M3-4-5 A34 Handout: What is Geometric Mechanics II?

Professor Darryl D Holm   9 January 2013
Imperial College London   d.holm@ic.ac.uk
`http://www.ma.ic.ac.uk/~dholm/`

Course meets $\boxed{\text{T, W, Th 11am @ Hux 658}}$

Text for the course M3-4-5 A34:

Geometric Mechanics II: Rotating, Translating & Rolling (aka GM2)
by Darryl D Holm, World Scientific: Imperial College Press, Singapore, Second edition (2011).
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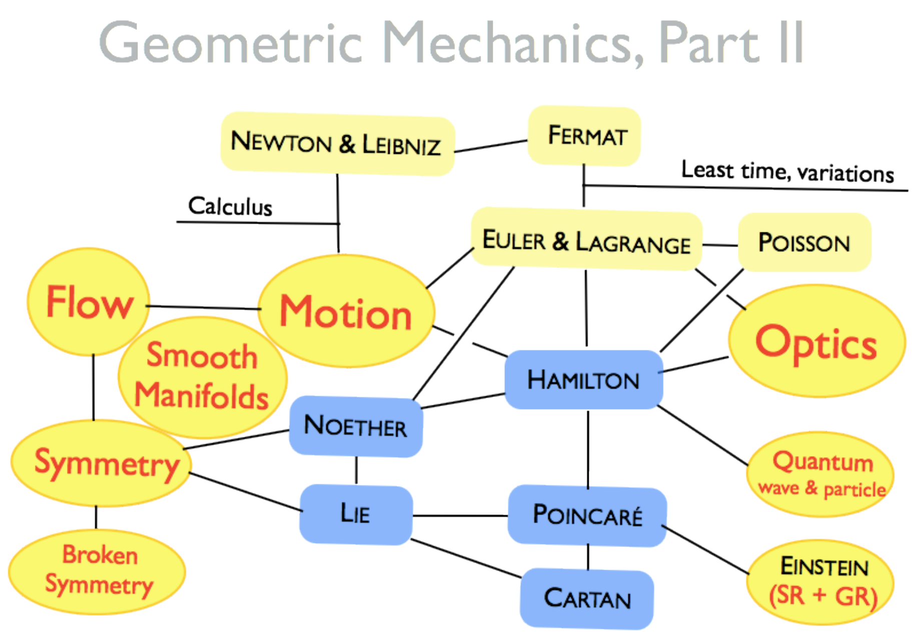

Figure 1: Geometric Mechanics has involved many great mathematicians!

Space, Time, Motion, . . . , Symmetry, Dynamics!

Background reading: Chapter 2, [Ho2011GM1].

Space

Space is taken to be a manifold Q with points $q \in Q$ (Positions, States, Configurations). The manifold Q will sometime be taken to be a Lie group G. We will do this when we consider rotation and translation, for example. In this case the group is $G = SE(3)$ the special Euclidean group in three dimensions.

Time

Time is taken to be a manifold T with points $t \in T$. Usually $T = \mathbb{R}$ (for real 1D time), but we will also consider $T = \mathbb{R}^2$ and maybe let T and Q both be complex manifolds

Motion

Motion is a map $\phi_t : T \to Q$, where subscript t denotes dependence on time t. For example, when $T = \mathbb{R}$, the motion is a curve $q_t = \phi_t \circ q_0$ obtained by composition of functions. The motion is called a *flow* if $\phi_{t+s} = \phi_t \circ \phi_s$, for $s, t \in \mathbb{R}$, and $\phi_0 = \mathrm{Id}$, so that $\phi_t^{-1} = \phi_{-t}$. Note that the composition of functions is associative, $(\phi_t \circ \phi_s) \circ \phi_r = \phi_t \circ (\phi_s \circ \phi_r) = \phi_t \circ \phi_s \circ \phi_r = \phi_{t+s+r}$, but it is not commutative, in general. Thus, we should anticipate Lie group actions on manifolds.

Velocity

Velocity is an element of the tangent bundle TQ of the manifold Q. For example, $\dot{q}_t \in T_{q_t}Q$ along a flow q_t that describes a smooth curve in Q.

