2 M3-4-5A16 Assessed Problems # 2: Do all four problems

Exercise 2.1 (Adjoint and coadjoint actions for $SE(2)$).

(A) Compute the the adjoint and coadjoint actions AD, Ad, ad, Ad* and ad* for $SE(2)$.

(B) Show that

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}^*_{(R_\theta(t),v(t))^{-1}}(\mu,\beta) = -\,\mathrm{ad}^*_{(\xi,\alpha)}(\mu,\beta)\,,$$

where one takes $\dot{R}_\theta(t)|_{t=0} = \xi \in \mathbb{R}$, $\dot{v}(t)|_{t=0} = \alpha \in \mathbb{R}^2$ and the pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle : se(2)^* \times se(2) \to \mathbb{R}$$

is given by the dot product of vectors in $\mathbb{R}^3$,

$$\Big\langle (\mu,\beta)\,,\,(\xi,\alpha)\Big\rangle = \mu\xi + \beta\cdot\alpha\,.$$

(C) Compute the equations of motion for the dynamics on $se(2)^*$ resulting from Hamilton's principle $\delta S = 0$ with $S = \int l(\xi,\alpha)\,dt$ for the Lagrangian

$$l(\xi,\alpha) = \frac{1}{2}A\xi^2 + \frac{1}{2}\alpha^T C\alpha$$

(D) Derive the corresponding Lie-Poisson bracket for the Hamiltonian description of dynamics on $se(2)^*$.

(E) Sketch the coadjoint orbits in coordinates $(\mu,\beta) \in \mathbb{R}^3$.

(F) Work out the cotangent-lift momentum maps for the action of $SE(2)$ on $\mathbb{R}^2$.

Answer.

(A) The special Euclidean group of the plane $SE(2) \simeq SO(2)\,\circledS\,\mathbb{R}^2$ acts on a vector $q = (q_1,\, q_2)^T \in \mathbb{R}^2$ in the plane by

$$(R_\theta(t), v(t))(q) = \begin{pmatrix} R_\theta(t) & v(t) \\ 0 & 1 \end{pmatrix} \begin{bmatrix} q \\ 1 \end{bmatrix} = \begin{bmatrix} R_\theta(t)q + v(t) \\ 1 \end{bmatrix},$$

where $v = (v_1,\, v_2)^T \in \mathbb{R}^2$ is a vector in the plane and R_θ is the 2×2 matrix for rotations of vectors in the plane by angle θ about the normal to the plane $\hat{z}$,

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The infinitesimal action is found by taking $\frac{d}{dt}|_{t=0}$ of this action, which yields

$$(R_\theta(t), v(t))(q) = \begin{pmatrix} -\xi\hat{z}\times & \alpha \\ 0 & 0 \end{pmatrix} \begin{bmatrix} q \\ 1 \end{bmatrix} = \begin{bmatrix} -\xi\hat{z}\times q + \alpha \\ 1 \end{bmatrix},$$

where $\xi = \dot{\theta}(0)$ and $\alpha = \dot{v}(0)$.

By following Section 6.2 of the text, one computes the actions AD, Ad, ad, Ad* and ad* for $SE(3)$. By specialising, one finds the $se(2)$ ad-action in vector notation,

$$\begin{aligned}\mathrm{ad}_{(\xi\,,\alpha)}(\tilde{\xi}\,,\,\tilde{\alpha}) &= [(\xi,\alpha)\,,\,(\tilde{\xi},\tilde{\alpha})\,] \\ &= \Big([\xi\,,\,\tilde{\xi}\,]\,,\,\xi\tilde{\alpha}-\tilde{\xi}\alpha\Big) \\ &= \Big(0\,,\,-\xi\hat{z}\times\tilde{\alpha}+\tilde{\xi}\hat{z}\times\alpha\Big)\,.\end{aligned}$$

This expression is useful in interpreting the ad and ad* actions as motion on $\mathbb{R}^3$. In particular, the pairing between the Lie algebra $se(2)$ and its dual $se(2)^*$ is given by the dot product of vectors in $\mathbb{R}^3$,

$$\Big\langle(\mu,\beta)\,,\,(\xi,\alpha)\Big\rangle = \mu\xi+\beta\cdot\alpha\,.$$

