfies

$$\dot{\boldsymbol{\Gamma}} := -\,\widehat{\Omega}\boldsymbol{\Gamma} = -\,\boldsymbol{\Omega}\times\boldsymbol{\Gamma}\,.$$

10. Rotating rigid body, $\widehat{\Omega} = O^{-1}\dot{O} \in T(SO(3) \simeq \mathfrak{so}(3)$:
$\ell(\boldsymbol{\Omega}) = \frac{1}{2}\boldsymbol{\Omega}\cdot I\boldsymbol{\Omega} \quad \text{with} \quad \boldsymbol{\Omega}\times = \widehat{\Omega}\,, \quad \text{that is,} \quad -\,\epsilon_{ijk}\Omega_k = \widehat{\Omega}_{ij}\,. \quad\Longrightarrow\quad I\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega}\times I\boldsymbol{\Omega} = 0\,.$

Exercise 1.2 For the Lagrangians in the previous exercise, compute the Legendre transforms

$$H(q,p) = \langle p,\, \dot q\rangle - L(q,\dot q) = T(p) + V(q) = KE + PE$$

and the canonical Hamiltonian equations for any five of the following simple mechanical systems.

1. Planar isotropic oscillator, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + \frac{k}{2}|\mathbf{x}|^2$

2. Planar anisotropic oscillator, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + \frac{k_1}{2}x_1^2 + \frac{k_2}{2}x_2^2$

3. Planar pendulum in polar coordinates, $(\theta, p_\theta) \in T^*S^1$: $\quad H = \frac{1}{2mR^2}p_\theta^2 + mgR(1-\cos\theta)$

4. Planar pendulum, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$, constrained to $S^1 = \{\mathbf{x}\in\mathbb{R}^2 : 1 - |\mathbf{x}|^2 = 0\}$:
   $H = \frac{1}{2m}|\mathbf{p}|^2 + mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} + \frac12\mu(1-|\mathbf{x}|^2)$

5. Charged particle in a magnetic field, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x})|^2 \quad \mathbf{p} := \partial L/\partial\dot{\mathbf{q}} = m\dot{\mathbf{x}} + \frac{e}{c}\mathbf{A}(\mathbf{x}) \in T^*M$

6. Kepler problem, $(r, p_r, \theta, p_\theta) \in T^*\mathbb{R}_+ \times T^*S^1$: $\quad H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r}$ with $p_\theta = r^2\dot\theta = const$

7. Free motion on a sphere, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x}\in\mathbb{R}^3 : 1 - |\mathbf{x}|^2 = 0\}$:
   $H = \frac{1}{2m}|\mathbf{p}|^2 - \mu(1-|\mathbf{x}|^2)$

8. Spherical pendulum (a), $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x}\in\mathbb{R}^3 : 1 - |\mathbf{x}|^2 = 0\}$:
   $H = \frac{1}{2m}|\mathbf{p}|^2 + mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} - \mu(1-|\mathbf{x}|^2)$

9. Spherical pendulum (b), $(O,\dot O) \in TSO(3)$, $\widehat{\xi} = O^{-1}\dot O \in T(SO(3) \simeq \mathfrak{so}(3)$, $\boldsymbol{\Pi} = \partial\ell/\partial\boldsymbol{\Omega} \in T^*(SO(3) \simeq \mathfrak{so}(3)^* \simeq \mathbb{R}^3 \quad H = \frac12\boldsymbol{\Pi}\cdot I^{-1}\boldsymbol{\Pi} + g\,\boldsymbol{\Gamma}\cdot\mathbf{x}_0$ with $\boldsymbol{\Pi} = \frac{\partial\ell}{\partial\boldsymbol{\Omega}} = I\boldsymbol{\Omega}$. Set $g=0$ to get freely rotating rigid body motion.

10. Rotating rigid body, $\boldsymbol{\Pi} \in T^*(SO(3) \simeq \mathfrak{so}(3)^* \simeq \mathbb{R}^3 \quad H = \frac12\boldsymbol{\Pi}\cdot I^{-1}\boldsymbol{\Pi}$ with $\boldsymbol{\Pi} = \frac{\partial\ell}{\partial\boldsymbol{\Omega}} = I\boldsymbol{\Omega}$.

