3 M3-4-5A16 Assessed Problems # 3: Do all four problems

Exercise 3.1 (Momentum map for the Heisenberg group).

The Heisenberg group is a subgroup of $SL(3,\mathbb{R})$ given by the ***upper triangular matrices***

$$\left\{ H = \begin{bmatrix} 1 & \xi_1 & \xi_3 \\ 0 & 1 & \xi_2 \\ 0 & 0 & 1 \end{bmatrix} \quad \xi_1, \xi_2, \xi_3 \in \mathbb{R} \right\}$$

Its matrix action on elements of $\mathbb{R}^2$ is computed as

$$\begin{bmatrix} 1 & \xi_1 & \xi_3 \\ 0 & 1 & \xi_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + \xi_1 y + \xi_3 \\ y + \xi_2 \\ 1 \end{bmatrix}$$

This defines the action of H on $\mathbb{R}^2$: $(x,y)^T \longrightarrow (x + \xi_1 y + \xi_3,\ y + \xi_2)^T$, with infinitesimal action $\Phi_\xi(x,y) = (\xi_1 y + \xi_3, \xi_2)$.

(A) (i) Linearise around the identity of the matrix Lie group H to find the matrix representation of its Lie algebra, $\mathfrak{h}$.

(ii) Write the isomorphism between the matrix representation of $\mathfrak{h}$ and vectors in $\mathbb{R}^3$.

(iii) Compute the ad-operation $\mathrm{ad}_\eta \xi$, $\mathrm{ad} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ on the matrix Lie algebra for $\eta, \xi \in \mathfrak{h}$.

(iv) Use the isomorphism $\mathfrak{h} \longleftrightarrow \mathbb{R}^3$ to write the ad-operation in $\mathfrak{h}$ as a vector operation in $\mathbb{R}^3$.

(B) Compute the dual operation $\mathrm{ad}^*_\eta \mu$, where $\mathrm{ad}^* : \mathfrak{h}^* \times \mathfrak{h} \to \mathfrak{h}^*$ for $\eta \in \mathfrak{h}$ and $\mu \in \mathfrak{h}^*$ by using the trace pairing for matrices.

(C) Compute the cotangent lift of the infinitesimal action $\Phi_\xi(x,y) = (\xi_1 y + \xi_3, \xi_2)$.

(D) Compute the cotangent-lift momentum map for the action of the Heisenberg Lie group on phase space $T^*\mathbb{R}^2$ with coordinates (x, y, p_x, p_y).

(E) Compute the Poisson brackets among the components of the cotangent-lift momentum map.

(F) Write the dynamics for the components of the cotangent-lift momentum map in $\mathbb{R}^3$ vector form and give a geometric interpretation of the motion in $\mathbb{R}^3$.

Answer.

(A) (i) Elements of the 3×3 matrix Lie algebra $\mathfrak{h}$ take the matrix form,

$$\xi = \begin{bmatrix} 0 & \xi_1 & \xi_3 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) The isomorphism between $\mathfrak{h}$ and $\mathbb{R}^3$ is given by

$$\xi = \begin{bmatrix} 0 & \xi_1 & \xi_3 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{bmatrix} \longleftrightarrow \boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

(iii) The matrix form of the ad-operation in $\mathfrak{h}$ is given by

$$\mathrm{ad}_\eta \xi = \begin{bmatrix} 0 & 0 & \eta_1\xi_2 - \eta_2\xi_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(iv) The ad-operation in $\mathfrak{h}$ as a vector operation in $\mathbb{R}^3$

$$\mathrm{ad}_\eta \xi = \begin{bmatrix} 0 & 0 & \eta_1\xi_2 - \eta_2\xi_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 \\ 0 \\ \eta_1\xi_2 - \eta_2\xi_1 \end{bmatrix} = (\hat{\boldsymbol{z}} \cdot \boldsymbol{\eta} \times \boldsymbol{\xi})\hat{\boldsymbol{z}}$$

(B) The inner product on the Heisenberg Lie algebra $\mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ is defined by the matrix trace pairing

$$\langle \eta,\, \xi \rangle = \mathrm{Tr}(\eta^T\, \xi) = \boldsymbol{\eta} \cdot \boldsymbol{\xi}\,.$$

