M 3-4-5 PA16 Notes on Geometric Mechanics

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Textbook

Geometric Mechanics I: Dynamics and Symmetry, by Darryl D Holm
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Marks

1. ***Assessed Homework:***

   - To help you prepare for the Final Exam, three Assessed Homework sets of 5 or 6 problems each will be handed out, spaced about three weeks apart, e.g., at week 3, week 6 and week 9. Each Assessed Homework set will be due about twelve days after it is assigned, although the due date for the last set may be delayed until immediately after the Winter Break, if desired.

2. ***Final Exam:*** Three of the five questions will be taken from the assessed homework assignments.

3. To help you prepare for the Assessed Homework sets, many Practice Problems and sketches of their solutions will be provided intermittently, as the lectures progress.

4. Lecture notes will be available online at `http://wwwf.imperial.ac.uk/~dholm/classnotes/`

Office hours

Arranged individually or in groups by appointment via email.

Class summary

This class explains, via many self-contained examples, a systematic framework for using ***Geometry*** in studying ***Mechanics***. Here, these terms mean the following.

- *Geometry* involves linear algebra, transformation theory, differential equations, variational calculus, Lie groups and their actions on manifolds.
- *Mechanics* means "the branch of physics concerned with the motion of bodies in a frame of reference". Usually this means differential equations, e.g., $\dot{X} = F(X,t)$.
- *Study* means "formulate and solve, so as to reveal the geometric meaning of the problem and thereby understand better how to obtain its solution".

For example, in the language of GM, Euler's rigid body dynamics becomes geodesic motion on the Lie group of 3D rotations $SO(3)$ with respect to the Riemannian metric given by the moment of inertia. The solution may also be represented as motion by smooth flows parameterised by time t that takes place along the intersections of two-dimensional surfaces in $\mathbb{R}^3$ that are level sets of the conservation laws for energy and angular momentum.

What is Geometric Mechanics?

GM lifts mechanics on a *manifold* M to mechanics on a *Lie group* G which *acts* (transitively) on M.

This sentence defining GM introduces three main concepts into classical Lagrangian and Hamiltonian mechanics:

- The configuration spaces are Manifolds. A manifold M is a space which admits differentiable transformations along curves (motions).
  In particular, manifolds admit the rules of calculus.
- Lie groups are groups of *transformations* which depend smoothly on a set of parameters, e.g. rotations or translations.
  In particular, Lie groups are groups which are also manifolds.
- A Group action is a transformation of a Lie group G which takes an initial point $q_0 \in M$ in a manifold M to another one along a smooth curve $q_t \in M$, denoted $q_t = g_t q_0$, for g_t a curve in Lie group G parameterised by t.

This class provides examples of how these concepts are used!

Lie groups describe the symmetries of Hamilton's principle $0 = \delta S$, with $S = \int_a^b L(q.\dot{q})\, dt$ for a Lagrangian $L : (q, \dot{q}) \in TM \to \mathbb{R}$, where TM (tangent bundle of M) is the union of the set of tangent vectors to M at all points $q \in M$.

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1 What is Geometric Mechanics? Where has it been applied?

1.1 Introduction to the Course and to Smooth Manifolds: How did GM develop?

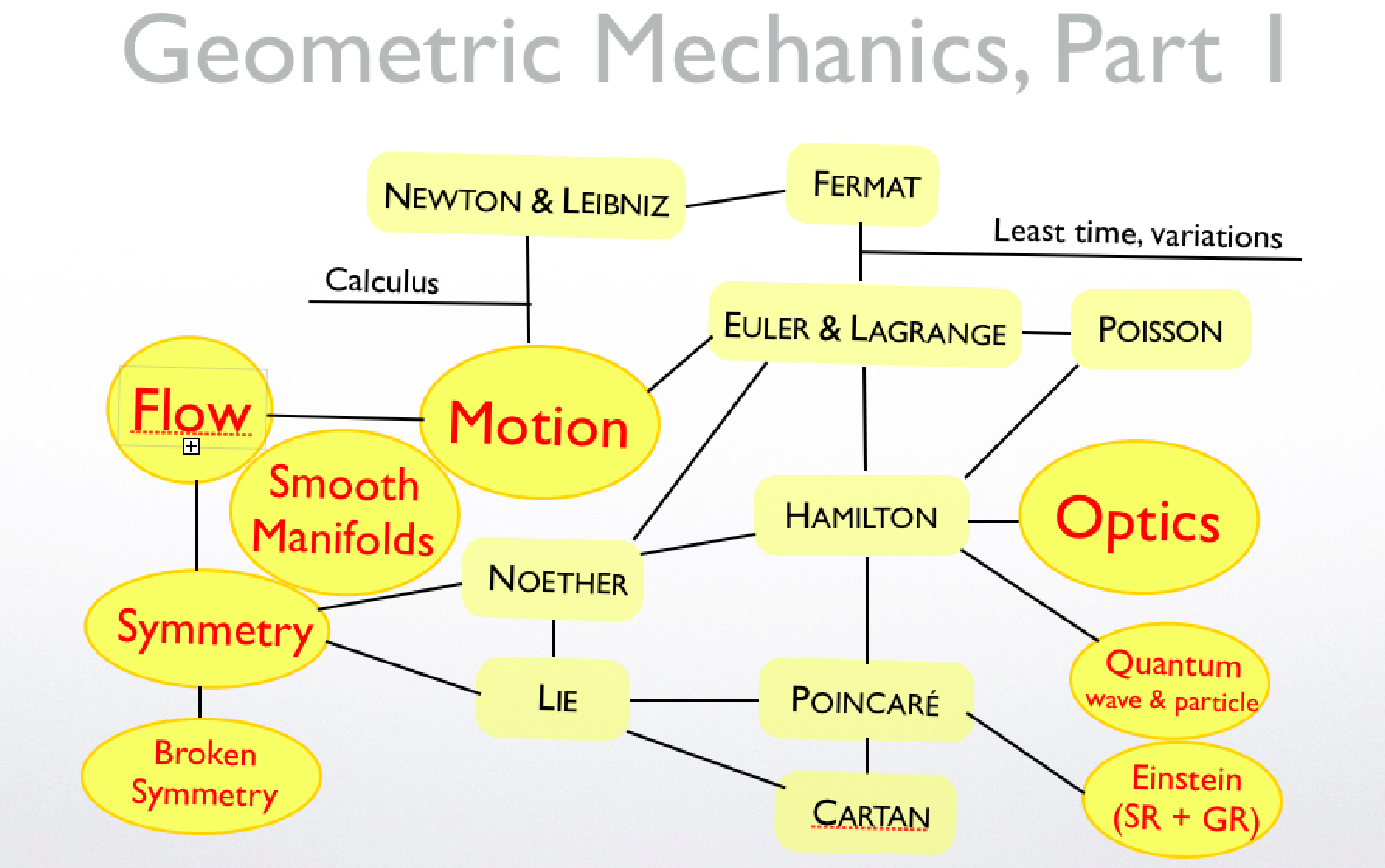


Figure 1: Geometric Mechanics has involved many great mathematicians!

1.2 Geometric Mechanics is a framework for many modern applications

Interplanetary missions
Variational integrators
Swimming fish
Bifurcations with symmetry
Lagrangian coherent structures
Euler-Poincaré theory
Multisymplectic formulation
Nonlinear stability
Under Water Vehicles
Geometric optimal control
Computational anatomy
Reduction by stages

Molecular oscillations
Astroid pairs
Satellites with tethers
Molecular strands
Elasticity
Image registration
Robotics
Peakons
Solitons
Fluid dynamics
Turbulence models
Complex fluids

Liquid crystals
Superfluids
Plasmas
Magnetohydrodynamics
Geophysical Fluid Dynamics
Global warming
General relativity
Field theory (GIMMSY!)
Lie groupoids and algebroids
Snakeboards
Swarming motion
Telecommunications

1.3 What are the next directions for GM?

Information geometry?
Nanoscience?
DNA folding?

Hybrid fluids/kinetic?
Information communication?
Salsa dancing robots?

Geometric quantum mechanics?
Data assimilation?
Things as yet un-named!

1.4 Newton (1686) The reduced Kepler problem: How conservation laws characterise solutions

Newton's equation (1686) for the ***reduced Kepler problem*** of planetary motion in the centre of mass frame is

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = 0\,, \tag{1}$$

in which μ is a constant and $r = |\mathbf{r}|$ with $\mathbf{r} \in \mathbb{R}^3$, is the distance between a planet and the Sun.

Scale invariance of this equation under the changes $R \to s^2 R$ and $T \to s^3 T$ in the units of space R and time T for any constant (s) means that it admits families of solutions whose space and time scales are related by $T^2/R^3 = const$. This is ***Kepler's third law***.

1. The scalar (resp. vector) product of equation (1) with $\mathbf{r}$ shows conservation of the energy E and (resp.) specific angular momentum $\mathbf{L}$, given by

$$\begin{aligned} E &= \frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{\mu}{r} \quad \text{(energy)}\,, \\ \mathbf{L} &= \mathbf{r} \times \dot{\mathbf{r}} \quad \text{(specific angular momentum)}\,. \end{aligned}$$

   Since $\mathbf{r} \cdot \mathbf{L} = 0$, the planetary motion in $\mathbb{R}^3$ takes place in a plane to which vector $\mathbf{L}$ is perpendicular. This is the orbital plane. Constancy of magnitude L means the orbit sweeps out equal areas in equal times (Kepler's second law). In the orbital plane, one may specify plane polar coordinates (r, θ) with unit vectors $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ in the plane and $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \widehat{\mathbf{L}}$ normal to it. In particular,

$$\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}} = r\hat{\mathbf{r}} \times (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) = r^2\dot{\theta}\,\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = r^2\dot{\theta}\,\widehat{\mathbf{L}}\,,$$

   so the magnitude of the angular momentum is $L = |\mathbf{L}| = r^2\dot{\theta}$.

2. The unit vectors for polar coordinates in the orbital plane are $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. These vectors satisfy

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\,\widehat{\mathbf{L}} \times \hat{\mathbf{r}} = \dot{\theta}\,\hat{\boldsymbol{\theta}} \quad \text{and} \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = \dot{\theta}\,\widehat{\mathbf{L}} \times \hat{\boldsymbol{\theta}} = -\,\dot{\theta}\,\hat{\mathbf{r}}\,, \quad \text{where} \quad \dot{\theta} = \frac{L}{r^2}\,.$$

Newton's equation of motion (1) for the Kepler problem may now be written equivalently using $\dot{\theta}/L = 1/r^2$ and $\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\,\dot{\theta}\,\hat{\mathbf{r}}$, as

$$0 = \ddot{\mathbf{r}} + \frac{\mu\mathbf{r}}{r^3} = \ddot{\mathbf{r}} + \frac{\mu}{L}\dot{\theta}\,\hat{\mathbf{r}} = \frac{d}{dt}\Big(\dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}}\Big)\,.$$

This equation implies conservation of the following vector *in the plane of motion*:

$$\mathbf{K} = \dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}} \quad (\textbf{\textit{Hamilton's vector}}) \text{ with } \mathbf{K}\cdot\mathbf{L} = 0\,.$$

The vector in the plane given by the cross product of the two conserved vectors **K** and **L**,

$$\mathbf{J} = \mathbf{K}\times\mathbf{L} = \dot{\mathbf{r}}\times\mathbf{L} - \mu\hat{\mathbf{r}} \quad (\textbf{\textit{Laplace–Runge–Lenz vector}})\,,$$

is also conserved. Note that the dimensions of **J** are given by $[J] = [\mu] = [r]^3[t]^{-2}$, the same as Kepler's Third Law!