Combining this definition of the pairing with the previous result yields an expression for the pairing of vectors using the dot product,

$$\Big\langle(\mu,\beta)\,,\,\mathrm{ad}_{(\xi\,,\alpha)}(\tilde{\xi}\,,\,\tilde{\alpha})\Big\rangle = -\alpha\times\beta\cdot\tilde{\xi}\hat{z}-\xi\hat{z}\times\beta\cdot\tilde{\alpha}\,,$$

which produces an expression for $\mathrm{ad}^*_{(\xi\,,\alpha)}(\mu,\beta)$,

$$\Big\langle\mathrm{ad}^*_{(\xi\,,\alpha)}(\mu,\beta)\,,\,(\tilde{\xi}\,,\,\tilde{\alpha})\Big\rangle = \big(-\,\alpha\times\beta,-\,\xi\hat{z}\times\beta\big)\cdot\big(\tilde{\xi}\hat{z}\,,\,\tilde{\alpha}\big)\,.$$

From here, one is able to write the Euler-Poincaré equation on $se(2)^*$ as

$$\left(\frac{d\mu}{dt},\,\frac{d\beta}{dt}\right) = \mathrm{ad}^*_{(\xi\,,\alpha)}(\mu,\beta) = \big(-\,\alpha\times\beta,-\,\xi\hat{z}\times\beta\big) \quad\text{with}\quad (\mu,\beta):=\left(\frac{\partial l}{\partial\xi},\,\frac{\partial l}{\partial\alpha}\right)$$

As we shall see, one may then Legendre transform over to the Lie-Poisson Hamiltonian formulation of motion on $se(2)^*$, by identifying

$$(\xi\hat{z}\,,\,\alpha) = \left(\frac{\partial h}{\partial\mu}\hat{z}\,,\,\frac{\partial h}{\partial\beta}\right).$$

The Casimirs of the Lie-Poisson bracket also determine the coadjoint orbits, which turn out to be concentric cylinders of radius $|\beta|$ centered on the μ-axis, plus fixed points on the μ-axis, as we discuss below.

(B) This is a special case of the following general result.

Co-Adjoint motion equation:
Let $g(t)$ be a path in a Lie group G and $\mu(t)$ be a path in $\mathfrak{g}^*$. Then

$$\frac{d}{dt}\mathrm{Ad}^*_{g(t)^{-1}}\mu(t) = \mathrm{Ad}^*_{g(t)^{-1}}\left[\frac{d\mu}{dt}-\mathrm{ad}^*_{\xi(t)}\mu(t)\right]\,, \qquad (2.1)$$

where $\xi(t)=g(t)^{-1}\dot{g}(t)$.

(C) The **Euler-Poincaré equation** on $se(2)^*$ is

$$\begin{aligned}\left(\frac{d\mu}{dt},\,\frac{d\beta}{dt}\right) &= \mathrm{ad}^*_{(\xi\,,\alpha)}(\mu,\beta) = \big(-\,\alpha\times\beta,-\,\xi\hat{z}\times\beta\big) \\ \text{with}\quad (\mu,\beta) &:= \left(\frac{\partial l}{\partial\xi},\,\frac{\partial l}{\partial\alpha}\right) = (A\xi,\,C\alpha)\end{aligned}$$

(D) Legendre transforming the Euler-Poincaré equation yields

$$\big(\dot{\mu}\hat{z}\,,\,\dot{\beta}\big) = \mathrm{ad}^*_{(\partial h/\partial\mu,\partial h/\partial\beta)}(\mu\hat{z},\beta) = \left(-\,\frac{\partial h}{\partial\beta}\times\beta\,,\,-\frac{\partial h}{\partial\mu}\hat{z}\times\beta\right).$$

After taking the time-derivative of an arbitrary function f of the Hamiltonian momentum variables $(\mu\hat{z}, \beta)$ this yields the Lie-Poisson bracket,

$$\Big\{f,\, h\Big\}(\mu\hat{z}, \beta) = -\beta\cdot\left(\frac{\partial f}{\partial\mu}\hat{z}\times\frac{\partial h}{\partial\beta} - \frac{\partial h}{\partial\mu}\hat{z}\times\frac{\partial f}{\partial\beta}\right)$$

(E) The Casimirs for this Lie-Poisson bracket are concentric cylinders of radius

$$|\beta| = \sqrt{\beta_1^2 + \beta_2^2}$$

centered on the μ-axis, plus fixed points on the μ-axis for which $\mu\hat{z}\cdot\beta = 0$.