Thus, elements of the dual Lie algebra $\mathfrak{h}^*(\mathbb{R})$ may be represented as ***lower triangular matrices***,[1]

$$\mu = \begin{bmatrix} 0 & 0 & 0 \\ \mu_1 & 0 & 0 \\ \mu_3 & \mu_2 & 0 \end{bmatrix} \in \mathfrak{h}^*(\mathbb{R})\,.$$

Likewise, the ad* operation of the Heisenberg Lie algebra $\mathfrak{h}$ on its dual $\mathfrak{h}^*$ is defined in terms of the matrix pairing by

$$\langle \mathrm{ad}^*_\eta \mu,\, \xi \rangle := \langle \mu,\, \mathrm{ad}_\eta \xi \rangle$$

$$\begin{aligned} \langle \mu,\, \mathrm{ad}_\eta \xi \rangle &= \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ \mu_1 & 0 & 0 \\ \mu_3 & \mu_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \eta_1\xi_2 - \xi_1\eta_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \\ &= \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ -\eta_2\mu_3 & 0 & 0 \\ 0 & \eta_1\mu_3 & 0 \end{bmatrix} \begin{bmatrix} 0 & \xi_1 & \xi_3 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{bmatrix} \right) \\ &= \langle \mathrm{ad}^*_\eta \mu,\, \xi \rangle. \end{aligned} \tag{3.1}$$

Thus, we have the formula for $\mathrm{ad}^*_\eta \mu$:

$$\dot{\mu} = \mathrm{ad}^*_\eta \mu = \begin{bmatrix} 0 & 0 & 0 \\ -\eta_2\mu_3 & 0 & 0 \\ 0 & \eta_1\mu_3 & 0 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} \dot{\mu}_1 \\ \dot{\mu}_2 \\ \dot{\mu}_3 \end{bmatrix} = \begin{bmatrix} 0 & \mu_3 & 0 \\ -\mu_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}.$$

This defines a Lie-Poisson bracket whose Casimir is μ_3. On restricting it to a level set of μ_3, it becomes canonical, as expected from the Marsden-Weinstein theorem.

(C) The cotangent lift of the infinitesimal action $\Phi_\xi(x, y) = (\xi_1 y + \xi_3, \xi_2)$ is computed from the Hamiltonian vector field for $J^\xi = p_x(\xi_1 y + \xi_3) + p_y\xi_2 = p_x y\xi_1 + p_y\xi_2 + p_x\xi_3$, namely

$$\{\,\cdot\,,\, J^\xi\} = (\xi_1 y + \xi_3)\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} - p_x\xi_1\frac{\partial}{\partial p_y}$$

Note that the flow of this Hamiltonian vector field leaves p_x invariant.

[1] The dual Lie algebra $\mathfrak{h}^*(\mathbb{R})$ may be represented equally well as ***symmetric matrices***, with arbitrary entries on the diagonal,

$$\mu = \begin{bmatrix} k_1 & \mu_1 & \mu_3 \\ \mu_1 & k_2 & \mu_2 \\ \mu_3 & \mu_2 & k_3 \end{bmatrix} \in \mathfrak{h}^*(\mathbb{R})\,.$$

but here we choose the equivalent representation by lower triangular matrices.

(D) The cotangent-lift momentum map J for the action of the Heisenberg Lie group on phase space $T^*\mathbb{R}^2$ is obtained from the formula

$$J^\xi = \langle J, \xi \rangle = \boldsymbol{J} \cdot \boldsymbol{\xi} = J_1\xi_1 + J_2\xi_2 + J_3\xi_3 = p_x y \xi_1 + p_y \xi_2 + p_x \xi_3 = \operatorname{tr}\left([J_1, J_2, J_3] \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \right)$$

Thus the momentum map is given by

$$\boldsymbol{J} = (J_1, J_2, J_3) = (p_x y, p_y, p_x)$$

(E) The Poisson brackets among the components of the cotangent-lift momentum map are given by

$$\{J_1, J_2\} = \{p_x y, p_y\} = p_x = J_3\,, \quad \{J_2, J_3\} = \{p_y, p_x\} = 0\,, \quad \{J_3, J_1\} = \{p_x, p_x y\} = 0$$

In tabular form, these Poisson brackets are

$$\{J_i,\, J_k\} = \begin{array}{|c|ccc|} \hline \{\,\cdot\,,\,\cdot\,\} & J_1 & J_2 & J_3 \\ \hline J_1 & 0 & J_3 & 0 \\ J_2 & -J_3 & 0 & 0 \\ J_3 & 0 & 0 & 0 \\ \hline \end{array}\,.$$

This Lie-Poisson bracket for (J_1, J_2, J_3) is the same as the one we had above for (μ_1, μ_2, μ_3), and J_3 is its Casimir.