3. From their definitions, these conserved quantities are related by

$$K^2 = 2E + \frac{\mu^2}{L^2} = \frac{J^2}{L^2}\,, \quad \text{upon using} \quad K^2 = \Big|\dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}}\Big|^2 = |\dot{\mathbf{r}}|^2 - \frac{2\mu}{L}\dot{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}} + \frac{\mu^2}{L^2} = |\dot{\mathbf{r}}|^2 - \frac{2\mu}{r} + \frac{\mu^2}{L^2} = 2E + \frac{\mu^2}{L^2}\,,$$

since $\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}$ and $L = r^2\dot{\theta}$. Equivalently,

$$L^2 + \frac{J^2}{(-2E)} = \frac{\mu^2}{(-2E)} \quad\Longrightarrow\quad -2E = \frac{\mu^2 - J^2}{L^2} \quad \text{and} \quad \mathbf{J}\cdot\mathbf{K}\times\mathbf{L} = K^2L^2 = J^2\,, \tag{2}$$

where $J^2 := |\mathbf{J}|^2$, etc. and $-2E > 0$ for bounded orbits.

4. Orient the conserved Laplace–Runge–Lenz vector $\mathbf{J}$ in the orbital plane to point along the reference line for the measurement of the polar angle θ, say from the centre of the orbit (Sun) to the perihelion (point of nearest approach, on midsummer's day). The scalar product of $\mathbf{r}$ and $\mathbf{J}$ then yields an elegant result for the Kepler orbit in plane polar coordinates:

$$\mathbf{r}\cdot\mathbf{J} = rJ\cos\theta = \mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{L} - \mu\,\mathbf{r}/r) = \mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{L}) - \mu\,r = L^2 - \mu\,r\,,$$

which implies

$$r(\theta) = \frac{L^2}{\mu + J\cos\theta} = \frac{l_\perp}{1 + e\cos\theta}\,, \tag{3}$$

As expected, the orbit $r(\theta)$ is a ***conic section*** whose origin is at one of the two foci. This is ***Kepler's first law***.

The Laplace–Runge–Lenz vector $\mathbf{J}$ is directed from the focus of the orbit to its perihelion (point of closest approach). The eccentricity of the conic section is $e = J/\mu = KL/\mu$ and its semi-latus rectum (normal distance from the line through the foci to the orbit) is $l_\perp = L^2/\mu$. The eccentricity vanishes ($e = 0$) for a circle and correspondingly $K = 0$ implies that $\dot{\mathbf{r}} = \mu\,\hat{\boldsymbol{\theta}}/L$. The eccentricity takes values $0 < e < 1$ for an ellipse, $e = 1$ for a parabola and $e > 1$ for a hyperbola.

5. One may use the conservation of $\mathbf{L}$ in $\mathbf{L}dt = \mathbf{r}\times d\mathbf{r}$ or L in $Ldt = r^2 d\theta$ to show that the constancy of magnitude $L = |\mathbf{L}|$ means the orbit sweeps out equal areas in equal times. This is ***Kepler's second law***. For an elliptical orbit, the integral $LT = \int_0^T Ldt = \int_0^{2\Pi} r(\theta)^2 d\theta = 2A$ yields the period in terms of angular momentum and the area.

6. One may use the result of part 5 and the geometric properties of ellipses to show that the period of the orbit is given by

$$\left(\frac{T}{2\Pi}\right)^2 = \frac{a^3}{\mu} = \frac{\mu^2}{(-2E)^3}\,.$$

The relation $T^2/a^3 = \textit{constant}$ is Kepler's third law. The constant is Newton's constant.

2 Geometric Mechanics involves motion on smooth manifolds

2.1 Definitions: Space, Time, Motion, . . . , Tangent space, Velocity, Motion equation

Space

Space is taken to be a smooth manifold Q with points $q \in Q$ (Positions, States, Configurations).

Let Q be a ***smooth manifold*** $\dim Q = n$. That is, Q is a smooth space that is locally $\mathbb{R}^n$. Operationally, a smooth manifold is a space on which the rules of calculus apply.

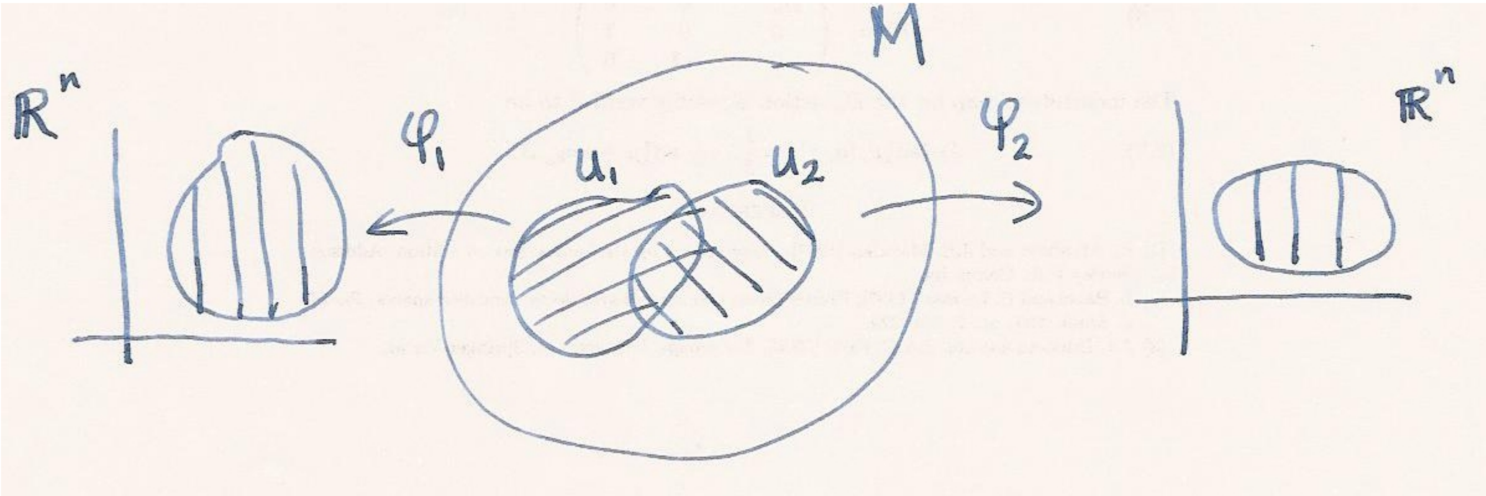


Figure 2: A manifold Q is defined by the disjoint union (or, atlas) of local charts, each of which is isomorphic to $\mathbb{R}^{\dim Q}$.

Examples of manifolds

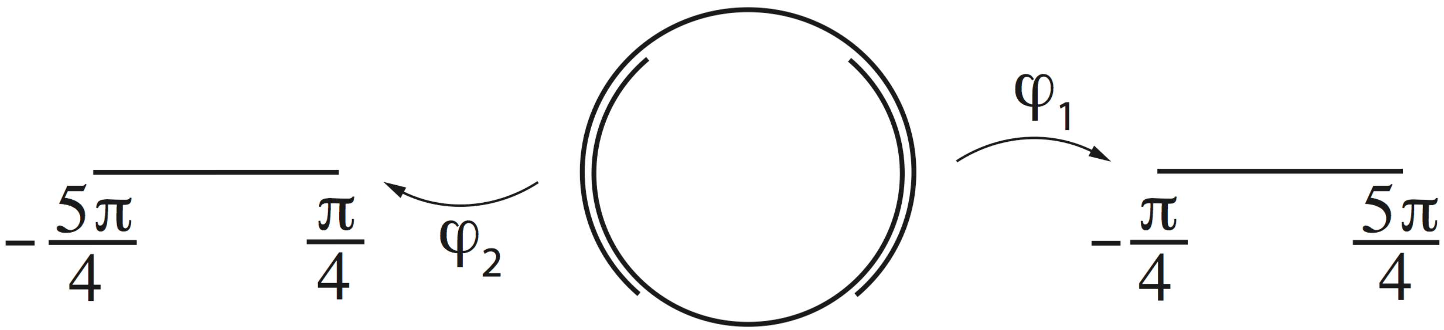


Figure 3: The circle S^1 is an example of a manifold that can be covered with two charts that are each locally $\mathbb{R}^1$.

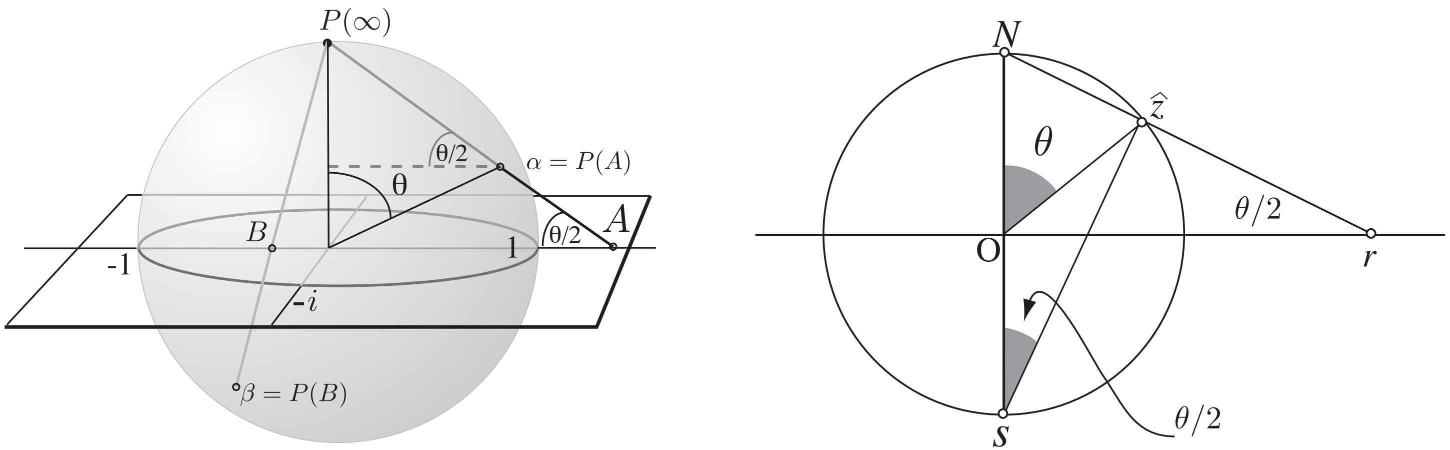
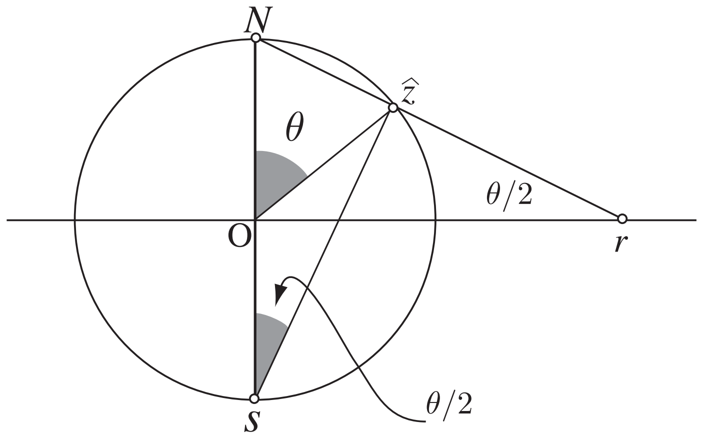


Figure 4: The Riemann map shows that the unit sphere S^2 is a manifold that can be covered with two charts that are each locally $\mathbb{R}^2$.