(F) The infinitesimal action of $SE(2)$ on coordinates $\mathbf{q}\in\mathbb{R}^2$ in the plane is

$$\mathbf{q}\to\mathbf{q}' = -\xi\hat{\mathbf{z}}\times\mathbf{q} + \alpha\,,$$

where $\alpha\in\mathbb{R}^2$. The cotangent lift of this infinitesimal action is

$$J^{(\xi,\alpha)} = \mathbf{p}\cdot\big(-\xi\hat{\mathbf{z}}\times\mathbf{q} + \alpha\big) = \mathbf{p}\times\mathbf{q}\cdot\xi\hat{\mathbf{z}} + \mathbf{p}\cdot\alpha = \Big\langle\big(\mathbf{p}\times\mathbf{q}\,,\,\mathbf{p}\big)\,,\,\big(\xi\hat{\mathbf{z}}\,,\,\alpha\big)\Big\rangle\,,$$

for $\mathbf{p}\in T^*\mathbb{R}^2$ at $\mathbf{q}\in\mathbb{R}^2$. That is, the momentum map $J^{(\xi,\alpha)}$ has 2 components.

- The component $\mu\,\hat{\mathbf{z}} = \mathbf{p}\times\mathbf{q}$ is the angular momentum of rotations in the plane. It points normal to the plane.
- The component $\mathbf{p} = \beta$ is the linear momentum in the plane.
- The Euler-Poincaré and Lie-Poisson formulations of the dynamics determines how these two components of the $SE(2)$ momentum map evolve for a given Lagrangian or Hamiltonian.

▲

Exercise 2.2 ($GL(n,\mathbb{R})$-invariant motions)**.**

Consider the Lagrangian

$$L = \frac{1}{2}\,\mathrm{tr}\Big(\dot{S}S^{-1}\dot{S}S^{-1}\Big) + \frac{1}{2}\,\dot{\mathbf{q}}\cdot S^{-1}\dot{\mathbf{q}}\,,$$

where S is an $n\times n$ symmetric matrix and $\mathbf{q}\in\mathbb{R}^n$ is an $n-$component column vector.

(A) Legendre transform to construct the corresponding Hamiltonian and canonical equations.

(B) Show that the Lagrangian and Hamiltonian are invariant under the group action

$$\mathbf{q}\to G\mathbf{q}\quad\text{and}\quad S\to GSG^T$$

for any constant invertible $n\times n$ matrix, G.

(C) Compute the infinitesimal generator for this group action and construct its corresponding momentum map. Is this momentum map equivariant? Prove it.

(D) Verify directly that this momentum map is a conserved $n\times n$ matrix quantity by using the equations of motion.

Answer.

(A) Legendre transform as

$$P = \frac{\partial L}{\partial \dot S} = S^{-1}\dot S S^{-1} \quad \text{and} \quad \mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = S^{-1}\dot{\mathbf{q}}$$

Thus, the Hamiltonian $H(Q,P)$ and its canonical equations are:

$$H(\mathbf{q},\mathbf{p},S,P) = \frac12 \mathrm{tr}\,\big(PS\cdot PS\big) + \frac12 \mathbf{p}\cdot S\mathbf{p}\,,$$

$$\dot S = \frac{\partial H}{\partial P} = SPS\,, \quad \dot P = -\frac{\partial H}{\partial S} = -\Big(PSP + \frac12 \mathbf{p}\otimes\mathbf{p}\Big),$$

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} = S\mathbf{p}\,, \quad \dot{\mathbf{p}} = \frac{\partial H}{\partial \mathbf{q}} = 0\,.$$

(B) Under the group action $\mathbf{q}\to G\mathbf{q}$ and $S\to GSG^T$ for any constant invertible $n\times n$ matrix, G, one finds $\dot S S^{-1} \to G\dot S S^{-1} G^{-1}$ and $\dot{\mathbf{q}}\cdot S^{-1}\dot{\mathbf{q}} \to \dot{\mathbf{q}}\cdot S^{-1}\dot{\mathbf{q}}$. Hence, $L\to L$. Likewise, $P\to G^{-T}PG^{-1}$ so $PS\to G^{-T}PSG^T$ and $\mathbf{p}\to G^{-T}\mathbf{p}$ so that $S\mathbf{p}\to GS\mathbf{p}$. Hence, $H\to H$, as well; so both L and H for the system are invariant.