Motion equation

The motion equation that determines $q_t \in Q$ takes the form

$$\dot{q}_t = f(q_t)$$

where $f(q)$ is a prescribed *vector field* over Q. For example, if the curve $q_t = \phi_t \circ q_0$ is a flow, then

$$\dot{q}_t = \dot{\phi}_t \phi_t^{-1} \circ q_t = f(q_t)$$

so that

$$\dot{\phi}_t = f \circ \phi_t =: \phi_t^* f$$

which defines the pullback of f by ϕ_t.

Optimal motion equation – Hamilton's principle

An *optimal* motion equation arises from Hamilton's principle,

$$\delta S[q_t] = 0 \quad \text{for} \quad S[q_t] = \int L(q_t, \dot{q}_t)\, dt\,,$$

in which variational derivatives are given by

$$\delta S[q_t] = \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} S[q_{t,\epsilon}]\,.$$

The introduction of a variational principle summons T^*Q, the cotangent bundle of Q. The cotangent bundle T^*Q is the dual space of the tangent bundle TQ, with respect to a pairing. That is, T^*Q is the space of real linear functionals on TQ with respect to the (real nondegenerate) pairing $\langle\,\cdot\,,\,\cdot\,\rangle$, induced by taking the variational derivative.

For example,

$$\text{if}\quad S = \int L(q,\dot q)\,dt\,,\quad \text{then}\quad \delta S = \int \Big\langle \frac{\partial L}{\partial \dot q_t}\,,\,\delta \dot q_t) \Big\rangle + \Big\langle \frac{\partial L}{\partial q_t}\,,\,\delta q_t) \Big\rangle\,dt = 0$$

leads to the *Euler-Lagrange equations*

$$-\,\frac{d}{dt}\frac{\partial L}{\partial \dot q_t} + \frac{\partial L}{\partial q_t} = 0\,.$$

The map $p := \frac{\partial L}{\partial \dot q_t}$ is called the *fibre derivative* of the Lagrangian $L : TQ \to \mathbb{R}$. The Lagrangian is called *hyperregular* if the velocity can be solved from the fibre derivative, as $\dot q_t = v(q,p)$. Hyperregularity of the Lagrangian is sufficient for invertibility of the *Legendre transformation*

$$H(q,p) := \langle\, p\,,\,\dot q\,\rangle - L(q,\dot q)$$

In this case, Hamilton's principle

$$0 = \delta \int \langle\, p\,,\,\dot q\,\rangle - H(q,p)\,dt\,,$$

gives *Hamilton's canonical equations*

$$\dot q = H_p \quad \text{and} \quad \dot p = -\,H_q\,,$$

whose solutions are equivalent to those of the Euler-Lagrange equations.

Symmetry

Lie group symmetries of the Lagrangian will be particularly important, both in reducing the number of independent degrees of freedom in Hamilton's principle and in finding conservation laws by Noether's theorem.

Dynamics!

Dynamics is the science of deriving, analysing, solving and interpreting the solutions of motion equations. GM2 will concentrate on dynamics in the case that the configuration space Q is a Lie group itself G and the Lagrangian $TG \to \mathbb{R}$ is transforms simply (e.g., is invariant) under the action of G. When the Lagrangian $TG \to \mathbb{R}$ is invariant under G, the problem reduces to a formulation on $TG/G \simeq \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G. With an emphasis on applications in mechanics, we will discuss a variety of interesting properties and results that are inherited from this formulation of dynamics on Lie groups.

What shall we study?

Figure 1 illustrates some of the relationships among the various accomplishments of the founders of geometric mechanics. We shall study these accomplishments and the relationships among them.

Hamilton: quaternions, AD, Ad, ad, Ad*, ad* actions, variational principles
Lie: Groups of transformations that depend smoothly on parameters
Poincaré: Mechanics on Lie groups, $SO(3)$, $SU(2)$, $Sp(2)$, $SE(3) \simeq SO(3) \circledS \mathbb{R}^3$
Noether: Implications of symmetry in variational principles
Cartan: Lie transformations of differential forms and fluid flows

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