The corresponding Lie-Poisson Hamiltonian equation is

$$\frac{df}{dt} = \frac{\partial f}{\partial J_i}\{J_i,\, J_k\}\frac{\partial h}{\partial J_k} = J_3 \underbrace{\left(\frac{\partial f}{\partial J_1}\frac{\partial h}{\partial J_2} - \frac{\partial h}{\partial J_1}\frac{\partial f}{\partial J_2} \right)}_{\text{Canonical}} = \frac{1}{2}\frac{\partial J_3^2}{\partial \mathbf{J}} \cdot \frac{\partial f}{\partial \mathbf{J}} \times \frac{\partial h}{\partial \mathbf{J}}$$

(F) **Geometric interpretation.** Upon expressing the Lie-Poisson bracket in vector form, the motion of $\mathbf{J} \in \mathbb{R}^3$ may be written as a cross product. Namely,

$$\frac{d\mathbf{J}}{dt} = -\frac{1}{2}\frac{\partial J_3^2}{\partial \mathbf{J}} \times \frac{\partial h}{\partial \mathbf{J}} = -J_3\,\hat{\mathbf{z}} \times \frac{\partial h}{\partial \mathbf{J}} = -\operatorname{ad}^*_{\partial h/\partial \mathbf{J}}\,(\hat{\mathbf{z}} J_3)$$

so the motion takes place in $\mathbb{R}^3$ along intersections of level sets of J_3^2 and the Hamiltonian $h(\mathbf{J})$. In components, this is

$$\begin{bmatrix} \dot{J}_1 \\ \dot{J}_2 \\ \dot{J}_3 \end{bmatrix} = \begin{bmatrix} 0 & J_3 & 0 \\ -J_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial h/\partial J_1 \\ \partial h/\partial J_2 \\ \partial h/\partial J_3 \end{bmatrix} = \begin{bmatrix} -J_3 \partial h/\partial J_2 \\ J_3\, \partial h/\partial J_1 \\ 0 \end{bmatrix}.$$

In the equivalent matrix form, this is

$$\dot{J} = \begin{bmatrix} 0 & 0 & 0 \\ \dot{J}_1 & 0 & 0 \\ \dot{J}_3 & \dot{J}_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ J_3\frac{\partial h}{\partial J_2} & 0 & 0 \\ 0 & -J_3\frac{\partial h}{\partial J_1} & 0 \end{bmatrix} = -\operatorname{ad}^*_{\partial h/\partial J} J_3.$$

▲

Exercise 3.2 (Quadratic Poisson brackets).

(A) Prove that the ***quadratic Poisson bracket*** on $\mathbb{R}^N$ given by

$$\{x_i, x_j\} = x_i x_j \left(\delta_{i,j+1} - \delta_{i+1,j}\right) \qquad 1 \le i, j \le N \quad \text{with} \quad x_0 = 0 = x_N$$

satisfies the Jacobi identity.

(B) Write out the quadratic Poisson structure for $N = 5$ as a 5×5 matrix.

(C) Does the quadratic Poisson bracket on $\mathbb{R}^N$ have a Casimir? If so, what is it?

(D) Prove the Jacobi identity for the quadratic Poisson structure on $\mathbb{R}^3$, by writing it as a Nambu bracket. Discuss the resulting motion as intersections of level sets of constants of motion for the case that the Hamiltonian is given by $h(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^2 = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$.

(E) Introduce the symmetric matrix

$$L_3 = \begin{pmatrix} 0 & x_1 & 0 & 0 \\ x_1 & 0 & x_2 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & 0 & x_3 & 0 \end{pmatrix}$$

and express the dynamical equation for the Hamiltonian in part (D) as a Lax pair in the form,

$$\frac{dL_3}{dt} = [L_3, \widetilde{J}_3]\,.$$

In particular, find the 4×4 skew symmetric matrix $\widetilde{J}_3$ by deforming the 3×3 skew symmetric matrix J_3. Hint: this matrix calculation is easy because the deformation of the matrix J_3 only involves inserting zeros.