Exercise. The figure illustrates Riemann's stereoscopic projection for the circle S^1. Use it to show that the circle is a manifold which may be covered by two charts. Derive the values of the stereoscopic projections x_N and x_S from the North and South poles onto the x-axis, respectively, of a point on the circle at polar angle θ. Explain the angle $\theta/2$. How are x_N and x_S related to each other? Hint: you may use trigonometry. ★

Answer. A point on the circle at polar angle θ from the North pole has height $z = \cos\theta$. The intersection of its stereographic projection with the x-axis is found from the proportion $r = \frac{x_N}{1} = \frac{\sin\theta}{1-\cos\theta} = \cot(\theta/2)\,,$ provided $\cos\theta \neq 1\,$. The corresponding stereographic projection from the South pole in the figure satisfies the proportion $\frac{x_S}{1} = \frac{\sin\theta}{1+\cos\theta}\,,$ provided $\cos\theta \neq -1\,$. Consequently, $x_S x_N = 1$, so that $x_S = 1/x_N = \tan(\theta/2)$ for $\theta \neq 0, \Pi$. ▲

Space

Space is taken to be a smooth manifold Q with points $q \in Q$ (Positions, States, Configurations).

Time

Time is taken to be a manifold T with points $t \in T$. Usually $T = \mathbb{R}$ (for real 1D time), but we will also consider $T = \mathbb{R}^2$, and the option to let T and Q both be complex manifolds is not out of the question.

Motion

Motion is a map $\phi_t : T \to Q$, where subscript t denotes dependence on time t. For example, when $T = \mathbb{R}$, the motion is a curve $q_t = \phi_t \circ q_0$ obtained by composition of functions. The motion is called a *flow* if $\phi_{t+s} = \phi_t \circ \phi_s$, for $s, t \in \mathbb{R}$, and $\phi_0 = \mathrm{Id}$, so that $\phi_t^{-1} = \phi_{-t}$. Note that the composition of functions is associative, $(\phi_t \circ \phi_s) \circ \phi_r = \phi_t \circ (\phi_s \circ \phi_r) = \phi_t \circ \phi_s \circ \phi_r = \phi_{t+s+r}$, but in general it is not commutative.

How Lie groups enter: the road to Geometric Mechanics

Recall that a Lie group is a group (of transformations) which depends smoothly on a set of parameters. In general, a Lie group is a group which is also a manifold.

When the motion is obtained from a Lie group action $G \times Q \to Q$, then it may be identified with a flow map $\phi_t : T \to G$, which we may regard as a *curve* on the Lie group G.

Thus, we should anticipate motion and mechanics to be lifted from configuration manifolds to Lie group manifolds. In this case, the motion on the configuration space Q may be obtained from the group action $G \times Q \to Q$ as $q_t = \phi_t q_0$ where $q_0 \in Q$ is the initial, or reference, configuration.

2.2 Curves on manifolds and their tangent spaces

The ***tangent space*** T_qQ contains vectors $v_q = \dot{q}(t) \in T_qQ$, tangent to curve $q(t) \in Q$ at point q. The coordinates are $(q, v_q) \in TQ_q$. Note, $\dim T_qQ = 2n$ and subscript q reminds us that v_q is an element of the tangent space at the point q and that on manifolds we must keep track of base points.

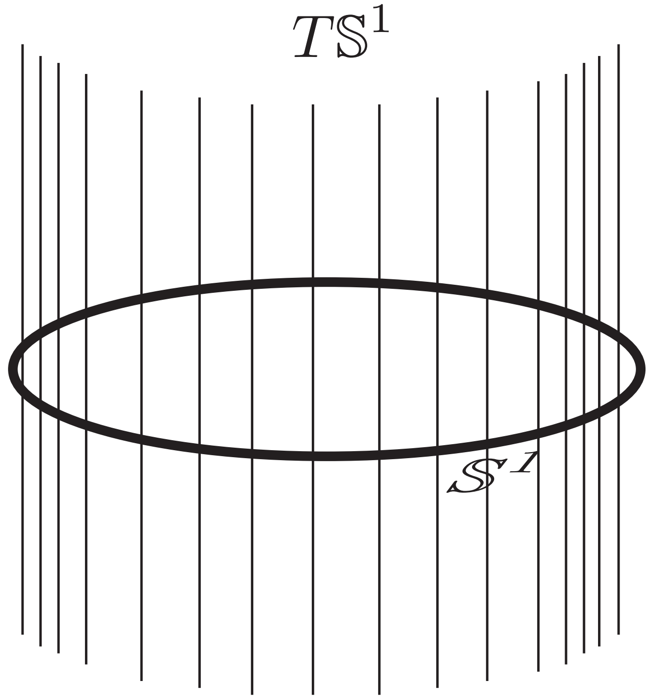


Figure 5: This is a sketch of the tangent bundle TS^1 of the circle S^1, $TS^1 = \{(\mathbf{x}, \mathbf{v}) \in T\mathbb{R}^2 : |\mathbf{x}|^2 = 1 \text{ and } \mathbf{x} \cdot \mathbf{v} = 0\}$.

The union of tangent spaces $TQ := \cup_{q \in Q} T_qQ$ is called the ***tangent bundle*** of the manifold Q.

The curve $q(t)$ describes the ***motion*** on manifold Q. The curve $\dot{q}(t) \in T_qQ$ is called the ***tangent lift*** of the curve $q(t) \in Q$.

2.3 Velocity and the Motion Equation

Velocity

The tangent lift vector $v_q = \dot{q}(t) \in T_qQ$ is called the *velocity* along a flow $q(t)$ that describes a smooth curve in Q.

Motion Equation

The *motion equation* that determines the flow $q_t \in Q$ takes the form

$$\dot{q}_t = f(q_t)$$

where the map $f : q \in M \to f(q) \in T_qM$ is a prescribed ***vector field*** over Q.

For example, if the curve $q_t = \phi_t \circ q_0$ is a flow, then

$$\dot{q}_t = \dot{\phi}_t \phi_t^{-1} \circ q_t = f(q_t)$$

so that

$$\dot{\phi}_t = f \circ \phi_t =: \phi_t^* f \quad (\phi_t^* f \text{ denotes the } \textit{pullback} \text{ of } f \text{ by } \phi_t)$$

2.4 After these definitions of the setting, we now define Variational Principles

smooth manifold	Riemannian metric	Legendre transformation
tangent space	geodesic	momentum
tangent bundle	Lagrangian	fibre derivative
tangent lift	Hamilton's principle	pairing
kinetic energy	variational derivative	

- Define **kinetic energy** $KE : TM \to \mathbb{R}$, via a *Riemannian metric* $g_q(\,\cdot\,,\,\cdot\,) : TM \times TM \to \mathbb{R}$. Explicitly, $KE = \frac{1}{2}g_q(\dot{q},\dot{q}) =: \frac{1}{2}\|\dot{q}\|^2$.
- Choose the **Lagrangian** $L : TM \to \mathbb{R}$. (For example, one could choose L to be KE.)

- **Hamilton's principle** is $\delta S = 0$ with $S = \int_a^b L(q, \dot{q})dt$, for a family of curves $q(t, \epsilon)$ parameterised smoothly by $(t, \epsilon) \in \mathbb{R} \times \mathbb{R}$. The linearisation

$$\delta S := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_a^b L(q(t,\epsilon), \dot{q}(t,\epsilon))dt \quad \text{with} \quad \delta q(t) := \frac{dq(t,\epsilon)}{d\epsilon}\bigg|_{\epsilon=0}$$

defines the **variational derivative** δS of S near the identity $\epsilon = 0$. The variations in q are assumed to vanish at endpoints in time, so that $q(a, \epsilon) = q(a)$ and $q(b, \epsilon) = q(b)$.

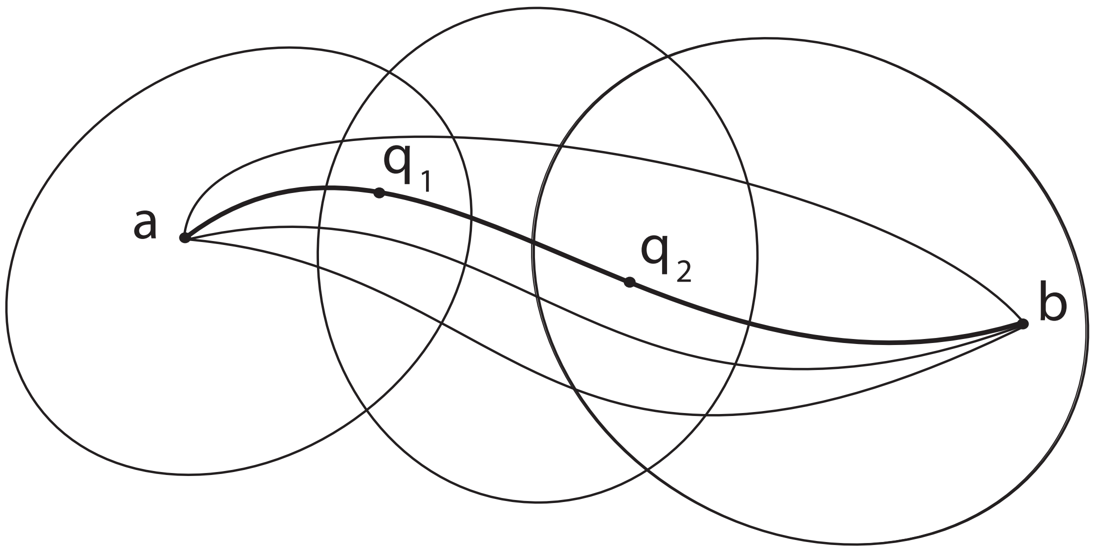


Figure 6: This is a sketch of variations of a family of curves on a manifold.