Exercise 1.3 (Two important examples of Noether's theorem)

(a) What conservation law does Noether's theorem imply for symmetries of the action principle given by $\delta\mathsf{S} = 0$ with

$$\mathsf{S} = \int_a^b L(\dot{\mathbf{q}}(t), \mathbf{q}(t), t)\,dt\,, \quad \text{for} \quad \mathbf{q}\in\mathbb{R}^3 \quad \text{and} \quad L : T\mathbb{R}^3 \to \mathbb{R}\,,$$

when the Lagrangian $L(\dot{\mathbf{q}}(t), \mathbf{q}(t), t)$ is invariant under infinitesimal azimuthal rotations about $\hat{\mathbf{z}}$ given by

$$\mathbf{q}(t,\epsilon) = \mathbf{q}(t) + \epsilon\,\hat{\mathbf{z}}\times\mathbf{q}(t) + O(\epsilon^2) \quad \text{so that} \quad \delta\mathbf{q} = \frac{d\mathbf{q}}{d\epsilon}\bigg|_{\epsilon=0} = \hat{\mathbf{z}}\times\mathbf{q}(t)$$

(b) What additional conservation law is implied by Noether's theorem when the Lagrangian in the form $L(\dot{\mathbf{q}}(t), \mathbf{q}(t))$ is translation invariant in time, t, so that $\partial_t L = 0$?

Exercise 1.4 (The free particle in $\mathbb{H}^2$)

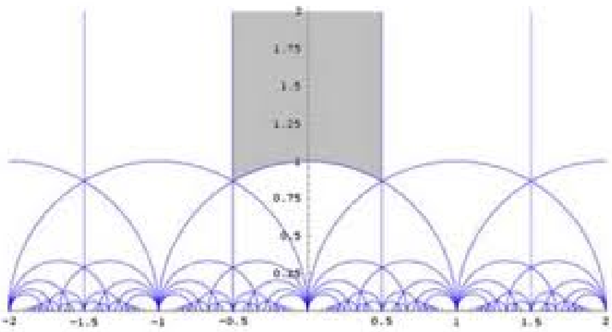

Figure 1: Geodesics on the Lobachevsky half-plane, $\mathbb{H}^2$.

In Appendix I of Arnold's book, *Mathematical Methods of Classical Mechanics*, page 303, we read.

> EXAMPLE. We consider the upper half-plane $y > 0$ of the plane of complex numbers $z = x + iy$ with the metric
> $$ds^2 = \frac{dx^2 + dy^2}{y^2}\,.$$
> It is easy to compute that the geodesics of this two-dimensional riemannian manifold are circles and straight lines perpendicular to the x-axis. Linear fractional transformations with real coefficients
> $$z \to \frac{az + b}{cz + d} \tag{1}$$
> are isometric transformations of our manifold ($\mathbb{H}^2$), which is called the *Lobachevsky plane.*[1]

Consider a free particle of mass m moving on the Lobachevsky half-plane $\mathbb{H}^2$. Its Lagrangian is the kinetic energy corresponding to the Lobachevsky metric. Namely,

$$L = \frac{m}{2}\left(\frac{\dot{x}^2 + \dot{y}^2}{y^2}\right). \tag{2}$$

(A) (1) Write the fibre derivatives (i.e., the momenta $\frac{\partial L}{\partial \dot{x}}$ and $\frac{\partial L}{\partial \dot{y}}$) of the Lagrangian (2) and

(2) compute its Euler-Lagrange equations.

These equations represent geodesic motion on $\mathbb{H}^2$.

(3) Evaluate the Christoffel symbols.

$\boxed{\textbf{\textit{Hint:}}}$ Geodesic equations look like $\ddot{q}^{\,c} + \Gamma^c_{be}(q)\dot{q}^b\dot{q}^e = 0$, where $\Gamma^c_{be}(q)$ are the Christoffel symbols.

(B) (1) Show that the quantities

$$u = \frac{\dot{x}}{y} \quad \text{and} \quad v = \frac{\dot{y}}{y} \tag{3}$$

are invariant under the quantities (3) are invariant under a subgroup the translations and scalings.

$$\begin{aligned}
T_\tau : (x, y) \mapsto (x + \tau, y) \qquad & \text{Flow of } X_T = \partial_x, \quad (\delta x, \delta y) = (1, 0), \quad [X_T, X_S] = X_T.\\
S_\sigma : (x, y) \mapsto (e^\sigma x, e^\sigma y) \qquad & \text{Flow of } X_S = x\partial_x + y\partial_y, \quad (\delta x, \delta y) = (x, y).
\end{aligned}$$

These transformations are translations T along the x axis and scalings S centered at $(x, y) = (0, 0)$.

[1] These isometric transformations of $\mathbb{H}^2$ have deep significance in physics. They correspond to the most general Lorentz transformation of space-time.

(C) (1) Use the invariant quantities (u, v) in (3) as new variables in Hamilton's principle.