Answer.

(A) The Jacobi identity is verified by a direct calculation, or maybe there is a smarter way . . .

(B) The quadratic Poisson structure on $\mathbb{R}^5$ is a *banded* matrix

$$J_5 = \begin{pmatrix} 0 & x_1x_2 & 0 & 0 & 0 \\ -x_1x_2 & 0 & x_2x_3 & 0 & 0 \\ 0 & -x_2x_3 & 0 & x_3x_4 & 0 \\ 0 & 0 & -x_3x_4 & 0 & x_4x_5 \\ 0 & 0 & 0 & -x_4x_5 & 0 \end{pmatrix}$$

whose bands are revealed clearly for $N = 5$.

(C) There seems to be no Casimir for the quadratic Poisson structure on $\mathbb{R}^N$ for $N > 3$.

(D) The quadratic Poisson structure on $\mathbb{R}^3$

$$J_3 = \begin{pmatrix} 0 & x_1x_2 & 0 \\ -x_1x_2 & 0 & x_2x_3 \\ 0 & -x_2x_3 & 0 \end{pmatrix} = x_2 \begin{pmatrix} 0 & x_1 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_3 & 0 \end{pmatrix} = x_2 \nabla(x_1x_3)\times$$

Thus, because x_2 factors out, the case $N = 3$ simplifies to a Nambu bracket and we may write the dynamical equations for the quadratic Poisson bracket in $\mathbb{R}^3$ as

$$\dot{\mathbf{x}} = J_3 \frac{\partial h}{\partial \mathbf{x}} = \{\mathbf{x}, h\} = x_2 \nabla(x_1x_3) \times \nabla h$$

For the case $N = 3$, the Casimir is $C_3 = x_1 x_3$.

When the Hamiltonian is

$$h(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^2 = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$$

then the motion for $N = 3$ takes place in $\mathbb{R}^3$ along intersections of level sets of $x_1 x_3$ (hyperbolic cylinders aligned with x_2) and h (coincident spheres with center at the origin). The level set $x_2 = 0$ is a plane of fixed points and the motion consists of heteroclinic orbits that connect points on the equator of the sphere to each other along the intersections with a family of hyperbolic cylinders, $x_1 x_3 = const.$

(E) For $N = 3$ the Hamiltonian form of the equations for $h = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$ may be written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = J_3(\mathbf{x})\frac{\partial h}{\partial \mathbf{x}} = \begin{pmatrix} 0 & x_1x_2 & 0 \\ -x_1x_2 & 0 & x_2x_3 \\ 0 & -x_2x_3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1x_2^2 \\ x_2(x_3^2 - x_1^2) \\ -x_3x_2^2 \end{pmatrix}$$

Remarkably, the matrix of quadratic quantities in the Hamiltonian matrix representation of these cubic dynamical equations plays a role in recognising their Lax pair, or commutator form.

We introduce the symmetric matrix

$$L_3 = \begin{pmatrix} 0 & x_1 & 0 & 0 \\ x_1 & 0 & x_2 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & 0 & x_3 & 0 \end{pmatrix}$$

and express its dynamical equation as a Lax pair in the form,

$$\frac{dL_3}{dt} = [L_3, \widetilde{J}_3]$$

with

$$\widetilde{J}_3(\mathbf{x}) = \begin{pmatrix} 0 & 0 & x_1x_2 & 0 \\ 0 & 0 & 0 & x_2x_3 \\ -x_1x_2 & 0 & 0 & 0 \\ 0 & -x_2x_3 & 0 & 0 \end{pmatrix}$$

The 4×4 matrix $\widetilde{J}_3(\mathbf{x})$ is a deformation of the 3×3 Hamiltonian matrix $J_3(\mathbf{x})$ obtained by replacing the zeros on the diagonal of J by the tridiagonal zeros of the matrix $\widetilde{J}$.

▲

Exercise 3.3 (Lie-Poisson brackets for the group $S \circledS (T \times T)$).

