Handout: Notes on Linear Symplectic Transformations

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Course meets T 11am, W 10am, Th 11am @ Hux 658

Contents

1 Plan of these lecture notes on $Sp(2)$

Introduce matrix Lie group $Sp(2)$ and its matrix Lie algebra $\mathfrak{sp}(2)$

Consider our goal: understanding coadjoint motion on $\mathfrak{sp}(2)^*$

Study the transformations in $Sp(2)$ and their infinitesimal generators, the Hamiltonian matrices $\{m_0, m_1, m_2, m_3\} \in \mathfrak{sp}(2)$. The latter are related to co-quaternions.

Write infinitesimal transformations of the matrix Lie algebra $\mathfrak{sp}(2)$ acting on $\mathbf{z} = (\mathbf{q}, \mathbf{p})^T \in T^*\mathbb{R}^2$. (Note that this action leaves $\mathbf{p} \times \mathbf{q}$ invariant.)

Study the Adjoint orbits $\mathrm{Ad} : Sp(2) \times \mathfrak{sp}(2) \to \mathfrak{sp}(2)$

Define the momentum map $\mathcal{J} : T^*\mathbb{R}^2 \to \mathfrak{sp}(2)^*$ and the Hamiltonian $\mathcal{J}^\xi = \langle \mathcal{J}, \xi \rangle$ for the infinitesimal transformations $\dot{\mathbf{z}} = \{\mathbf{z}, \mathcal{J}^\xi\}$.

Derive the momentum map $\mathcal{J}$ from Hamilton's principle $\delta S = 0$ with $S = \int \ell(\xi) dt$, constrained by the action of $T^*\mathbb{R}^2$ under $Sp(2)$, thereby finding with right invariant $\xi = \dot{M}M^{-1}$

$$\frac{\partial \ell}{\partial \xi} = \mathcal{J}(\mathbf{z}) \qquad \dot{\mathbf{z}} = \xi \mathbf{z} \quad \Longrightarrow \quad \frac{d\mathcal{J}}{dt} + \mathrm{ad}^*_\xi \mathcal{J} = 0 \quad \text{or} \quad \mathrm{Ad}^*_{M^{-1}(t)} \mathcal{J}(t) = \mathcal{J}(0)$$

Write the components of the momentum map $\mathcal{J}(\mathbf{z}) \in \mathfrak{sp}(2)^*$ in terms of $\mathbf{Y} \in \mathbb{R}^3$, with

$$\mathcal{J} = \mathbf{z} \otimes \mathbf{z}^T J = 2 \begin{bmatrix} Y_3 & -Y_1 \\ Y_2 & -Y_3 \end{bmatrix} \qquad \mathbf{Y} = (Y_1, Y_2, Y_3) = (|\mathbf{q}|^2, |\mathbf{p}|^2, \mathbf{p} \cdot \mathbf{q})$$

noting that $S^2(\mathbf{Y}) := Y_1 Y_2 - Y_3^2 = |\mathbf{p} \times \mathbf{q}|^2$ in which $\mathbf{p} \times \mathbf{q}$ is invariant under $Sp(2)$.

Explain how $\mathbb{R}^3$-bracket dynamics of $\dot{\mathbf{Y}} = \nabla S^2 \times \nabla H$ is related to coadjoint motion on $\mathfrak{sp}(2)^*$.

2 The matrix Lie group $Sp(2)$ and its matrix Lie algebra $\mathfrak{sp}(2)$

Symplectic 2×2 matrices $M(s) \in Sp(2)$ depending smoothly on a real parameter $s \in \mathbb{R}$ satisfy

$$M(s) J M(s)^T = J \tag{1}$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

Set $\xi(s) = \dot{M}(s) M^{-1}(s)$, so that $\dot{M}(s) = \xi(s) M(s)$ is the reconstruction equation. Take d/ds of the defining relation

$$\begin{aligned} \frac{d}{ds}\left(M(s) J M(s)^T\right) &= \xi J + J \xi^T \\ &= \xi J + (\xi J^T)^T \\ &= \xi J - (\xi J)^T = 0 \end{aligned}$$

Thus, $\xi J = (\xi J)^T \in sym$ is symmetric.

- Conjugation by J shows that $J\xi = (J\xi)^T \in sym$ is also symmetric.
- Replacing $M \leftrightarrow M^T$ gives the corresponding result for left invariant $\Xi(s) := M^{-1}(s)\dot{M}(s)$.