3 Euler–Lagrange equation

Theorem 1 (Hamilton 1835, Euler 1750, Lagrange 1756). *Hamilton's principle $\delta S = 0$ with $S = \int_a^b L(q, \dot{q})dt$ implies the* ***Euler–Lagrange (EL) equation:***

$$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} = \frac{\partial L(q,\dot{q})}{\partial q}\,, \quad \textit{for any } L(q,\dot{q})\,.$$

Proof 1 Vary coordinates $(q, v) \in TQ$, subject to the constraint $v = \frac{dq}{dt}$ (tangent lift) using the linearisation

$$\delta S := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_a^b L(q(t,\epsilon), v(t,\epsilon))dt \quad \text{with, e.g.,} \quad \delta q(t) := \frac{dq(t,\epsilon)}{d\epsilon}\bigg|_{\epsilon=0}$$

Apply Hamilton's principle,

$$\begin{aligned} 0 = \delta S &= \delta \int_a^b L(q,v) + \left\langle p, \frac{dq}{dt} - v \right\rangle dt \\ &= \int_a^b \left\langle \frac{\partial L}{\partial v} - p, \delta v \right\rangle + \left\langle \frac{\partial L}{\partial q} - \frac{dp}{dt}, \delta q \right\rangle + \Big\langle \delta p, \dot{q} - v \Big\rangle dt + \Big\langle p, \delta q \Big\rangle \bigg|_a^b \end{aligned}$$

Then assemble the EL equation from the various stationary conditions, and evaluate $\frac{\partial L}{\partial v}\big|_{v=\dot{q}}$.

Proof 2 Vary the curve $q(t)$ in the family $q(t,\epsilon) \in \mathcal{C}(Q)$ using the linearisation

$$\delta S := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_a^b L(q(t,\epsilon), \dot{q}(t,\epsilon))dt \quad \text{with} \quad \delta q(t) := \frac{dq(t,\epsilon)}{d\epsilon}\bigg|_{\epsilon=0}$$

and set $\delta \frac{dq}{dt} = \frac{d}{dt}\delta q$ in the variation of the action S as

$$\begin{aligned}
0 = \delta S = \delta \int_a^b L(q,\dot{q})dt = \int_a^b \delta L(q,\dot{q})dt &= \int_a^b \left\langle \frac{\partial L}{\partial \dot{q}}, \delta\dot{q} \right\rangle + \underbrace{\left\langle \frac{\partial L}{\partial q}, \delta q \right\rangle}_{\text{Pairing}} dt \\
&= \int_a^b \left\langle \frac{\partial L}{\partial \dot{q}}, \frac{d}{dt}\delta q \right\rangle + \left\langle \frac{\partial L}{\partial q}, \delta q \right\rangle dt \\
&= \int_a^b \left\langle \underbrace{- \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q}}_{\text{EL equation}}, \delta q \right\rangle dt \; + \; \underbrace{\left\langle \frac{\partial L}{\partial \dot{q}}, \delta q \right\rangle \bigg|_a^b}_{\text{Endpoint term } = 0}
\end{aligned}$$

$\square$

3.1 Lie group symmetries and Noether's theorem

- ***Introduction of Lie group symmetries:***
  - A ***group*** is a set of elements with an associative binary product that has a unique inverse and identity element.
  - A ***Lie group*** G is a group whose transformations depends smoothly on a set of parameters in $\mathbb{R}^{\dim(G)}$. (A Lie group is also a smooth manifold, so it is an ideal arena for geometric mechanics, e.g., rigid body motion on $SO(3)$.)

- ***Noether's theorem:*** Suppose $q(t,\epsilon) = q_\epsilon(t) = \phi_\epsilon \circ q(t)$ represents a Lie group, i.e., group of transformations of $q(t)$ that depends smoothly on a set of parameters ϵ. Its linearisation is computed from a Taylor series as

$$q(t) \to q_\epsilon(t) = q(t) + \epsilon \frac{dq(t,\epsilon)}{d\epsilon}\bigg|_{\epsilon=0} + O(\epsilon^2) = q(t) + \epsilon\delta q(t) + O(\epsilon^2),$$

where the linear term is a vector field on Q

$$\delta q(t) := \frac{dq(t,\epsilon)}{d\epsilon}\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} (\phi_\epsilon \circ q_0) =: \Phi(q)\,, \quad \text{called the } \boxed{\textit{infinitesimal transformation}}$$

That is, $\Phi(q)$ is the ***linearisation*** of the flow map ϕ_ϵ at the point $q \in Q$.

Suppose also that the Lagrangian $L(q,\dot q)$ in Hamilton's principle $\delta S = 0$ with $S = \int_a^b L(q,\dot q)dt$ is *invariant* under these infinitesimal transformations, so that $\delta S = 0$ as a consequence of this invariance. Then the endpoint term above $\langle \frac{\partial L}{\partial \dot q}, \delta q\rangle = \langle p, \delta q\rangle$ is a *constant of the motion*. That is, the quantity $\langle \frac{\partial L}{\partial \dot q}, \delta q\rangle = \langle p, \delta q\rangle$ is a constant, whenever $q(t)$ is a solution of the EL equations for this invariant Lagrangian. This argument proves the following.

Theorem 2 (Noether, 1918)**.**
To each Lie symmetry of the Lagrangian, $(\frac{d}{d\epsilon}\big|_{\epsilon=0} L(q,\dot{q})) = 0$*, there corresponds a conservation law,* $\langle \frac{\partial L}{\partial \dot{q}}, \Phi(q) \rangle = \langle p, \Phi(q) \rangle$.

Example: *Ignorable coordinates*: For $L(q,\dot{q},\dot{\theta})$ invariant under $\theta \to \theta + \epsilon$, $\delta\theta = \epsilon$, we have $\frac{d}{dt}\Big\langle \frac{\partial L}{\partial \dot{\theta}},\, \epsilon \Big\rangle = \Big\langle \frac{\partial L}{\partial \theta},\, \epsilon \Big\rangle = 0$.

3.2 Exercise: Euler-Lagrange equations for Hamilton's principle for Simple Mechanical Systems

Lagrangians for Simple Mechanical Systems take the form, $L(q,\dot q) = T(\dot q) - V(q) = KE - PE$.

1. Planar isotropic oscillator, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$: $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - \frac{k}{2}|\mathbf{x}|^2 \implies \ddot{\mathbf{x}} = -\,\omega^2\mathbf{x}$ with $\omega^2 = k/m$

2. Planar anisotropic oscillator, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$: $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - \frac{k_1}{2}x_1^2 - \frac{k_2}{2}x_2^2 \implies \ddot{x}_i = -\,\omega_i^2 x_i$ with $\omega_i^2 = k_i/m \quad i = 1,2$

3. Planar pendulum in polar coordinates, $(\theta,\dot\theta) \in TS^1$: $L = \frac{m}{2}R^2\dot\theta^2 - mgR(1-\cos\theta) \implies \ddot\theta = -\omega^2\sin\theta$ with $\omega^2 = g/R$

4. Planar pendulum, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$, constrained to $TS^1 = \{\mathbf{x},\dot{\mathbf{x}} \in T\mathbb{R}^2 \,|\, 1-|\mathbf{x}|^2 = 0 \;\&\; \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}$: $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} + \mu(1-|\mathbf{x}|^2)$

5. Charged particle in a magnetic field, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$: $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 + \frac{e}{c}\dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}) \implies \ddot{\mathbf{x}} = \frac{e}{mc}\dot{\mathbf{x}}\times\mathbf{B}$ with $\mathbf{B} = \operatorname{curl}\mathbf{A}$

6. Kepler problem, $(r,\dot r,\theta,\dot\theta) \in T\mathbb{R}_+ \times TS^1$: $L = \frac{m}{2}\left(\dot r^2 + r^2\dot\theta^2\right) + \frac{GMm}{r} \implies \ddot r = -\frac{GM}{r^2} + \frac{J^2}{r^3}$ with $J = r^2\dot\theta = const$

7. Free motion on a sphere, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$: $L = \frac{1}{2}|\dot{\mathbf{x}}|^2 + \mu(1-|\mathbf{x}|^2)$

8. Spherical pendulum (a), $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$: $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} + \mu(1-|\mathbf{x}|^2)$

9. Spherical pendulum (b), setting $\mathbf{x}(t) = O(t)\mathbf{x}_0\,,\quad \dot{\mathbf{x}}(t) = \dot O(t)\mathbf{x}_0$ for $(O,\,\dot O) \in TSO(3)\,,$ where $\mathbf{x}_0 = \mathbf{x}(0)$ is the initial position of the particle and $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x}$, without the need for the constraint $|\mathbf{x}|^2 = 1$, since rotations preserve length. Set $g = 0$ to get free motion on the sphere.

10. Rotating rigid body, $\widehat{\xi} = O^{-1}\dot O \in T(SO(3) \simeq \mathfrak{so}(3)$ $\quad \ell = \frac{1}{2}\boldsymbol{\Omega}\cdot I\boldsymbol{\Omega}$ with $\boldsymbol{\Omega}\times = \widehat{\xi}\,,$ that is, $-\,\epsilon_{ijk}\Omega_k = \widehat{\xi}_{ij}$

3.3 Practice problem – Hamilton's Principle for geodesics (covariant derivatives)

- ***Geodesics:*** When $L = KE = \frac{1}{2}g_q(\dot{q},\dot{q}) =: \frac{1}{2}\|\dot{q}\|^2$, the solution $q(t)$ of the EL equations that passes from point $q(a)$ to $q(b)$ is called the *geodesic path* with respect to the metric $g_q : TM \times TM \to \mathbb{R}$. The geodesic represents the path of shortest distance $q(a) \to q(b)$ measured by

$$ds^2 := dq^a g_{ab}(q) dq^b = g_q(\dot{q},\dot{q})dt^2 = \|\dot{q}\|^2\, dt^2$$

- ***Exercise:*** Compute the EL equations for a geodesic with respect to the metric $g_q : TM \times TM \to \mathbb{R}$. That is, compute the EL equations for $L = KE = \frac{1}{2}g_q(\dot{q},\dot{q}) =: \frac{1}{2}\|\dot{q}\|^2$.

- ***Answer:*** The KE Lagrangian is

$$L(q,\,\dot{q}) = \frac{1}{2}\dot{q}^b g_{bc}(q)\dot{q}^c\,.$$

Its partial derivatives are given by

$$\frac{\partial L}{\partial \dot{q}^a} = g_{ac}(q)\dot{q}^c \quad \text{and} \quad \frac{\partial L}{\partial q^a} = \frac{1}{2}\frac{\partial g_{bc}(q)}{\partial q^a}\dot{q}^b\dot{q}^c\,.$$

Consequently, its Euler–Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} \;=\; g_{ae}(q)\ddot{q}^{\,e} + \frac{\partial g_{ae}(q)}{\partial q^b}\dot{q}^b\dot{q}^e - \frac{1}{2}\frac{\partial g_{be}(q)}{\partial q^a}\dot{q}^b\dot{q}^e = 0\,.$$

Symmetrising the coefficient of the middle term and contracting with co-metric g^{ca} satisfying $g^{ca}g_{ae} = \delta^c_e$ yields

$$\boxed{\ddot{q}^{\,c} + \Gamma^c_{be}(q)\dot{q}^b\dot{q}^e = 0} \quad \text{with} \quad \Gamma^c_{be}(q) = \frac{1}{2}g^{ca}\left[\frac{\partial g_{ae}(q)}{\partial q^b} + \frac{\partial g_{ab}(q)}{\partial q^e} - \frac{\partial g_{be}(q)}{\partial q^a}\right], \tag{4}$$

in which the Γ^c_{be} are called the ***Christoffel symbols*** for the Riemannian metric g_{ab}.