(C) The infinitesimal actions for $G(\epsilon) = Id + \epsilon A + O(\epsilon^2)$, where $A\in gl(n)$ are

$$X_A\mathbf{q} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} G(\epsilon)\mathbf{q} = A\mathbf{q}$$

and

$$X_A S = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\Big(G(\epsilon)SG(\epsilon)^T\Big) = AS + SA^T$$

The defining relation for the corresponding momentum map yields

$$\begin{aligned}\langle J, A\rangle = \langle (Q,P), X_A\rangle &= \mathrm{tr}\,(PX_AS) + \mathbf{p}\cdot X_A\mathbf{q}\\ &= \mathrm{tr}\,\big(P(AS+SA^T)\big) + \mathbf{p}\cdot A\mathbf{q}\end{aligned}$$

Hence, $\langle J, A\rangle := \mathrm{tr}\,\big(JA^T\big) = \mathrm{tr}\,\big((2SP + \mathbf{q}\otimes\mathbf{p})A\big)$, so

$$J = (2PS + \mathbf{p}\otimes\mathbf{q})$$

This momentum map is a cotangent lift, so it is equivariant.

(D) Conservation of the momentum map is verified directly by:

$$\dot J = (2\dot P S + 2P\dot S + \mathbf{p}\otimes\dot{\mathbf{q}}) = 0$$

▲

Exercise 2.3 (Canonical variables for the rigid body on $SO(n)$)**.**

The Euler-Lagrange equation for the rigid body on $SO(n)$ are given in matrix commutator form

$$\frac{dM}{dt} = [\,M\,,\,\Omega\,] \quad \text{with} \quad M = \mathbb{A}\Omega + \Omega\mathbb{A}\,, \tag{2.2}$$

where the $n\times n$ matrices M, Ω are skew-symmetric. The tangent lift of the *right* action of the group $SO(n)$ on itself is given by

$$Q_t = Q_0 O_t \Longrightarrow \dot Q_t = Q_t\Omega_t \quad \text{with} \quad \Omega_t = O_t^{-1}O_t$$

where $\Omega_t = O_t^{-1}O_t$ is *left*-invariant under $O\to UO$, with $U\in SO(n)$.

(A) Show that equation (2.2) may be derived from Hamilton's principle $\delta S = 0$ whose action integral is constrained by the tangent lift of the right-action of the group $SO(n)$ on itself. That is, for

$$\begin{aligned} S(\Omega, Q, P) &= \int_a^b l(\Omega) + \Big\langle P, \dot{Q} - Q\Omega \Big\rangle \, dt \\ &= \int_a^b l(\Omega) + \mathrm{tr}\Big(P^T \left(\dot{Q} - Q\Omega \right) \Big) \, dt \,, \end{aligned} \tag{2.3}$$

derive equation (2.2), in which $M = \delta l / \delta\Omega = \frac{1}{2}(Q^T P - P^T Q)$, and the $Q, P \in SO(n)$ satisfy the following equations,

$$\dot{Q} = Q\Omega \quad \text{and} \quad \dot{P} = P\Omega \,, \tag{2.4}$$

as a result of the constraints.

(B) Write these equations in Hamiltonian form and show that they recover the motion equation (2.2). What is the Hamiltonian in these variables?

(C) Compute the Poisson bracket for functions of M by making the change of variables $M : T^*Q \to so(n)^*$ given by

$$M = \frac{1}{2}(Q^T P - P^T Q).$$

Answer.