Consider a semidirect-product Lie group $S \circledS (T \times T)$ comprising a radial scaling transformation S in the xy-plane and two affines translations (shears) $T \times T$ of z depending linearly on x and y. The matrix representation of its action on $\mathbb{R}^3$ is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{\epsilon_1} & 0 & 0 \\ 0 & e^{\epsilon_1} & 0 \\ \epsilon_2 & \epsilon_3 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

This defines the Lie group of lower-triangular 3×3 matrices

$$\begin{pmatrix} e^{\epsilon_1} & 0 & 0 \\ 0 & e^{\epsilon_1} & 0 \\ \epsilon_2 & \epsilon_3 & 1 \end{pmatrix} \in G_{\triangleright}$$

(A) Compute the group product and inverse element for the matrix Lie group $G_\triangleright$.

(B) Find the matrix representation of its Lie algebra $\mathfrak{g}_\triangleright$ and explicitly compute the adjoint operation. Write the formula for $\mathrm{ad}_\xi \eta$ in matrix form for $\xi, \eta \in \mathfrak{g}_\triangleright$.

(C) Compute the coadjoint action of its Lie algebra on its dual Lie algebra. Write the formula for $\mathrm{ad}^*_\xi \mu$ in matrix form for $\xi \in \mathfrak{g}_\triangleright$ and $\mu \in \mathfrak{g}^*_\triangleright$.

(D) Write the Euler-Poincaré equation

$$\dot{\mu} = \mathrm{ad}^*_\xi \mu \quad \text{with} \quad \mu = \frac{\partial l}{\partial \xi},$$

in which $\xi := g_t^{-1}\dot{g}_t$ for a Lagrangian $l(\xi)$ that is invariant under $S \circledS (T \times T)$.

(E) Legendre transform this equation to the (Lie-Poisson) Hamiltonian side. What infinitesimal transformations are generated, when the Lie-Poisson structure is regarded as a matrix operator acting on $\nabla h \in \mathbb{R}^3$? Hint: think of the Lie-Poisson form as a Hamiltonian vector field.

(F) Does the final Poisson bracket have a Casimir? If so, express it as a function on $\mathbb{R}^3$.

(G) Describe the solution when the Hamiltonian is given by $h = \frac{1}{2}|\mathbf{x}|^2$. Does the dynamics have a plane of fixed points?

Answer.

(A) The group product is

$$g_1 g_2 = \begin{pmatrix} e^{\epsilon_1} & 0 & 0 \\ 0 & e^{\epsilon_1} & 0 \\ \epsilon_2 & \epsilon_3 & 1 \end{pmatrix} \begin{pmatrix} e^{\beta_1} & 0 & 0 \\ 0 & e^{\beta_1} & 0 \\ \beta_2 & \beta_3 & 1 \end{pmatrix} = \begin{pmatrix} e^{\epsilon_1+\beta_1} & 0 & 0 \\ 0 & e^{\epsilon_1+\beta_1} & 0 \\ \epsilon_2 e^{\beta_1} + \beta_2 & \epsilon_3 e^{\beta_1} + \beta_3 & 1 \end{pmatrix}$$

The inverse is

$$g_1^{-1} = \begin{pmatrix} e^{-\epsilon_1} & 0 & 0 \\ 0 & e^{-\epsilon_1} & 0 \\ -\epsilon_2 e^{-\epsilon_1} & -\epsilon_3 e^{-\epsilon_1} & 1 \end{pmatrix}$$

(B) The matrix representation of its Lie algebra $\mathfrak{g}_\triangleright$ is given by

$$g_t = \begin{pmatrix} e^{\epsilon_1} & 0 & 0 \\ 0 & e^{\epsilon_1} & 0 \\ \epsilon_2 & \epsilon_3 & 1 \end{pmatrix} \quad \text{and} \quad \xi := g_t^{-1}\dot{g}_t = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_1 & 0 \\ \xi_2 & \xi_3 & 0 \end{pmatrix}$$

(C) The adjoint and coadjoint actions of the group $S \circledS (T \times T)$. The Lie algebra commutator is given for two Lie algebra elements ξ and η in $\mathfrak{g}_\triangleright$ by

$$\mathrm{ad}_\xi \eta = [\xi, \eta] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \xi_2\eta_1 - \eta_2\xi_1 & \xi_3\eta_1 - \eta_3\xi_1 & 0 \end{pmatrix}$$