These Euler–Lagrange equations are the ***geodesic equations*** of a free particle moving in a Riemannian space. They are often written as

$$\ddot{q} + \nabla_{\dot{q}}\dot{q} = 0,$$

in terms of the ***covariant derivative*** $\nabla_{\dot{q}}$.

3.4 Practice problem - The isoperimetric problem (what Lagrange wrote to Euler about). Work out the details of the calculation here.

This problem is to find the curve between two points (x_1, y_1) and (x_2, y_2), of specified length, that maximises the area integral $\int_{x_1}^{x_2} y(x)dx$.

In this example the length of the curve is

$$L[y] = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx,$$

which takes the specified value $l = const$. The area is

$$A[y] = \iint dx \wedge dy = \int_{x_1}^{x_2} y(x)dx.$$

We look for extrema of the modified functional

$$S[y] = \int_{x_1}^{x_2} ydx - \lambda \int_{x_1}^{x_2} (\sqrt{1 + y'^2}dx - l\,),$$

where λ is a scalar constant (Lagrange multiplier), to be determined. The Euler-Lagrange equation is

$$\lambda \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1 + y'^2}} \right) + 1 = 0\,. \tag{5}$$

Hence, a first integration yields $\frac{y'}{\sqrt{1+y'^2}} = -(x - x_0)/\lambda$, giving the parametric solution, after solving for y'^2,

$$x = x_0 \pm \lambda \sin(\theta), \qquad y = y_0 \pm \lambda \cos(\theta), \tag{6}$$

so $(x - x_0)^2 + (y - y_0)^2 = \lambda^2$ and the extremum is the arc of a circle of radius λ.

The variational problem satisfied by a soap bubble is analogous to the isoperimetric problem. For the soap bubble, the surface area is extremised, holding the volume integral constant. The Lagrange multiplier is the pressure, p.

3.5 Legendre transform

- *LT* $: (q, \dot{q}) \in TM \to (q, p) \in T^*M$ defines *momentum* p as the *fibre derivative* of L, namely

$$p := \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \in T^*M \qquad \textit{(fibre derivative)}\,.$$

The LT is *invertible* for $\dot{q} = f(q, p)$, provided the *Hessian* $\partial^2 L(q, \dot{q})/\partial\dot{q}\partial\dot{q}$ has nonzero determinant. Note, $\dim T^*M = 2n$.

- In terms of the LT, the ***Hamiltonian*** $H : T^*M \to \mathbb{R}$ is defined by

$$H(q, p) = \Big\langle p, \dot{q} \Big\rangle - L(q, \dot{q})$$

in which the expression $\langle p, \dot{q}\rangle$ in this calculation identifies a *pairing* $\langle\,\cdot\,,\,\cdot\,\rangle : T^*M \times TM \to \mathbb{R}$.

Taking the differential of this definition yields

$$\begin{aligned} dH &= \Big\langle H_p, dp \Big\rangle + \Big\langle H_q, dq \Big\rangle \\ &= \Big\langle dp, \dot{q} \Big\rangle + \Big\langle p - L_{\dot{q}}, d\dot{q} \Big\rangle - \Big\langle L_q, dq \Big\rangle \end{aligned}$$

from which Hamilton's principle $\delta S = 0$ for $S = \int_a^b \langle p, \dot{q}\rangle - H(q, p)\, dt$ produces *Hamilton's canonical equations* on phase space T^*M,

$$H_p = \dot{q} \quad \text{and} \quad H_q = -\,L_q = -\,\dot{p}\,.$$

- Hamilton's principle $\delta S = 0$ for $S = \int_a^b \langle p, \dot{q} \rangle - H(q,p)\,dt$ produces *Hamilton's canonical equations* on phase space T^*M,

$$H_p = \dot{q} \quad \text{and} \quad H_q = -\,L_q = -\,\dot{p}\,.$$

Exercise. Verify the previous statement. That is, compute the results of the following Phase-space form of Hamilton's principle on T^*M, given by $\delta S = 0$ with $S = \int_a^b \langle p, \dot{q} \rangle - H(q,p)\,dt$. ★

- **Answer.** One computes

$$\delta S = \delta \int_a^b \langle p, \dot{q} \rangle - H(q,p)\,dt = \int_a^b \delta \langle p, \dot{q} \rangle - \delta H(q,p)\,dt$$
$$= \int_a^b \Big\langle \delta p\,, \dot{q} - H_p \Big\rangle - \Big\langle \dot{p} + H_q, \delta q \Big\rangle dt + \underbrace{\Big\langle p, \delta q \Big\rangle \Big|_a^b}_{\text{Endpoint term}}$$

▲

Remark 3. *We will return to the endpoint term in formulating Noether's theorem on phase space, that is, on* T^*M.

4 Hamilton's equations

4.1 Practice problem – Legendre transforms for the simple mechanical systems in §3.2

- Legendre transform: $H(q,p) = \langle p,\, \dot q\rangle - L(q,\dot q) = T(p) + V(q) = KE + PE.$

For example,

1. Planar isotropic oscillator, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + \frac{k}{2}|\mathbf{x}|^2$
2. Planar anisotropic oscillator, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + \frac{k_1}{2}x_1^2 + \frac{k_2}{2}x_2^2$
3. Planar pendulum in polar coordinates, $(\theta, p_\theta) \in T^*S^1$: $\quad H = \frac{1}{2mR^2}p_\theta^2 + mgR(1-\cos\theta)$
4. Planar pendulum, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$, constrained to $S^1 = \{\mathbf{x}\in\mathbb{R}^2 : 1 - |\mathbf{x}|^2 = 0\}$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + mg\,\hat{\mathbf{e}}_2\cdot\mathbf{x} - \mu(1-|\mathbf{x}|^2)$
5. Charged particle in a magnetic field, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x})|^2 \quad \mathbf{p} := \partial L/\partial\dot{\mathbf{q}} = m\dot{\mathbf{x}} + \frac{e}{c}\mathbf{A}(\mathbf{x}) \in T^*M$
6. Kepler problem, $(r, p_r, \theta, p_\theta) \in T^*\mathbb{R}_+ \times T^*S^1$: $\quad H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r}$ with $p_\theta = r^2\dot\theta = const$
7. Free motion on a sphere, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x}\in\mathbb{R}^3 : 1 - |\mathbf{x}|^2 = 0\}$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 - \mu(1-|\mathbf{x}|^2)$
8. Spherical pendulum (a), $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x}\in\mathbb{R}^3 : 1 - |\mathbf{x}|^2 = 0\}$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} - \mu(1-|\mathbf{x}|^2)$
9. Spherical pendulum (b), $(O,\dot O) \in TSO(3)$, $\widehat{\xi} = O^{-1}\dot O \in T(SO(3) \simeq \mathfrak{so}(3)$, $\mathbf{\Pi} = \partial\ell/\partial\mathbf{\Omega} \in T^*(SO(3) \simeq \mathfrak{so}(3)^* \simeq \mathbb{R}^3$
   $H = \frac12\mathbf{\Pi}\cdot I^{-1}\mathbf{\Pi} + g\,\mathbf{\Gamma}\cdot\mathbf{x}_0$ with $\mathbf{\Pi} = \frac{\partial\ell}{\partial\mathbf{\Omega}} = I\mathbf{\Omega}$. Set $g=0$ to get freely rotating rigid body motion.
10. Rotating rigid body, $\mathbf{\Pi} \in T^*(SO(3) \simeq \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ $\quad H = \frac12\mathbf{\Pi}\cdot I^{-1}\mathbf{\Pi}$ with $\mathbf{\Pi} = \frac{\partial\ell}{\partial\mathbf{\Omega}} = I\mathbf{\Omega}$.

4.2 Canonical Poisson bracket

- The Hamiltonian dynamics of a phase-space function is given by

$$\frac{d}{dt}F(q,p) = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial p}\dot{p} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} := \{F,\, H\}$$

The operation $\{F,\, H\}$ is called the *canonical Poisson bracket* of F with H on the phase space T^*M.

The canonical Poisson bracket operation $\{\cdot\,,\,\cdot\}$ is a map among smooth real functions $\mathcal{F}(T^*M) : T^*M \to \mathbb{R}$

$$\{\cdot\,,\,\cdot\} : \mathcal{F}(T^*M) \times \mathcal{F}(T^*M) \to \mathcal{F}(T^*M)\,, \tag{7}$$

so that Hamiltonian dynamics *on phase space* T^*M is given by $\dot{F} = \{F\,,\,H\}$ for any $F \in \mathcal{F}(T^*M)$.

Definition 4 (Poisson bracket)**.** *A **Poisson bracket operation** $\{\cdot\,,\,\cdot\}$ is defined by its properties listed below:*

- *It is **bilinear**.*
- *It is **skew-symmetric**, $\{F\,,\,H\} = -\{H\,,\,F\}$.*
- *It satisfies the **Leibniz rule** (product rule),*

$$\{FG\,,\,H\} = \{F\,,\,H\}G + F\{G\,,\,H\}\,,$$

  for the product of any two functions F and G on M.
- *It satisfies the **Jacobi identity**,*

$$\{F\,,\,\{G\,,\,H\}\} + \{G\,,\,\{H\,,\,F\}\} + \{H\,,\,\{F\,,\,G\}\} = 0\,, \tag{8}$$

  for any three functions F, G and H on M.

Remark. The Leibniz rule associates Poisson brackets with differential operators on smooth functions $F \in \mathcal{F}(T^*M)$.

The differential operator or ***Hamiltonian vector field*** generated by the canonical Poisson bracket with F is

$$X_F := \{\,\cdot\,,\,F\,\} = \frac{\partial F}{\partial p}\frac{\partial}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial}{\partial p}\,.$$

- **Exercise:** What is ***Noether's theorem for Hamilton's principle in phase-space, on T^*M***?
- **Answer:** For an infinitesimal transformation $(\delta q\,,\delta p)$ that induces $\delta L = \delta\big(\langle p, \dot q\rangle - H(q,p)\big)$ we have

$$\delta S = \delta \int_a^b \Big\langle p, \dot q \Big\rangle - H(q,p)\,dt = \int_a^b \delta\Big\langle p, \dot q\Big\rangle - \delta H(q,p) = \int_a^b \Big\langle \delta p\,, \dot q - H_p \Big\rangle - \Big\langle \dot p + H_q, \delta q\Big\rangle dt + \underbrace{\Big\langle p, \delta q\Big\rangle\Big|_a^b}_{\text{Endpoint}}$$

4.3 Cotangent lift and Noether's theorem on the Hamiltonian side

- Suppose the variations due to the infinitesimal transformations on a manifold M take the form $\delta q = \xi_M(q)$. Then the corresponding Hamiltonian for these infinitesimal transformations is

$$J^\xi := \Big\langle p, \xi_M(q)\Big\rangle \quad \text{so that} \quad \delta q = \frac{\partial J^\xi}{\partial p} = \xi_M(q) \quad \text{and} \quad \delta p = -\,\frac{\partial J^\xi}{\partial q} = -\,\xi_M'(q)^T p$$

The last expression is called the ***cotangent lift*** to T_q^*M of the infinitesimal transformation $q \to q_\epsilon = q + \epsilon \xi_M(q)$ on M.