2.1 Examples of matrices in $Sp(2)$

One easily checks that the following are $Sp(2)$ matrices

$$M_1(\tau_1) = \begin{bmatrix} 0 & -1 \\ -2\tau_1 & 1 \end{bmatrix}, \qquad M_3(\tau_3) = \begin{bmatrix} e^{\tau_3} & 0 \\ 0 & e^{-\tau_3} \end{bmatrix}, \qquad M(\omega) = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}.$$

The product is also an $Sp(2)$ matrix

$$M(\tau_1, \tau_3, \omega) = M(\omega) M_3(\tau_3) M_1(\tau_1) = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} e^{\tau_3} & 0 \\ 0 & e^{-\tau_3} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -2\tau_1 & 1 \end{bmatrix} \tag{3}$$

This is a general result, called the ***Iwasawa decomposition*** of the symplectic matrix group, usually written as

$$Sp(2, \mathbb{R}) = \mathsf{KAN}.$$

The rightmost matrix factor represents the nilpotent subgroup N. The middle factor is the abelian subgroup A. The leftmost factor is the maximal compact subgroup K.

The KAN decomposition (3) also follows by exponentiation

$$M(\tau_1, \tau_3, \omega) = M(\omega) M_3(\tau_3) M_1(\tau_1) = e^{\frac{1}{2}\omega(m_1 + m_2)} e^{\tau_3 m_3} e^{\tau_1 m_1} \tag{4}$$

of the 2×2 Hamiltonian matrices

$$m_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad m_1 = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad m_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5}$$

These are related to the Pauli matrices and thus to the quaternions by

$$m_0 = \sigma_0, \quad m_1 = -\sigma_1 + i\sigma_2, \quad m_2 = \sigma_1 + i\sigma_2, \quad m_3 = \sigma_3, \tag{6}$$

so that $\frac{1}{2}(m_1 + m_2) = i\sigma_2$.

2.2 Hamiltonian matrices and co-quaternions

One may define a co-quaternion C and its conjugate C^* as

$$C = wm_0 + xm_1 + ym_2 + zm_3 = \begin{bmatrix} w+z & 2y \\ -2x & w-z \end{bmatrix} \qquad C^* = \begin{bmatrix} w-z & -2y \\ 2x & w+z \end{bmatrix}$$

Then $\operatorname{tr} CC^* = 2(w^2 - z^2) + 8xy$ is a hyperbolic relation that leads to co-quaternions whose multiplication law is obtained from the relation (6) between the Hamiltonian basis matrices and the Pauli matrices.

2.3 Infinitesimal transformations

Definition 1. The infinitesimal transformation of the $Sp(2)$ matrix Lie group acting on the manifold $T^*\mathbb{R}^2$ is a vector field $\xi_M(\mathbf{z}) \in T\mathbb{R}^2$ that may be expressed as the derivative of the group transformation, evaluated at the identity,

$$\xi_M(\mathbf{z}) = \frac{d}{ds}\big[\exp(s\xi)\mathbf{z}\big]\Big|_{s=0} = \xi\mathbf{z}\,. \tag{7}$$

Here, the diagonal action of the Hamiltonian matrix (ξ) and the two-component real multi-vector $\mathbf{z} = (\mathbf{q},\, \mathbf{p})^T$ (denoted as $\xi\mathbf{z}$) has components given by $(\xi_{kl}q_l, \xi_{kl}p_l)^T$, with $k, l = 1, 2$. The matrix ξ is any linear combination of the traceless constant Hamiltonian matrices (5).

Examples The action of $Sp(2)$ on $T^*\mathbb{R}^2$ is obtained from the infinitesimal actions, expressed in terms of the Hamiltonian matrices, as

$$m_1: \quad \frac{d}{d\tau_1}\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} 0 \\ -2\mathbf{q} \end{bmatrix} \quad \Longrightarrow \begin{bmatrix} \mathbf{q}(\tau_1) \\ \mathbf{p}(\tau_1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2\tau_1 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{q}(0) \\ \mathbf{p}(0) \end{bmatrix}$$