(D) An element of the dual Lie algebra is represented by the transpose matrix

$$\mu = \begin{pmatrix} \mu_1 & 0 & \mu_2 \\ 0 & \mu_1 & \mu_3 \\ 0 & 0 & 0 \end{pmatrix}$$

The coadjoint action of its Lie algebra on its dual Lie algebra is computed, as follows.

$$\begin{aligned}
\langle \mu,\, \mathrm{ad}_\xi \eta \rangle &= \frac{1}{2}\mathrm{trace}(\mu\, \mathrm{ad}_\xi \eta) \\
&= \mu_2(\xi_2\eta_1 - \eta_2\xi_1) + \mu_3(\xi_3\eta_1 - \eta_3\xi_1) \\
&= (\mu_2\xi_2 + \mu_3\xi_3,\, -\mu_2\xi_1,\, -\mu_3\xi_1)\cdot(\eta_1,\, \eta_2,\, \eta_3)^T \\
&= \frac{1}{2}\mathrm{trace}\,(\mathrm{ad}^*_\xi \mu\, \eta) \\
&= \langle \mathrm{ad}^*_\xi \mu,\, \eta \rangle
\end{aligned}$$

In matrix form, the formula for $\mathrm{ad}^*_\xi \mu$ is

$$\mathrm{ad}^*_\xi \mu = \begin{pmatrix} \mu_2\xi_2 + \mu_3\xi_3 & 0 & -\mu_2\xi_1 \\ 0 & \mu_2\xi_2 + \mu_3\xi_3 & -\mu_3\xi_1 \\ 0 & 0 & 0 \end{pmatrix}$$

This formula is the ingredient needed for writing the Euler-Poincaré equation

$$\dot{\mu} = \mathrm{ad}^*_\xi \mu \quad \text{with} \quad \mu = \frac{\partial l}{\partial \xi},$$

in which $\xi := g_t^{-1}\dot{g}_t$ for a Lagrangian $l(\xi)$ that is invariant under $S\circledS(T\times T)$. In components, the Euler-Poincaré equation is

$$\begin{aligned}
\dot{\mu}_1 &= \mu_2\xi_2 + \mu_3\xi_3 \\
\dot{\mu}_2 &= -\mu_2\xi_1 \\
\dot{\mu}_3 &= -\mu_3\xi_1
\end{aligned}$$

(E) After Legendre transforming to the corresponding Hamiltonian, $h(\mu)$, with $\xi_k = \partial h/\partial \mu_k$ and rearrangement into a matrix product form, this set of formulas becomes

$$\begin{pmatrix} \dot{\mu}_1 \\ \dot{\mu}_2 \\ \dot{\mu}_3 \end{pmatrix} = \begin{pmatrix} 0 & \mu_2 & \mu_3 \\ -\mu_2 & 0 & 0 \\ -\mu_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial h/\partial \mu_1 \\ \partial h/\partial \mu_2 \\ \partial h/\partial \mu_3 \end{pmatrix}$$

Upon identifying $\mu = (x,\, y,\, z)$, this becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & y & z \\ -y & 0 & 0 \\ -z & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial h/\partial x \\ \partial h/\partial y \\ \partial h/\partial z \end{pmatrix} = \begin{pmatrix} y\partial h/\partial y + z\partial h/\partial z \\ -y\partial h/\partial x \\ -z\partial h/\partial x \end{pmatrix} = \begin{pmatrix} y\partial_y + z\partial_z \\ -y\partial_x \\ -z\partial_x \end{pmatrix} h$$

These are the infinitesimal transformations of $S\circledS(T\times T)$, represented as a vector field. This make sense, because it means that given a Hamiltonian Lie-Poisson structure one may convert it to an Euler-Poincaré formulation, by identifying the infinitesimal transformations associated with the Lie-Poisson structure.

In our case, the scaling transformation in our case leaves invariant the ratio y/z for any Hamiltonian; so $C = y/z$ will be the Casimir in the Hamiltonian formulation.

(F) Our Poisson bracket expresses the dynamics in $\mathbb{R}^3$ as

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\} = z^2 \nabla \frac{y}{z} \times \nabla h$$

so $C = y/z$ is its Casimir.