The cotangent lift specifies the infinitesimal transformation of $p \in T_q^*M$, given the infinitesimal transformation of $q \in M$.

$$q \to q_\epsilon = q + \epsilon\xi_M(q) \text{ on } M \quad \Longrightarrow \quad (q,p) \to (q_\epsilon, p_\epsilon) = (q + \epsilon\xi_M(q),\, p - \,\epsilon\xi_M'(q)^T p) \text{ on } T_q^*M\,.$$

The time derivative of $J^\xi(q,p)$ is given by

$$\frac{d}{dt}J^\xi(q,p) = \frac{\partial J^\xi}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial J^\xi}{\partial p}\frac{\partial H}{\partial q} = -\,\frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q = -\,\delta H = \{J^\xi, H\} = -\,\{H, J^\xi\} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} H(q,p).$$

In the last step we defined the infinitesimal transformation of H under canonical transformations generated by $J^\xi := \langle p, \xi_M(q)\rangle := \langle p, \delta q\rangle$, the conserved endpoint term in Noether's theorem. This calculation proves the following.

Corollary 5. *On the Hamiltonian side, Noether's theorem for conservation of the endpoint term $J^\xi := \langle p, \xi_M(q)\rangle := \langle p, \delta q\rangle$ follows from Lie symmetry of the Hamiltonian function under $\delta H = \{H, J^\xi\} = 0$.*

The differential operator or **Hamiltonian vector field** generated by the canonical Poisson bracket with J^ξ is defined by

$$\frac{d}{d\epsilon} = X_{J^\xi} := \{\,\cdot\,,\, J^\xi\,\} = \frac{\partial J^\xi}{\partial p}\frac{\partial}{\partial q} - \frac{\partial J^\xi}{\partial q}\frac{\partial}{\partial p} = \xi_M(q)\frac{\partial}{\partial q} - \xi\,'(q)^T p\frac{\partial}{\partial p} = \delta q\frac{\partial}{\partial q} + \delta p\frac{\partial}{\partial p}\,.$$

4.4 Example: Angular velocity and angular momentum

Let $G \times M \to M$ with $G = SO(3)$ and $M = \mathbb{R}^3$. That is, $SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$.
Let $q(\epsilon) = O(\epsilon)q(0)$ with $O \in SO(3)$, so that $O^TO = Id$ and $q \in \mathbb{R}^3$. Then the infinitesimal transformation is[1]

$$\delta q := q'(\epsilon)\Big|_{\epsilon=0} = \big[O'(\epsilon)q(0)\big]_{\epsilon=0} = \big[O'(\epsilon)O^{-1}(\epsilon)q(\epsilon)\big]_{\epsilon=0} := \widehat{\xi}q = \xi \times q \quad \text{with} \quad \widehat{\xi}_{ab} = -\epsilon_{abc}\xi^c.$$

Remark 6 (***Hat map***)**.** The components of any 3×3 skew matrix $\widehat{\xi}$ may be identified with the corresponding components of a vector $\boldsymbol{\xi} \in \mathbb{R}^3$, by the linear invertible relation,

$$\widehat{\xi} = \begin{pmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{pmatrix} \quad \text{with} \quad \widehat{\xi}_{ab} = -\epsilon_{abc}\xi^c\,, \qquad (9)$$

for $a, b, c = 1, 2, 3$. This is an ***isomorphism*** (one-to-one invertible map) between 3×3 skew-symmetric matrices and vectors in $\mathbb{R}^3$.

Remark 7 (***Hat map***)**.** The overscript hat $(\,\widehat{}\,)$ applied to a vector in $\mathbb{R}^3$ identifies that vector with a 3×3 skew-symmetric matrix. For example, the unit vectors in the Cartesian basis set, $\{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3\}$, are associated with the basis elements of $\mathfrak{so}(3)$, by $\widehat{\mathbf{e}}_a$, or in matrix components,

$$(\widehat{\mathbf{e}}_a)_{bc} = -\delta^d_a\epsilon_{dbc} = -\epsilon_{abc} = (\mathbf{e}_a\times)_{bc}\,.$$

This equation introduces the convenient notation $\widehat{\boldsymbol{e}}$ that denotes the basis for the 3×3 skew-symmetric matrices $\widehat{e}_a$, with $a = 1, 2, 3$ as a vector of matrices. One may check the commutator $[\widehat{e}_a, \widehat{e}_b] = \epsilon_{abc}\widehat{e}_c$; so that

$$[\,\widehat{\xi},\, \widehat{\eta}\,] = \boldsymbol{\xi} \times \boldsymbol{\eta} \cdot \widehat{\boldsymbol{e}} =: (\boldsymbol{\xi} \times \boldsymbol{\eta})\,\widehat{}$$

for $\widehat{\xi} = \xi^a\widehat{e}_a$ and $\widehat{\eta} = \eta^b\widehat{e}_b$.

[1] The matrix $\widehat{\xi} = \dot{O}O^{-1} = \dot{O}O^T$ is skew, since $\frac{d(OO^T)}{dt} = \frac{d(Id)}{dt} = \dot{O}O^T + O\dot{O}^T = \dot{O}O^T + (\dot{O}O^T)^T = \widehat{\xi} + \widehat{\xi}^T = 0.$

4.4.1 Back to the canonical angular momentum example

The Hamiltonian

$$J^\xi(q,p) = q \times p \cdot \xi = p \cdot \xi_M(q) = p \cdot \xi \times q$$

generates infinitesimal $SO(3)$ rotations around the vector $\xi \in \mathfrak{so}(3) \simeq \mathbb{R}^3$, as we may compute

$$\delta q = \big\{ q,\, J^\xi(q,p) \big\} = \xi \times q(t), \qquad \delta p = \big\{ p,\, J^\xi(q,p) \big\} = \xi \times p(t),$$

using the canonical Poisson bracket $\big\{\,\cdot\,,\,\cdot\,\big\}$.

Thus, the **_cotangent lift_** of an infinitesimal rotation of q given by $\delta q = \xi_M(q) = \xi \times q$ is an infinitesimal rotation of p given by $\delta p = -\,\xi'_M(q)^T p = \xi \times p$. These equations imply the following variation for $J(q,p) = q \times p \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3$

$$\delta J = \xi \times J(t) \quad \text{for } \xi \in \mathfrak{so}(3) \simeq \mathbb{R}^3 \text{ and } J \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3\,,$$

as obtained by using the product rule for the Poisson bracket and the Jacobi identity for the cross product of vectors in $\mathbb{R}^3$. [2]

The quantity $J(q,p) = q \times p \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ is called the **_angular momentum_**.

The map $J(q,p) = q \times p : T_q^*M \to \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ is the **_cotangent lift momentum map_** for the action of the Lie group of spatial rotations $G = SO(3)$ on the manifold $M = \mathbb{R}^3$.

[2] Thus, $\delta J = \{q \times p, J^\xi(q,p)\} = \delta(q \times p) = \delta q \times p + q \times \delta p = (\xi \times q) \times p + q \times (\xi \times p) = p \times (q \times \xi) + q \times (\xi \times p) = -\,\xi \times (p \times q) = \xi \times (q \times p) = \xi \times J$.

Remark 8 (An example of the Marsden-Weinstein reduction theorem (1974)). *Being an isomorphism, the hat map $\xi \in \mathbb{R}^3 \to \widehat{\xi} \in \mathfrak{so}(3)$ allows us to identify all the spaces where the various quantities live with $\mathbb{R}^3$*

$$\xi \in \mathbb{R}^3 \simeq \mathfrak{so}(3) \quad \textit{and} \quad J(q,p) = q \times p \in \mathbb{R}^3 \simeq \mathbb{R}^{3^*} \simeq \mathfrak{so}(3)^*.$$

If we remove the degeneracy of this notation, we find that $J^\xi(q,p) = q \times p \cdot \xi = p \cdot \xi \times q = p \cdot \xi_M(q)$ involves two different pairings which happen to coincide for $\mathbb{R}^3$, because of the permutability of the triple scalar product. Namely,

$$J^\xi(q,p) = \big\langle J(q,p)\,,\,\xi\big\rangle_{\mathfrak{so}(3)^*\times\mathfrak{so}(3)} = \big\langle p\,,\,\xi_M(q)\big\rangle_{T^*M\times TM}$$

The equivariant, Poisson, cotangent lift momentum map $J : T^*M \to \mathfrak{so}(3)^*$ reduces the problem of rigid rotational motion to fewer dimensions, plus a reconstruction equation, as sketched in the following commutative diagram of maps.

$$\frac{dz}{dt} = \{z,\, H(z)\}_{can}, \;\; \dim T^*\mathbb{R}^3 = 6$$

$$\begin{array}{ccc}
z = (q,p) \in T^*\mathbb{R}^3 & \longrightarrow & T^*\mathbb{R}^3 \\
J(0) = q(0) \times p(0) \Big\downarrow & \textit{Equivariance} & \Big\downarrow J(t) = q(t) \times p(t) \\
J \in \mathbb{R}^3 \simeq \mathfrak{so}(3)^* & \longrightarrow & J \in \mathbb{R}^3 \simeq \mathfrak{so}(3)^*
\end{array}$$

$$\frac{dJ}{dt} = \{J,\, H(J)\}_{LP} = -\,J \times \frac{\partial H}{\partial J}, \quad \textit{conserved } |J|^2 \implies \textit{reduction } \mathbb{R}^6 \to \mathbb{R}^3 \to S^2 \textit{ and } \dim S^2 = 2$$

In the final step, the area element of the sphere S^2 provides the symplectic canonical Poisson bracket for the reduced motion, which is thus completely integrable. This last remark is an example of the Marsden-Weinstein reduction theorem (1974).