$$m_2: \quad \frac{d}{d\tau_2}\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} 2\mathbf{p} \\ 0 \end{bmatrix} \quad \Longrightarrow \begin{bmatrix} \mathbf{q}(\tau_2) \\ \mathbf{p}(\tau_2) \end{bmatrix} = \begin{bmatrix} 1 & 2\tau_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{q}(0) \\ \mathbf{p}(0) \end{bmatrix}$$

$$m_3: \quad \frac{d}{d\tau_3}\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ -\mathbf{p} \end{bmatrix} \quad \Longrightarrow \begin{bmatrix} \mathbf{q}(\tau_3) \\ \mathbf{p}(\tau_3) \end{bmatrix} = \begin{bmatrix} e^{\tau_3} & 0 \\ 0 & e^{-\tau_3} \end{bmatrix}\begin{bmatrix} \mathbf{q}(0) \\ \mathbf{p}(0) \end{bmatrix}$$

$$\tfrac{1}{2}(m_1 + m_2): \quad \frac{d}{d\omega}\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ -\mathbf{q} \end{bmatrix} \quad \Longrightarrow \begin{bmatrix} \mathbf{q}(\tau_3) \\ \mathbf{p}(\tau_3) \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}\begin{bmatrix} \mathbf{q}(0) \\ \mathbf{p}(0) \end{bmatrix}$$

The last is obtained easily by writing $\frac{d}{d\omega}(\mathbf{q} + i\mathbf{p}) = -i(\mathbf{q} + i\mathbf{p})$.

Thus, as claimed, the KAN decomposition of a symplectic matrix M may be written in terms of the Hamiltonian matrices as

$$M = e^{\frac{1}{2}\omega(m_1+m_2)}e^{\tau_3 m_3}e^{\tau_1 m_1}\,. \tag{8}$$

Remark 1. Notice that all of these $Sp(2)$ transformations leave invariant the cross-product $\mathbf{p}\times\mathbf{q}$. The quantity $\mathbf{p}\times\mathbf{q}$ is called Lagrange's invariant in the study of geometric optics, for which the linear symplectic transformations play a key role.

Remark 2. Under the matrix commutator $[m_i, m_j] := m_i m_j - m_j m_i$, the Hamiltonian matrices m_i with $i = 1, 2, 3$ close among themselves, as

$$[m_1, m_2] = 4m_3\,, \quad [m_2, m_3] = -\,2m_2\,, \quad [m_3, m_1] = -\,2m_1\,. \tag{9}$$

3 Adjoint orbits of the action of $Sp(2)$ on $\mathfrak{sp}(2)$

We consider the Hamiltonian matrix

$$\begin{aligned} m(\omega,\gamma,\tau) &= \frac{\omega}{2}(m_1+m_2) + \frac{\gamma}{2}(m_2-m_1) + \tau m_3 \\ &= \begin{bmatrix} \tau & \gamma+\omega \\ \gamma-\omega & -\tau \end{bmatrix}, \end{aligned} \tag{10}$$

which may be regarded as a pure co-quaternion.

- The eigenvalues of the Hamiltonian matrix (10) are determined from

$$\lambda^2 + \Delta = 0\,, \quad \text{with} \quad \Delta = \det m = \omega^2 - \gamma^2 - \tau^2\,. \tag{11}$$

Consequently, the eigenvalues come in pairs, given by

$$\lambda^{\pm} = \pm\sqrt{-\Delta} = \pm\sqrt{\tau^2+\gamma^2-\omega^2}\,. \tag{12}$$

- Adjoint orbits in the space $(\gamma+\omega, \gamma-\omega, \tau) \in \mathbb{R}^3$ obtained from the action of a symplectic matrix $M(\tau_i)$ on a Hamiltonian matrix $m(\omega,\gamma,\tau)$ by matrix conjugation

$$m \to m' = M(\tau_i) m M^{-1}(\tau_i) \quad \text{(no sum on } i = 1,2,3)$$

may alter the values of the parameters (ω,γ,τ) in (10).

- Since Adjoint action preserves eigenvalues, it preserves the value of the determinant Δ. This means the Adjoint orbits of the Hamiltonian matrices lie on the level sets of the determinant Δ.

- The Adjoint orbits of the Hamiltonian matrices corresponding to these eigenvalues change type, depending on whether $\Delta < 0$ (hyperbolic), $\Delta = 0$ (parabolic), or $\Delta > 0$ (elliptic), as illustrated in Figure 1 and summarised in the table below.