$\boxed{\textbf{\textit{Hint:}}}$ the transformed Lagrangian is

$$\ell(u, v) = \frac{m}{2}(u^2 + v^2)\,.$$

(2) Find the corresponding conserved Noether quantities.

(D) Transform the Euler-Lagrange equations from x and y to the variables u and v that are invariant under the symmetries of the Lagrangian.

Then:

(1) Show that the resulting system conserves the kinetic energy expressed in these variables.

(2) Discuss its integral curves and critical points in the uv plane.

(3) Show that the u and v equations can be integrated explicitly in terms of sech and tanh.

$\boxed{\textbf{\textit{Hint:}}}$ In the u, v variables, the Euler-Lagrange equations for the Lagrangian (2) are expressed as

$$\frac{d}{dt}\frac{u}{y} = 0 \quad \text{and} \quad \frac{d}{dt}\frac{v}{y} + \frac{u^2 + v^2}{y} = 0\,.$$

Expanding these equations using $u = \dot{x}/y$ and $v = \dot{y}/y$ yields

$$\dot{u} = uv\,, \qquad \dot{v} = -\,u^2 \tag{4}$$

(E) (1) Legendre transform the Lagrangian (2) to the Hamiltonian side, obtain the canonical equations and

(2) derive the Poisson brackets for the variables u and v. $\boxed{\textbf{\textit{Hint:}}}$ $\{yp_x, yp_y\} = yp_x$.

Exercise 1.5 (Poisson brackets for the Hopf map)

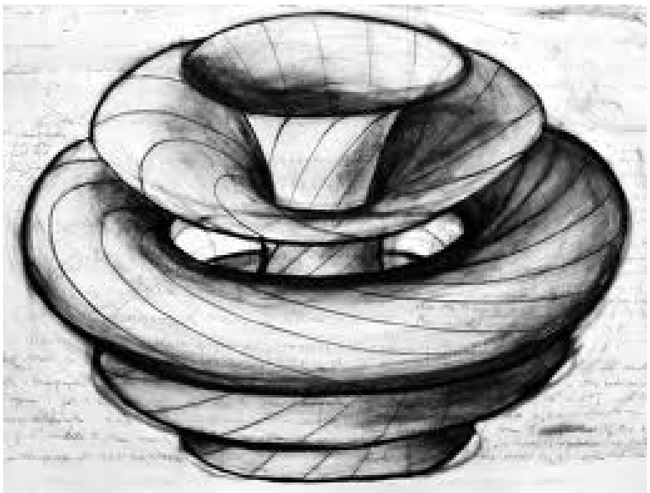

Figure 2: The Hopf map.

In coordinates $(a_1, a_2) \in \mathbb{C}^2$*, the Hopf map* $\mathbb{C}^2/S^1 \to S^3 \to S^2$ *is obtained by transforming to the four quadratic* S^1*-invariant quantities*

$$(a_1, a_2) \to Q_{jk} = a_j a_k^* \,, \quad \textit{with} \quad j,k = 1,2\,.$$

Let the $\mathbb{C}^2$ *coordinates be expressed as*

$$a_j = q_j + ip_j$$

in terms of canonically conjugate variables satisfying the fundamental Poisson brackets

$$\{q_k, p_m\} = \delta_{km} \quad \textit{with} \quad k, m = 1, 2.$$

(A) Compute the Poisson brackets $\{a_j, a_k^*\}$ *for* $j, k = 1, 2$.

(B) Is the transformation $(q, p) \to (a, a^*)$ *canonical? Explain why or why not.*

$\boxed{\textbf{\textit{Hint:}}}$ *a map* $(q, p) \to (Q, P)$ *whose Poisson bracket is* $\{Q, P\} = c\{q, p\}$ *with a constant factor* c *is still regarded as being canonical.*

(C) Compute the Poisson brackets among Q_{jk}*, with* $j, k = 1, 2$.

(D) Make the linear change of variables,

$$X_0 = Q_{11} + Q_{22}\,, \quad X_1 + iX_2 = 2Q_{12}\,, \quad X_3 = Q_{11} - Q_{22}\,,$$

and compute the Poisson brackets among (X_0, X_1, X_2, X_3).

(E) Express the Poisson bracket $\{F(\mathbf{X}), H(\mathbf{X})\}$ *in vector form among functions* F *and* H *of* $\mathbf{X} = (X_1, X_2, X_3)$.

(F) Show that the quadratic invariants (X_0, X_1, X_2, X_3) *themselves satisfy a quadratic relation.*

How is this relevant to the Hopf map?