*The **reconstruction equation** is*

$$\dot{O} = \widehat{\omega}(t)O(t),$$

whose solution is derived from the solution for $J(t) \in \mathfrak{so}(3)^*$ via integration, after inverting the Legendre transformation to obtain $\widehat{\omega}(t) = \dot{O}O^{-1}(t) \in \mathfrak{so}(3)$ from $\omega := \partial H/\partial J$. The motion in phase space $T^*M = \mathbb{R}^3 \times \mathbb{R}^{3^*}$ is then found from

$$\begin{aligned}
\dot{q} &= \omega(t) \times q(t) = \widehat{\omega}(t)q(t) = \dot{O}O^{-1}(t)q(t), \quad \textit{and} \\
\dot{p} &= \omega(t) \times p(t) = \widehat{\omega}(t)p(t) = \dot{O}O^{-1}(t)p(t).
\end{aligned}$$

4.5 The angular momentum map generalises the notion of Poisson brackets for $SO(3)$.

- **Exercise:** Show for vectors $\xi, \eta \in \mathbb{R}^3$ that for angular momentum $J(q,p) = q \times p \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ and $J^\xi = \xi \cdot J(q,p)$ that

$$\Big\{ J^\xi ,\, J^\eta \Big\} = J^{\xi \times \eta}.$$

- **Answer:** The proof follows by a direct calculation using Jacobi's identity for vector cross products:[3]

$$\Big\{ J^\xi ,\, J^\eta \Big\} = \Big\{ J \cdot \xi ,\, J \cdot \eta \Big\} = \Big\{ q \times p \cdot \xi ,\, q \times p \cdot \eta \Big\}_{can} = (q \times p) \cdot (\xi \times \eta) = J \cdot (\xi \times \eta) = J^{\xi \times \eta}.$$

Hence, for functions $F(J(q,p)) = F \circ J$ and $H(J(q,p)) = H \circ J$ of the angular momentum map J we have

$$\Big\{ J_k ,\, J_l \Big\} = \epsilon_{kl}{}^m J_m \quad \text{and} \quad \Big\{ F(J) ,\, H(J) \Big\} = J \cdot \frac{\partial F}{\partial J} \times \frac{\partial H}{\partial J} \quad \text{so that} \quad \frac{dJ}{dt} = \Big\{ J ,\, H(J) \Big\} = -J \times \frac{\partial H}{\partial J} \,.$$

Thus, the angular momentum map $J(q,p) : T^*\mathbb{R}^3 \to \mathbb{R}^3$ is **Poisson**, which means that $\{ F \circ J \,, H \circ J \} = \{ F ,\, H \} \circ J$.

Upon denoting $\mathbf{J} \in \mathbb{R}^3$ the Poisson bracket becomes $\{F,\, H\} = \nabla C \cdot \nabla F \times \nabla H$ with motion equation $\dot{\mathbf{J}} = -\nabla C \times \nabla H$ where $C(\mathbf{J}) = \frac{1}{2}|\mathbf{J}|^2$. This means the motion takes place on **spheres** along intersections of level sets of C and H.

[3] Using the calculation in the previous footnote, $\Big\{ q \times p \cdot \xi ,\, q \times p \cdot \eta \Big\}_{can} = -\eta \cdot \{ q \times p ,\, J^\xi(q,p) \}_{can} = -\eta \cdot \xi \times (q \times p) = -J \cdot \eta \times \xi = J \cdot \xi \times \eta.$

4.6 Rigid body rotation – Clebsch Hamilton's principle

Review of the Clebsch Hamilton's principle for the Euler-Lagrange equations

First, before deriving the Lagrangian and Hamiltonian formulations of rigid body dynamics, let's recall our earlier derivation of the Euler-Lagrange equations from the constrained Hamilton's principle, in which we varied coordinates $(q, v) \in TQ$, subject to the constraint $v = \frac{dq}{dt}$ (tangent lift). In this case, the constrained action integral was varied according to

$$\delta S = \delta \int_a^b L(q, v) + \left\langle p, \frac{dq}{dt} - v \right\rangle dt \; = \int_a^b \left\langle \frac{\partial L}{\partial v} - p, \delta v \right\rangle + \left\langle \frac{\partial L}{\partial q} - \frac{dp}{dt}, \delta q \right\rangle + \left\langle \delta p, \dot{q} - v \right\rangle dt + \left\langle p, \delta q \right\rangle \Big|_a^b .$$

Then we assembled the EL equation

$$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} = \frac{\partial L(q,\dot{q})}{\partial q}$$

from the various stationary conditions, and evaluated $\frac{\partial L}{\partial v}\Big|_{v=\dot{q}} = \frac{\partial L(q,\dot{q})}{\partial \dot{q}}$.

In our next theorem, we are going to do the same sort of variational calculation when $Q \in SO(3)$ and derive the equations for a rigidly rotating body described by a curve in $SO(3)$, for the case $G \times M \to M$ when both $G = SO(3)$ and $M = SO(3)$. That is, $SO(3) \times SO(3) \to SO(3)$, with flow $Q(t + s) = Q(t)Q(s)$ and $Q(t - t) = Q(0) = Id$, obtained from the rotation group $SO(3)$ acting on itself.

Theorem 9 (Clebsch form of Hamilton's principle for the rigid body)**.**
For $Q \in SO(3)$, the Euler-Lagrange equations become Euler-Poincaré rigid-body equations in matrix commutator form,

$$\frac{d}{dt}\frac{\partial l}{\partial \widehat{\Omega}} = -\left[\widehat{\Omega},\, \frac{\partial l}{\partial \widehat{\Omega}}\right] \quad \textit{or, for} \quad \widehat{\Pi} := \frac{\partial l}{\partial \widehat{\Omega}}\,, \quad \textit{equivalently} \quad \frac{d\widehat{\Pi}}{dt} = -\widehat{\Omega}\widehat{\Pi} + \widehat{\Pi}\widehat{\Omega} =: -\big[\widehat{\Omega},\, \widehat{\Pi}\big], \tag{10}$$

with (body, left-invariant) angular velocity $\widehat{\Omega} = Q^{-1}\dot{Q} = -\widehat{\Omega}^T \in \mathfrak{so}(3) = T_e SO(3)$ and body angular momentum $\widehat{\Pi} := \partial l/\partial\widehat{\Omega}$. The commutator equation (10) *emerges from the constrained Hamilton's principle, $\delta S = 0$ with constrained action integral*

$$S(\widehat{\Omega}, Q, P) = \int_a^b l(\widehat{\Omega}) + \Big\langle P\,,\, \dot{Q} - Q\widehat{\Omega}\Big\rangle\, dt = \int_a^b l(\widehat{\Omega}) + \mathrm{tr}\Big(P^T\big(\dot{Q} - Q\widehat{\Omega}\big)\Big)\, dt = \int_a^b l(\widehat{\Omega}) + \mathrm{tr}\Big((Q^TP)^T\big(Q^{-1}\dot{Q} - \widehat{\Omega}\big)\Big)\, dt\,, \tag{11}$$

for $(Q,P) \in T^*SO(3)$. Stationarity ($\delta S = 0$) leads to the following variational conditions

$$\widehat{\Pi} = \frac{\delta l}{\delta\widehat{\Omega}} = \frac{1}{2}(P^TQ - Q^TP) \in \mathfrak{so}(3)^*\,, \quad \Big\langle P\,,\, \dot{Q} - Q\widehat{\Omega}\Big\rangle := \mathrm{tr}\Big(P^T\,\big(\dot{Q} - Q\widehat{\Omega}\big)\Big) = \mathrm{tr}\Big((Q^TP)^T\big(Q^{-1}\dot{Q} - \widehat{\Omega}\big)\Big)\,,$$

and the quantities $(Q,P) \in T^*SO(3)$ satisfy the following symmetric equations,

$$\dot{Q} = Q\widehat{\Omega} \quad \textit{and} \quad \dot{P} = P\widehat{\Omega}\,, \tag{12}$$

as a result of the constraints. These equations have Lie-Poisson Hamiltonian form,

$$\frac{dF}{dt} = \Big\{F\,,\, H\Big\} = -\Big\langle \Pi\,,\, \Big[\frac{\partial F}{\partial \Pi}\,,\, \frac{\partial H}{\partial \Pi}\Big]\Big\rangle\,. \tag{13}$$

Proof. The variations of the constrained action S in (11) are given by

$$\begin{aligned}
\delta S &= \int_a^b \Big\langle \frac{\delta l}{\delta\widehat{\Omega}}\,,\, \delta\widehat{\Omega}\Big\rangle - \Big\langle P\,,\, Q\,\delta\widehat{\Omega}\Big\rangle + \Big\langle \delta P\,,\, \dot{Q} - Q\widehat{\Omega}\Big\rangle + \Big\langle P\,,\, \delta\dot{Q} - (\delta Q)\,\widehat{\Omega}\Big\rangle\, dt \\
&= \int_a^b \Big\{ \mathrm{tr}\Big[\Big(\widehat{\Pi}^T - \frac{1}{2}(P^TQ - Q^TP)\Big)\delta\widehat{\Omega}\Big] \\
&\qquad + \mathrm{tr}\Big[\delta P^T\Big(\dot{Q} - Q\widehat{\Omega}\Big)\Big] - \mathrm{tr}\Big[\Big(\dot{P}^T + \widehat{\Omega}P^T\Big)\delta Q\Big]\Big\} dt + \Big\langle P\,,\, \delta Q\Big\rangle\Big|_a^b.
\end{aligned}$$

Thus, stationarity of this ***implicit variational principle*** implies the following set of equations

$$\widehat{\Pi} = \frac{\delta l}{\delta \widehat{\Omega}} = \frac{1}{2}(P^T Q - Q^T P)\,, \quad \dot{Q} = Q\widehat{\Omega} \quad \text{and} \quad \dot{P} = P\widehat{\Omega}\,. \tag{14}$$

The commutator form of the rigid-body equations in (10) emerges from these, upon elimination of Q and P, as

$$\begin{aligned} \frac{d\widehat{\Pi}}{dt} &= \frac{1}{2}(\dot{P}^T Q + P^T \dot{Q} - \dot{Q}^T P - Q^T \dot{P}) \\ &= \frac{1}{2}\widehat{\Omega}(Q^T P - P^T Q) - \frac{1}{2}(P^T Q - Q^T P)\widehat{\Omega} \\ &= -\,\widehat{\Omega}\,\widehat{\Pi} + \widehat{\Pi}\,\widehat{\Omega} = -\left[\widehat{\Omega},\, \widehat{\Pi}\right]. \end{aligned}$$

These are Euler's equations for the rigid body on $T^*SO(3) \simeq so(3)^*$. Now, by Legendre transforming to

$$H(\widehat{\Pi}) = \left\langle \widehat{\Pi}\,,\, \widehat{\Omega} \right\rangle - l(\widehat{\Omega}) \quad \text{with} \quad dH = \left\langle d\widehat{\Pi}\,,\, \widehat{\Omega} \right\rangle + \left\langle \widehat{\Pi} - \frac{\partial l}{\partial \widehat{\Omega}}\,,\, d\widehat{\Omega} \right\rangle$$

and using $\widehat{\Omega} = \partial H/\partial \widehat{\Pi}$ and $\widehat{\Omega}^T = -\widehat{\Omega}$ we find the following *Lie-Poisson bracket* for the Hamiltonian formulation of the rigid body dynamics,

$$\begin{aligned} \frac{dF}{dt} = \left\langle \frac{\partial F}{\partial \widehat{\Pi}}\,,\, \frac{d\widehat{\Pi}}{dt} \right\rangle &= \left\langle \frac{\partial F}{\partial \widehat{\Pi}}\,,\, \left[\widehat{\Pi}\,,\, \frac{\partial H}{\partial \widehat{\Pi}}\right] \right\rangle \\ &= \mathrm{tr}\left(\frac{\partial F}{\partial \widehat{\Pi}} \left[\widehat{\Pi}\,,\, \frac{\partial H}{\partial \widehat{\Pi}}\right]^T \right) = \mathrm{tr}\left(\widehat{\Pi}^T \left[\frac{\partial F}{\partial \widehat{\Pi}}\,,\, \frac{\partial H}{\partial \widehat{\Pi}}\right] \right) \\ &= -\left\langle \widehat{\Pi}\,,\, \left[\frac{\partial F}{\partial \widehat{\Pi}}\,,\, \frac{\partial H}{\partial \widehat{\Pi}}\right] \right\rangle \\ \text{(By the hat map)} \qquad &= -\,\mathbf{\Pi} \cdot \frac{\partial F}{\partial \mathbf{\Pi}} \times \frac{\partial H}{\partial \mathbf{\Pi}} =: \Big\{ F\,,\, H \Big\}. \end{aligned}$$

The Poisson bracket defined this way satisfies all of the required properties. In particular, it satisfies the Jacobi identity, since it is a linear functional of the commutator of skew-symmetric matrices, which satisfies the Jacobi identity; since, as we know, the skew-symmetric matrices form $T_eSO(3)$; which, as we shall see, is the Lie algebra $\mathfrak{so}(3)$ for infinitesimal rotations.