(A) The constraint implies an angular velocity $Q^{-1}(t)\dot{Q}(t) = \Omega(t)$, that is left-invariant under $Q \to UQ$, for any fixed $U \in SO(n)$. Thus, the Lagrangian in (2.3) is left-invariant under this action, too. This invariance sets up the transformation from the $(Q,\, P)$ equations to the M equation. In terms of the trace pairing for skew-symmetric matrices,

$$\langle\, A \,,\, B \,\rangle := \mathrm{tr}(A^T B),$$

we find

$$\begin{aligned} \delta S(\Omega, Q, P) &= \int_a^b \Big\langle \frac{\partial l}{\partial \Omega} - Q^T P \,,\, \delta\Omega \Big\rangle \\ &\quad + \Big\langle \delta P \,,\, \dot{Q} - Q\Omega \Big\rangle - \Big\langle \dot{P} - P\Omega \,, \delta Q \Big\rangle \, dt \,, \end{aligned}$$

after using the identity

$$-\,\delta\langle P,\, Q\Omega\rangle = \langle \delta P,\, -\,Q\Omega\rangle + \langle P\Omega,\, \delta Q\rangle + \langle -\,Q^T P,\, \delta\Omega\rangle.$$

Given Ω, setting $\delta S = 0$ produces the canonical equations

$$\dot{Q} = Q\Omega = \frac{\partial J^\Omega}{\partial P} \quad \text{and} \quad \dot{P} = P\Omega = -\,\frac{\partial J^\Omega}{\partial Q}$$

for the Hamiltonian $J^\Omega = \langle Q^T P,\, \Omega\rangle$ with variations

$$\begin{aligned} \delta J^\Omega &= \langle (\delta Q^T) P + Q^T \delta P,\, \Omega \rangle \\ &= \mathrm{tr}\Big(P^T \delta Q \Omega + (\delta P^T) Q\Omega \Big) \\ &= \mathrm{tr}\Big((P\Omega^T)^T \delta Q + (\delta P^T) Q\Omega \Big) \\ &= \langle P\Omega^T,\, \delta Q\rangle + \langle \delta P,\, Q\Omega\rangle \\ &= \langle -\,P\Omega,\, \delta Q\rangle + \langle \delta P,\, Q\Omega\rangle \end{aligned}$$

Thus, the vector field consisting of the tangent and cotangent lifts

$$(\dot{Q},\, \dot{P}) = (Q\Omega,\, P\Omega)$$

is the Hamiltonian vector field

$$X_{J^\Omega} := \{\,\cdot\,,\, J^\Omega\}$$

for the Hamiltonian

$$J^\Omega = \langle Q^T P,\, \Omega\rangle = \langle \tfrac{1}{2}(Q^T P - P^T Q),\, \Omega\rangle$$

at fixed Ω. The quantity $M = \frac{1}{2}(Q^T P - P^T Q)$ is called the *momentum map* $M : T^*SO(n) \to so(n)^*$ for the right-action of $SO(n)$ on itself.

Extra credit:
Compute the momentum map for the *left*-action of $SO(n)$ on itself.

(B) To find the Hamiltonian form, set $M := \partial l/\partial\Omega = \frac{1}{2}(Q^T P - P^T Q)$ and take $\xi = -\xi^T \in so(n)$. Compute the pairing

$$\begin{aligned}
\big\langle \dot{M}, \xi \big\rangle &= -\big\langle \dot{P}^T Q + P^T \dot{Q}, \xi \big\rangle \\
&= -\big\langle (P\Omega)^T Q + P^T (Q\Omega), \xi \big\rangle \\
&= -\big\langle -\Omega P^T Q + P^T Q \Omega, \xi \big\rangle \\
&= \big\langle -\Omega M + M\Omega, \xi \big\rangle \\
&= \big\langle -\big[\Omega,\, M\big], \xi \big\rangle
\end{aligned}$$

The Hamiltonian in these variables is found from the Legendre transform,

$$h(M) = \big\langle M, \Omega \big\rangle - l(\Omega)$$

which satisfies

$$\delta h(M) = \Big\langle \frac{\partial h}{\partial M}\,,\, \delta M \Big\rangle = \Big\langle M - \frac{\partial l}{\partial \Omega}\,, \delta\Omega \Big\rangle + \big\langle \delta M, \Omega \big\rangle.$$

Hence, we have the dual velocity-momentum relations

$$\frac{\partial h}{\partial M} = \Omega \quad \text{and} \quad M = \frac{\partial l}{\partial \Omega}$$

and the equation of motion $\dot{M} = -[\Omega,\, M]$ becomes

$$\dot{M} = -\left[\frac{\partial h}{\partial M}\,,\, M\right] = \{M,\, h\}_{LP}$$

with Lie-Poisson brackets given by

$$\frac{df}{dt} = -\left\langle \frac{\partial f}{\partial M}\,,\, \left[\frac{\partial h}{\partial M}\,,\, M\right]\right\rangle = -\left\langle M\,,\, \left[\frac{\partial f}{\partial M}\,,\, \frac{\partial h}{\partial M}\right]\right\rangle = \{f,\, h\}_{LP}$$