Harmonic (elliptic) orbit	*Trajectories: Ellipses*
$\Delta = 1\,, \quad \lambda^{\pm} = \pm\, i$	$m_H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$
Free (parabolic) orbit	*Trajectories: Straight lines*
$\Delta = 0\,, \quad \lambda^{\pm} = 0$	$m_H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$
Repulsive (hyperbolic) orbit	*Trajectories: Hyperbolas*
$\Delta = -1\,, \quad \lambda^{\pm} = \pm\, 1$	$m_H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

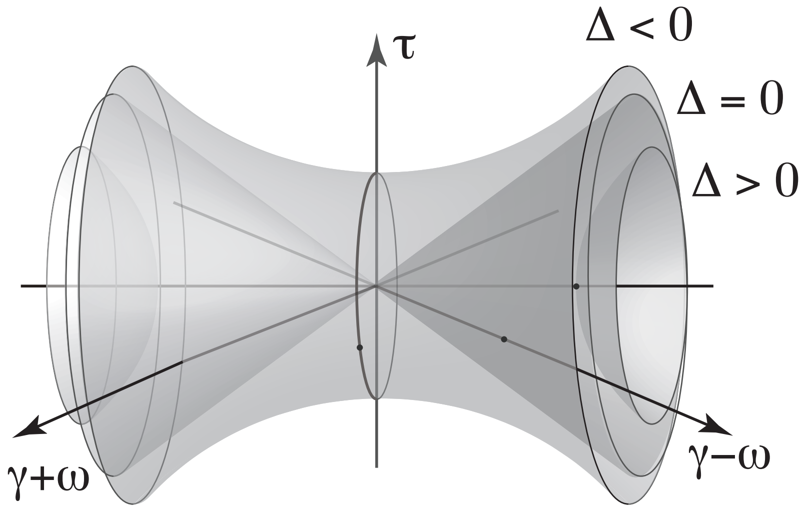

Figure 1: The action by matrix conjugation of a symplectic matrix on a Hamiltonian matrix changes its parameters $(\omega, \gamma, \tau) \in \mathbb{R}^3$, while preserving the value of the discriminant $\Delta = \omega^2 - \gamma^2 - \tau^2$. Three families of orbits emerge, that are hyperbolic ($\Delta < 0$), parabolic ($\Delta = 0$) and elliptic ($\Delta > 0$).

4 The momentum map and coadjoint orbits

Definition 2. *The momentum map* $\mathcal{J} : T^*\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2 \to \mathfrak{sp}(2)^*$ *is defined by*

$$
\begin{aligned}
\mathcal{J}^\xi(\mathbf{z}) \;&:=\; \Big\langle \mathcal{J}(\mathbf{z}),\, \xi \Big\rangle_{sp(2,\mathbb{R})^* \times sp(2,\mathbb{R})} \\
&=\; \Big(\mathbf{z},\, J\xi\mathbf{z} \Big)_{\mathbb{R}^2 \times \mathbb{R}^2} \\
&:=\; z_k (J\xi)_{kl} z_l \\
&=\; \mathbf{z}^T \cdot J\xi\mathbf{z} \\
&=\; \mathrm{tr}\Big((\mathbf{z} \otimes \mathbf{z}^T J)\xi \Big) \,,
\end{aligned}
\tag{13}
$$

where $\mathbf{z} = (\mathbf{q},\, \mathbf{p})^T \in \mathbb{R}^2 \times \mathbb{R}^2$.

Remark 3. The map $\mathcal{J}(\mathbf{z})$ given in (13) by

$$
\mathcal{J}(\mathbf{z}) = (\mathbf{z} \otimes \mathbf{z}^T J) \in \mathfrak{sp}(2)^*
\tag{14}
$$

sends $\mathbf{z} = (\mathbf{q},\, \mathbf{p})^T \in \mathbb{R}^2 \times \mathbb{R}^2$ to $\mathcal{J}(\mathbf{z}) = (\mathbf{z} \otimes \mathbf{z}^T J) \in \mathfrak{sp}(2)^*$.