(G) On the plane $z = 0$, the dynamics reduces to

$$\dot{x} = y\partial h/\partial y\,, \quad \dot{y} = -y\partial h/\partial x\,, \quad \dot{z} = 0\,,$$

so the plane $z = 0$ is an invariant plane, but not a plane of fixed points.

The x-axis $z = 0 = y$ is a *line* of fixed points.

When $\nabla h = \mathbf{x}$ the dynamics becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} y^2 + z^2 \\ -xy \\ -xz \end{pmatrix}$$

The x-axis $y = 0 = z$ is a line of fixed points. We have $y/z = const$ by construction, and the motion off the x-axis moves in planes whose yz-orientation remains constant. Cylindrical polar coordinates in one of these planes are given by

$$r = \sqrt{y^2 + z^2}\,, \quad \tan\theta = y/z$$

Then we have

$$\dot{x} = r^2\,, \quad \dot{r} = -xr\,, \quad \dot{\theta} = 0$$

So the line of fixed points along the x-axis is attracting for $x > 0$ and repelling for $x < 0$. The motion is in the positive x direction and eventually approaches the x-axis in a plane that stays oriented at a constant angle θ.

▲

Exercise 3.4 (Canonical variables for the rigid body on $SU(n)$).

(A) Compute the Euler-Poincaré equation for the inverse AD-action, $Q_t = AD_{U_t^{-1}} Q_0 = U_t^{-1} Q_0 U_t$, of the matrix Lie group $SU(n)$ on itself.

(B) Specialise to $n = 2$ and write the equations explicitly as 2×2 matrices.

(C) Transform to the Lie-Poisson Hamiltonian formulation for the case of $SU(n)$.

Answer.

(A) The *tangent lift* of the AD-action is found by taking the time derivative of the AD-action, from which (suppressing subscript t's)

$$\dot{Q} = -\,[\Omega, Q] \quad \text{with} \quad \Omega := U^{-1}\dot{U} \in su(n)\,,$$

in which the left-invariant $\Omega := U^{-1}\dot{U} \in su(n)$ is skew-Hermitian,

$$\Omega^\dagger + \Omega = 0\,.$$

This skew-Hermitian property may be seen by expanding the unitary condition near the identity of the $SU(n)$ matrices,

$$Id = U^\dagger U = (Id + s\Omega^\dagger)(Id + s\Omega) = Id + s(\Omega^\dagger + \Omega) + O(s^2).$$

From Hamilton's principle $\delta S = 0$ with action integral

$$\begin{aligned} S(\Omega, Q, P) &= \int_a^b l(\Omega) + \Big\langle P,\, \dot{Q} + [\Omega,\, Q] \Big\rangle \\ &= \int_a^b l(\Omega) + \mathrm{tr}\Big(P\, \big(\dot{Q} + [\Omega,\, Q] \big) \Big)\, dt\,, \end{aligned}$$

constrained by the tangent lift relation $\dot{Q} + [\Omega,\, Q] = 0$, we have

$$\begin{aligned} \delta S &= \int_a^b \Big\{ \Big\langle \frac{\delta l}{\delta \Omega} - [P,\, Q]\,,\, \delta\Omega \Big\rangle \\ &\qquad + \Big\langle \delta P\,,\, \dot{Q} + [\Omega,\, Q] \Big\rangle + \Big\langle \delta Q\,,\, \dot{P} + [\Omega,\, P] \Big\rangle \Big\}\, dt\,, \end{aligned}$$

for which $\delta l/\delta\Omega = [P,\, Q]$ and $Q, P \in SU(n)$ satisfy the following equations,

$$\dot{Q} = -\,[\Omega,\, Q] \quad \text{and} \quad \dot{P} = -\,[\Omega,\, P]\,, \tag{3.2}$$

as a result of the constraints.

This expands to the Euler-Poincaré equation

$$\dot{M} = \mathrm{ad}^*_\Omega M = -\,[\Omega,\, M]\,, \tag{3.3}$$

with $M = \delta l/\delta\Omega = [P,\, Q]$.