(C) A direct computation using the canonical brackets $\{Q_i,\, P_j\} = \delta_{ij}$ gives the Lie-Poisson bracket in terms of matrix components of $M \in so(n)^*$,

$$\begin{aligned}
\{M_{ij},\, M_{kl}\} &= \tfrac{1}{4}\{P_i Q_j - Q_i P_j,\, P_k Q_l - Q_k P_l\} \\
&= \tfrac{1}{2}\Big(- M_{jk}\delta_{il} + M_{ik}\delta_{jl} - M_{il}\delta_{jk} + M_{jl}\delta_{ik}\Big).
\end{aligned}$$

The motion equation is then obtained from

$$\begin{aligned}\dot{M}_{ij} &= \{M_{ij},\, h\} = \{M_{ij},\, M_{kl}\}\frac{\partial h}{\partial M_{kl}} =: \{M_{ij},\, M_{kl}\}\Omega_{kl} \\ &= (M\Omega)_{ij} - (M\Omega)_{ji} = [M,\, \Omega]_{ij}\end{aligned}$$

where the angular velocity components are defined as

$$\Omega_{kl} := \frac{\partial h}{\partial M_{kl}} \,.$$

▲

Exercise 2.4 (Euler-Poincaré equation EPDiff in one dimension)**.**

The EPDiff($\mathbb{R}$) equation for the H^1 norm of the velocity u is obtained from the Euler-Poincaré reduction theorem for a right-invariant Lagrangian, when one defines the Lagrangian to be half the square of the H^1 norm $\|u\|_{H^1}$ of the vector field of velocity $u = \dot{g}g^{-1} \in \mathfrak{X}(\mathbb{R})$ on the real line $\mathbb{R}$ with $g \in \mathrm{Diff}(\mathbb{R})$. Namely,

$$l(u) = \frac{1}{2}\|u\|^2_{H^1} = \frac{1}{2}\int_{-\infty}^{\infty} u^2 + u_x^2 \; dx \,.$$

(Assume $u(x)$ vanishes as $|x| \to \infty$.)

(A) Derive the EPDiff equation on the real line in terms of its velocity u and its momentum $m = \delta l/\delta u = u - u_{xx}$ in one spatial dimension for this Lagrangian.

Hint: Prove a Lemma first, that $u = \dot{g}g^{-1}$ implies $\delta u = \xi_t - \mathrm{ad}_u\xi$ with $\xi = \delta g g^{-1}$.

(B) Use the Clebsch constrained Hamilton's principle

$$S(u,p,q) = \int l(u)\, dt + \int p(t)\big(\dot{q}(t) - u(q(t),t)\big)dt$$

to derive the peakon singular solution $m(x,t)$ of EPDiff as a momentum map in terms of canonically conjugate variables $q_i(t)$ and $p_i(t)$, with $i = 1, 2, \dots, N$.

Answer.

(A) ***Lemma***

$u = \dot{g}g^{-1}$ implies $\delta u = \xi_t - \mathrm{ad}_u\xi$ with $\xi = \delta g g^{-1}$.

Proof. Write $\xi = \dot{g}g^{-1}$ and $\eta = g'g^{-1}$ in natural notation and express the partial derivatives $\dot{g} = \partial g/\partial t$ and $g' = \partial g/\partial \epsilon$ using the right translations as

$$\dot{g} = \xi \circ g \quad \text{and} \quad g' = \eta \circ g \,.$$

By the chain rule, these definitions have mixed partial derivatives

$$\dot{g}' = \xi' = \nabla\xi \cdot \eta \quad \text{and} \quad \dot{g}' = \dot{\eta} = \nabla\eta \cdot \xi \,.$$

The difference of the mixed partial derivatives implies the desired formula,

$$\xi' - \dot{\eta} = \nabla\xi \cdot \eta - \nabla\eta \cdot \xi =: -\mathrm{ad}_\xi\eta \,.$$

□

Deriving the EPDiff equation on the real line:

The EPDiff(H^1) equation is written on the real line in terms of its velocity u and its momentum $m = \delta l/\delta u$ in one spatial dimension as

$$m_t + um_x + 2mu_x = 0\,, \quad \text{where} \quad m = u - u_{xx}$$

where subscripts denote partial derivatives in x and t.