4.1 Deriving the momentum map (13) from Hamilton's principle

Proposition 1. *The momentum map (13) may be derived from Hamilton's principle.*

Proof. Let $\xi \in \mathfrak{sp}(2)$ act on $\mathbf{z} \in T^*\mathbb{R}^2$ as in formula (7), namely,

$$
\frac{d\mathbf{z}}{dt} = \xi\mathbf{z} \,,
$$

and let $\ell(\xi)$ be a reduced Lagrangian in Hamilton's principle $\delta S = 0$ with $S = \int \ell(\xi)dt$, constrained by the action of $T^*\mathbb{R}^2$ under $Sp(2)$. Taking the variations yields

$$
\begin{aligned}
0 = \delta S &= \delta \int \ell(\xi) + \mathbf{z}^T \cdot J(\dot{\mathbf{z}} - \xi \mathbf{z})\, dt \\
&= \int \left\langle \frac{\partial \ell}{\partial \xi} - \mathbf{z} \otimes \mathbf{z}^T J \,,\, \delta\xi \right\rangle + \delta \mathbf{z}^T \cdot J(\dot{\mathbf{z}} - \xi\mathbf{z}) - \delta\mathbf{z} \cdot J(\dot{\mathbf{z}} - \xi\mathbf{z})^T\, dt
\end{aligned}
$$

thereby finding $\frac{\partial \ell}{\partial \xi} = \mathbf{z} \otimes \mathbf{z}^T J = \mathcal{J}(\mathbf{z})$. □

Proposition 2. *Suppose the Lagrangian $\ell(\xi)$ is hyper-regular, so that one may solve $\frac{\partial \ell}{\partial \xi}$ for ξ. Then, applying the time derivative to $\mathcal{J}(\mathbf{z})$ and using the Lie algebra action shows that the momentum map $\mathcal{J}(\mathbf{z})$ undergoes coadjoint motion,*

$$
\frac{d\mathbf{z}}{dt} = \xi \mathbf{z} \quad \Longrightarrow \quad \frac{d\mathcal{J}}{dt} + \mathrm{ad}^*_\xi \mathcal{J} = 0 \quad \textit{or, equivalently,} \quad \mathrm{Ad}^*_{M^{-1}(t)} \mathcal{J}(t) = \mathcal{J}(0).
$$

Proof. The proof of this proposition is not given in these lecture notes. Instead, we shall show that each component of the momentum map undergoes coadjoint motion. □

4.2 Components of the momentum map

The map $\mathcal{J} : T^*\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2 \to \mathfrak{sp}(2)^*$ in (13) for $Sp(2,\mathbb{R})$ acting diagonally on $\mathbb{R}^2 \times \mathbb{R}^2$ in Equation (14) may be expressed in matrix form as

$$
\begin{aligned}
\mathcal{J} &= (\mathbf{z} \otimes \mathbf{z}^T J) \\
&= 2 \begin{pmatrix} \mathbf{p}\cdot\mathbf{q} & -|\mathbf{q}|^2 \\ |\mathbf{p}|^2 & -\mathbf{p}\cdot\mathbf{q} \end{pmatrix} \\
&= 2 \begin{pmatrix} Y_3 & -Y_1 \\ Y_2 & -Y_3 \end{pmatrix} ,
\end{aligned}
\tag{15}
$$

consisting of rotation-invariant components,

$$
T^*\mathbb{R}^2 \to \mathbb{R}^3 : (\mathbf{q},\mathbf{p})^T \to \mathbf{Y} = (Y_1, Y_2, Y_3)\,,
$$

defined as

$$
Y_1 = |\mathbf{q}|^2 \ge 0\,, \quad Y_2 = |\mathbf{p}|^2 \ge 0\,, \quad Y_3 = \mathbf{p}\cdot\mathbf{q}\,. \tag{16}
$$

Remark 4. Under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{sp}(2)^* \times \mathfrak{sp}(2) \to \mathbb{R}$ given by the trace of the matrix product, one finds the Hamiltonian, or phase-space function,

$$
\Big\langle \mathcal{J}(\mathbf{z}),\, \xi \Big\rangle = \mathrm{tr}\big(\mathcal{J}(\mathbf{z})\,\xi\big)\,, \tag{17}
$$

for $\mathcal{J}(\mathbf{z}) = (\mathbf{z} \otimes \mathbf{z}^T J) \in \mathfrak{sp}(2)^*$ and $\xi \in \mathfrak{sp}(2)$.