Momentum map: The vector field

$$(\dot{Q},\, \dot{P}) = (-\,[\Omega,\, Q],\, -\,[\Omega,\, P]) = \left(\frac{\partial J^\Omega}{\partial P},\, -\frac{\partial J^\Omega}{\partial Q} \right)$$

is the Hamiltonian vector field

$$(\dot{Q},\, \dot{P}) = \left(\frac{\partial J^\Omega}{\partial P},\, -\frac{\partial J^\Omega}{\partial Q} \right)$$

with Hamiltonian J^Ω given for fixed Ω by

$$J^\Omega = \Big\langle [P,\, Q]\,,\, \Omega \Big\rangle =: \Big\langle J\,,\, \Omega \Big\rangle$$

with variations at fixed Ω given by

$$\delta J^\Omega = \Big\langle [\delta P,\, Q],\, \Omega \Big\rangle + \Big\langle [P,\, \delta Q]\,,\, \Omega \Big\rangle = \Big\langle -\,[\Omega,\, Q],\, \delta P \Big\rangle + \Big\langle [\Omega,\, P],\, \delta Q \Big\rangle$$

obtained, for example, from the $n \times n$ matrix trace pairing,

$$\Big\langle [\delta P,\, Q],\, \Omega \Big\rangle = \mathrm{tr}\Big((\delta P) Q \Omega - Q \delta P \Omega \Big) = \Big\langle \delta P,\, [Q,\, \Omega] \Big\rangle.$$

The corresponding *momentum map* $J : T^*SO(n) \to su^*(n)$ is given by J above, namely,

$$J = [P,\, Q]\,.$$

(B) **Pauli matrices:**
For the case of 2×2 matrices in $su(2)$, the commutator $[\Omega,\, M]$ can be written as a vector cross product, by using the property of the (skew-Hermitian) Pauli matrices,

$$\sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \tag{3.4}$$

that their matrix commutator $[\sigma_a,\, \sigma_b] := \sigma_a\sigma_b - \sigma_b\sigma_a$ obeys

$$[\sigma_a,\, \sigma_b] = -2\epsilon_{abc}\sigma_c, \qquad a, b, c \in \{1, 2, 3\}.$$

This is the basis for identifying $su(2)$ and $su(2)^*$ with $\mathbb{R}^3$. By writing the vector of Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, so that

$$\Omega = \boldsymbol{\Omega} \cdot \boldsymbol{\sigma} \quad \text{and} \quad M = \boldsymbol{M} \cdot \boldsymbol{\sigma}$$

one finds $[\Omega,\, M] = \boldsymbol{\Omega} \times \boldsymbol{M} \cdot \boldsymbol{\sigma}$, so that

$$0 = \dot{M} + [\Omega,\, M] = (\dot{\boldsymbol{M}} + \boldsymbol{\Omega} \times \boldsymbol{M}) \cdot \boldsymbol{\sigma}.$$

Geodesic motion:
For geodesic motion on $SU(2)$ the Lagrangian is $l = \frac{1}{2}\langle \Omega,\, \mathbb{I}\Omega\rangle$, where $\Omega \in su(2)$ with $\Omega^\dagger = -\,\Omega$ and $M = \mathbb{I}\Omega \in su(2)^*$ with $\mathbb{I}^T = \mathbb{I}$ a real symmetric matrix. Consequently, the Lie-algebra isomorphism $su(2) \simeq \mathbb{R}^3$ implies that geodesic motion on $SU(2)$ satisfies the $\mathbb{R}^3$ vector equation

$$\dot{\boldsymbol{M}} + \boldsymbol{\Omega} \times \boldsymbol{M} = 0 \quad \text{with} \quad \boldsymbol{M} = \mathbb{I}\boldsymbol{\Omega},$$

in the *same form* as Euler's rigid body equations.

(C) The Hamiltonian form is found by taking the time derivative of a smooth function F of M,

$$\begin{aligned}
\frac{d}{dt}F(M) &= \Big\langle \frac{\partial F}{\partial M}\,,\, \dot{M} \Big\rangle \\
&= \Big\langle \frac{\partial F}{\partial M}\,,\, \mathrm{ad}^*_{\partial H/\partial M} M \Big\rangle \\
&= -\Big\langle M\,,\, \Big[\frac{\partial F}{\partial M},\, \frac{\partial H}{\partial M}\Big] \Big\rangle
\end{aligned}$$

Hence, the Poisson bracket is given by the Lie-Poisson form,

$$\{F,\, H\} = -\Big\langle M\,,\, \Big[\frac{\partial F}{\partial M},\, \frac{\partial H}{\partial M}\Big] \Big\rangle$$

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