Proof. This equation is derived from the variational principle with $l(u) = \frac{1}{2}\|u\|^2_{H^1}$ as follows.

$$\begin{aligned}
0 = \delta S &= \delta \int l(u) dt = \frac{1}{2}\,\delta \iint u^2 + u_x^2 \, dx\, dt \\
&= \iint (u - u_{xx})\, \delta u \, dx \, dt =: \iint m\, \delta u \, dx\, dt \\
&= \iint m\, (\xi_t - \mathrm{ad}_u \xi)\, dx\, dt \\
&= \iint m\, (\xi_t + u\xi_x - \xi u_x)\, dx\, dt \\
&= -\iint (m_t + (um)_x + mu_x)\, \xi \, dx\, dt \\
&= -\iint (m_t + \mathrm{ad}^*_u m)\, \xi\, dx\, dt\,,
\end{aligned}$$

where $u = \dot{g}g^{-1}$ implies $\delta u = \xi_t - \mathrm{ad}_u\xi$ with $\xi = \delta g g^{-1}$. □

Hamiltonian structure for EPDiff:

Legendre transformation:

$$h(m) = \int mu \, dx - l(u)$$

so

$$\delta h = \int u \delta m \, dx + \int (m - u + u_{xx})\, \delta u \, dx$$

Thus, $u = \delta h/\delta m$, $m = \delta l/\delta u - u - u_{xx}$ and

$$m_t = -\mathrm{ad}^*_{\delta h/\delta m} m = -(\partial_x m + m\partial_x)\frac{\delta h}{\delta m}$$

The corresponding ***Lie-Poisson bracket*** is

$$\{f,\, h\}(m) = -\int \frac{\delta f}{\delta m}(\partial_x m + m\partial_x)\frac{\delta h}{\delta m}\, dx$$

with ***Casimir***

$$C = \int \sqrt{m}\, dx$$

(B) The constrained Clebsch action integral is given as

$$S(u,p,q) = \int l(u)\, dt + \int p(t)\big(\dot{q}(t) - u(q(t),t)\big) dt$$

whose variation in u is gotten by inserting a delta function, so that

$$\begin{aligned}
0 = \delta S &= \int \left(\frac{\delta l}{\delta u} - p(t)\delta(x - q(t)) \right) \delta u \, dx\, dt \\
&\quad - \int \left(\dot{p}(t) + \frac{\partial u}{\partial q}\, p(t) \right) \delta q - \delta p \Big(\dot{q}(t) - u(q(t),t) \Big) dt\,.
\end{aligned}$$

The singular momentum solution $m(x,t)$ of EPDiff(H^1) is written as the ***momentum map***

$$m(x,t) = \delta l/\delta u = p(t)\delta(x - q(t))$$

$$\int m(x,t)u(x,t)\,dx = \int p(t)\delta(x-q(t))u(x,t)\,dx = p(t)u(q(t),t)$$

Consequently, the variables (q,p) satisfy canonical Hamiltonian equations,

$$\dot q(t) = u(q(t),t) = \frac{\partial h}{\partial p}\,, \qquad \dot p(t) = -\frac{\partial u}{\partial q}\,p(t) = -\frac{\partial h}{\partial q}\,,$$

with $u(q(t),t) = p(t)G(q(t))$ where $G(x)$ is the Green's function for the Helmholtz operator $1 - \partial_x^2$. That is,

$$G(x) = \frac{1}{2}e^{-|x|}$$

Consequently, one may write the Hamiltonian for the canonical parameters of the singular solution explicitly as

$$h(p,q) = \frac{1}{2}p^2 G(q) = \frac{1}{4}p(t)^2 e^{-|q(t)|}$$

Note that all of this calculation goes through just the same for the multi-particle case. E.g., for N particles,

$$S(u,\{p\},\{q\}) = \int l(u)\,dt + \sum_{A=1}^{N}\int p_A(t)\big(\dot q_A(t) - u(q_A(t),t)\big)dt$$

▲