Applying the momentum map $\mathcal{J}$ to the vector of Hamiltonian matrices $\mathbf{m} = (m_1,\, m_2,\, m_3)$ in equation (5) yields the individual components in (15), as

$$
\mathcal{J}\cdot\mathbf{m} = 2\mathbf{Y} \quad \Longleftrightarrow \quad \mathbf{Y} = \frac{1}{2}\, z_k (J\mathbf{m})_{kl} z_l\,. \tag{18}
$$

Remark 5.

(a) The components of the vector $\mathbf{Y} = (Y_1,\, Y_2,\, Y_3)$ satisfy the relation $S^2 = Y_1Y_2 - Y_3^2 = |\mathbf{p}\times\mathbf{q}|^2$ in which $\mathbf{p}\times\mathbf{q}$ is invariant under $Sp(2)$. Thus, coadjoint motion lies on level sets of S^2.

(b) The canonical Poisson brackets among the components $(Y_1,\, Y_2,\, Y_3)$ reflect the matrix commutation relations in (9),

$$
[m_i,\, m_j] = \begin{array}{|c|ccc|}\hline [\,\cdot\,,\,\cdot\,] & m_1 & m_2 & m_3 \\ \hline m_1 & 0 & 4m_3 & 2m_1 \\ m_2 & -4m_3 & 0 & -2m_2 \\ m_3 & -2m_1 & 2m_2 & 0 \\ \hline \end{array} = c_{ij}^k m_k\,. \tag{19}
$$

In tabular form, these Poisson brackets are

$$
\{Y_i,\, Y_j\} = \begin{array}{|c|ccc|}\hline \{\,\cdot\,,\,\cdot\,\} & Y_1 & Y_2 & Y_3 \\ \hline Y_1 & 0 & 4Y_3 & 2Y_1 \\ Y_2 & -4Y_3 & 0 & -2Y_2 \\ Y_3 & -2Y_1 & 2Y_2 & 0 \\ \hline \end{array} = -\,\epsilon_{ijk}\frac{\partial S^2}{\partial Y_k}\,. \tag{20}
$$

That is, the canonical Poisson brackets of the components of the momentum map close among themselves.

$$
\{Y_1, Y_2\} = 4Y_3\,,\quad \{Y_2, Y_3\} = -\,2Y_2\,,\quad \{Y_3, Y_1\} = -\,2Y_1 \quad\text{and}\quad \{Y_i, S^2\} = 0 \quad\text{for}\quad i = 1,2,3\,.
$$

In a moment we will show that these Poisson brackets among the components $(Y_1,\, Y_2,\, Y_3)$ satisfy the Jacobi identity.

(c) The Poisson bracket relations (20) among the components of the momentum map $\mathcal{J}$ imply the following for smooth functions F and H of the vector $\mathbf{Y} = (Y_1,\, Y_2,\, Y_3)$

$$
\frac{dF}{dt} = \{F, H\} = -\nabla S^2\cdot\nabla F\times\nabla H(\mathbf{Y}) = Y_k\, c_{ij}^k\,\frac{\partial F}{\partial Y_i}\frac{\partial H}{\partial Y_i} = -\left\langle \mathbf{Y}\,,\,\mathrm{ad}_{\frac{\partial H}{\partial \mathbf{Y}}}\frac{\partial F}{\partial \mathbf{Y}}\right\rangle = -\left\langle \mathrm{ad}^*_{\frac{\partial H}{\partial \mathbf{Y}}}\mathbf{Y}\,,\,\frac{\partial F}{\partial \mathbf{Y}}\right\rangle
$$

which is the Lie-Poisson bracket on $\mathfrak{sp}(2)^*$. (The Lie-Poisson bracket satisfies the Jacobi identity by virtue of being dual to a Lie algebra.)

(d) This means that the momentum map (15) is Poisson.

(e) In particular, the components of the momentum map in vector form $\mathbf{Y}$ satisfy the equation for coadjoint motion

$$
\frac{d\mathbf{Y}}{dt} = \nabla S^2\times\nabla H(\mathbf{Y}) = -\,\mathrm{ad}^*_{\frac{\partial H}{\partial \mathbf{Y}}}\mathbf{Y}
$$

for any choice of Hamiltonian $H(\mathbf{Y})$ that depends on the momentum map components.

(f) Thus, each component of the momentum map in (15) undergoes coadjoint motion.