□

4.7 Lie algebras

We shall see that, if G is a Lie group, then T_eG (the tangent space at the identity) is an interesting vector space $T_eG = \mathfrak{g}$ possessing a remarkable structure called its **Lie algebra**.

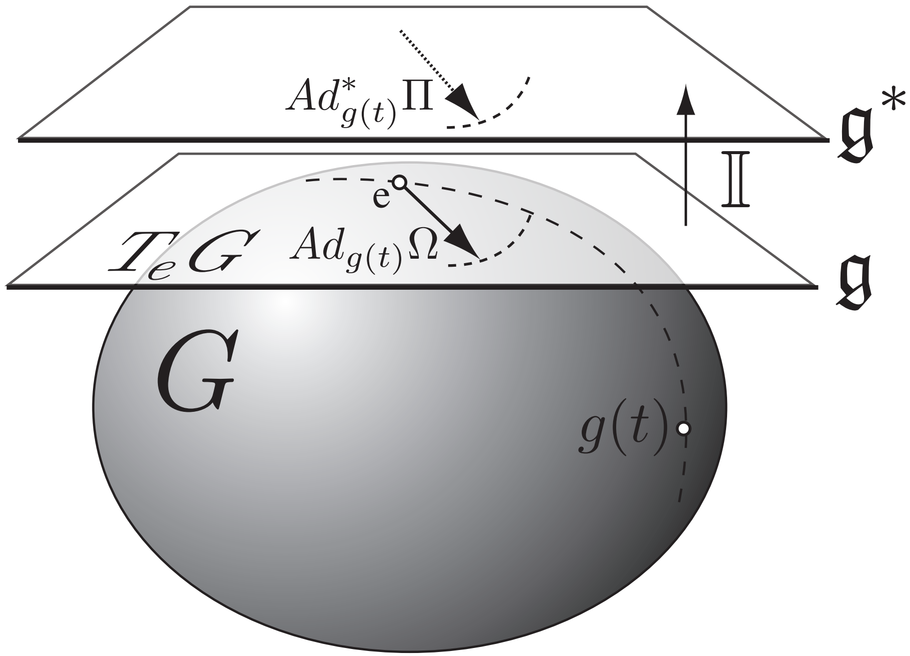


Figure 7: The Ad and Ad* operations of $g(t)$ act, respectively, on the Lie algebra $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ and on its dual $\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$.

Lemma. *Let G be a matrix Lie group and let $g \in G$. Then there exists a map $G \times T_eG \to T_eG$*

$$\xi \in T_eG \quad \Rightarrow \quad g\xi g^{-1} \in T_eG,$$

where the expression $g\xi g^{-1}$ makes sense as a matrix expression.

Proof. Let $c(t) \in G$ be a curve in the matrix Lie group G, such that

$$c(0) = e \text{ and } \dot{c}(0) = \xi.$$

Define the product

$$\gamma(t) = gc(t)g^{-1} \in G.$$

Then

$$\gamma(0) = e \text{ and } \dot{\gamma}(0) = g\xi g^{-1} \in T_eG,$$

which proves the Lemma. ■

Proposition *Let G be a matrix Lie group and let $\xi, \eta \in T_eG$. Then also the matrix commutator $\xi\eta - \eta\xi \in T_eG$.*

Proof Let $c(t) \in G$ be a curve such that

$$c(0) = e \text{ and } \dot{c}(0) = \xi.$$

Also define

$$b(t) = c(t)\eta c(t)^{-1} \in T_eG, \quad \text{by the Lemma.}$$

Then (since T_eG, being a vector space, is closed under differentiation)

$$\dot{b}(t) \in T_eG$$

and (recalling the formula for the derivative of the inverse matrix)

$$\begin{aligned}
\dot{b}(0) &= \dot{c}(0)\eta c(0)^{-1} + c(0)\eta \,\frac{d}{dt}\big(c(t)^{-1}\big)\Big|_{t=0} \\
&= \dot{c}(0)\eta c(0)^{-1} - c(0)\eta c(0)^{-1}\dot{c}(0)c(0)^{-1} \\
&= \xi\eta - \eta\xi \in T_eG. \qquad ■
\end{aligned}$$

Definition (Lie algebra) A **Lie algebra** is a vector space V endowed with a **commutator** (or *Lie bracket*), defined as a bilinear map

$$[\,\cdot\,,\,\cdot\,]\,:\,V\times V\to V$$

such that

1. $[B,A]=-[A,B] \qquad \forall A,B\in V$ (the commutator is skew-symmetric)
2. $[[A,B],C]+[[B,C],A]+[[C,A],B]=0 \quad \forall A,B,C\in V.$ (and satisfies the **Jacobi identity**)

Theorem (without proof) *Let G be a matrix Lie group. Then T_eG is a Lie algebra (denoted by $\mathfrak{g}$), with commutator given by the matrix commutator.*

Examples

1. The Lie algebra $\mathfrak{gl}(n,\mathbb{R}):=T_e\mathrm{GL}(n,\mathbb{R})$ is the vector space of real square $n\times n$ matrices (with commutator).
2. The Lie algebra $\mathfrak{sl}(n,\mathbb{R}):=T_e\mathrm{SL}(n,\mathbb{R})$ is the vector space of real *traceless* square matrices.

   Proof: Take $g(t)\in\mathrm{SL}(n,\mathbb{R})$; then $\det g(t)=1$. Now take $g(t)$ such that $g(0)=e$ and $\dot g(0)=\xi\in\mathfrak{sl}(n,\mathbb{R})$. Then, by using the formula for the derivative of the determinant,

   $$0=\frac{\mathrm{d}}{\mathrm{d}t}\Big(\det g(t)\Big)\Big|_{t=0}=\det g(0)\ \mathrm{Tr}\,\big(g(0)^{-1}\dot g(0)\big)=\mathrm{Tr}\,\xi.$$

3. The Lie algebra $\mathfrak{so}(3)=T_e\mathrm{SO}(3)$ is the vector space of skew-symmetric matrices.

Two important properties:

$$\dot g(t)\in T_{g(t)}G\left\{\begin{array}{lll}\Rightarrow & (1) & g^{-1}\dot g(t)\in T_eG\simeq\mathfrak{g}\\ \Rightarrow & (2) & \dot g g^{-1}(t)\in T_eG\simeq\mathfrak{g}\end{array}\right. .$$

This means the tangent lifts $\dot g(t)\in T_{g(t)}G$ to the curve $g(t)\in G$ can be pulled back to T_eG by either left or right translations.

4.8 Actions of a matrix Lie group on itself and on its matrix Lie algebra

Definition (conjugation, or AD action) Let $g \in G$. Then the operation

$$\begin{aligned} \mathrm{AD}: G \times G &\to G \\ h &\mapsto ghg^{-1} \qquad \forall h \in G \end{aligned}$$

is called the **conjugation, or AD action**, of G on itself.

Take an arbitrary curve $h(t) \in G$ such that $h(0) = e$. Then, upon denoting

$$\xi = \dot{h}(0) \in T_e G$$

we define

$$\mathrm{Ad}_g\, \xi := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} I_g h(t) = g\xi g^{-1} \in T_e G.$$

Definition (Adjoint and coAdjoint actions of G on $\mathfrak{g}$ and $\mathfrak{g}^*$) *The* ***Adjoint action*** *of the matrix group G on $\mathfrak{g}$ is a map*

$$\mathrm{Ad}:\ G \times \mathfrak{g} \to \mathfrak{g} \quad \textit{which for matrices is} \quad \mathrm{Ad}_g\, \xi = g\xi g^{-1}\,.$$

The dual map Ad^* is defined in terms of a (nondegenerate) pairing $\langle \cdot\,,\,\cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ where $\langle\, \mu\,,\, \xi\, \rangle = \mathrm{tr}(\mu^T \xi)$ for matrices as

$$\mathrm{Ad}^*:\ G \times \mathfrak{g}^* \to \mathfrak{g}^*\,, \quad \textit{where} \quad \big\langle \mathrm{Ad}^*_g\, \mu,\ \xi \big\rangle = \big\langle \mu,\ \mathrm{Ad}_g\, \xi \big\rangle\,.$$

The dual map Ad^* is called the ***coAdjoint action*** *of G on $\mathfrak{g}^*$.*

Exercise. Show that the computation

$$\mathrm{Ad}_g \mathrm{Ad}_h \xi = g(h\xi h^{-1})g^{-1} = (gh)\xi(gh)^{-1} = \mathrm{Ad}_{gh}\xi\,,$$

for $g, h \in G$ and $\xi \in \mathfrak{g}$, combined with the definition $\langle \mathrm{Ad}^*_g\,\mu,\ \xi\rangle = \langle \mu,\ \mathrm{Ad}_g\,\xi\rangle$ implies

$$\mathrm{Ad}^*_{g^{-1}}\mathrm{Ad}^*_{h^{-1}}\mu = \mathrm{Ad}^*_{h^{-1}g^{-1}}\mu = \mathrm{Ad}^*_{(gh)^{-1}}\mu\,,$$

for any $\mu \in \mathfrak{g}^*$. In other words, the actions of Ad_g on $\mathfrak{g}$ and $\mathrm{Ad}^*_{g^{-1}}$ on $\mathfrak{g}^*$ form ***representations*** of the Lie group G. These are called the Adjoint and coAdjoint representations of the Lie group G, respectively. ★

Exercise. Consider a curve $g(t) \in G$ such that $g(0) = e$ and denote $\eta = \dot{g}(0) \in T_eG$.
Show that

$$\mathrm{ad}_\eta\,\xi := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Ad}_{g(t)}\,\xi \quad \text{for all } \xi \in \mathfrak{g}.$$

★

Definition (adjoint and coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ and $\mathfrak{g}^*$) *The **adjoint action** of the matrix Lie algebra on itself is given as a map*

$$\begin{aligned} \mathrm{ad} : \mathfrak{g}\times\mathfrak{g} &\to \mathfrak{g} \\ \mathrm{ad}_\eta\,\xi &= [\eta, \xi]. \end{aligned}$$

The dual map $\mathrm{ad}^* : \mathfrak{g}^*\times\mathfrak{g} \to \mathfrak{g}^*$ *is defined as usual via the pairing* $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^*\times\mathfrak{g} \to \mathbb{R}$, *as*

$$\big\langle \mathrm{ad}^*_\eta\,\mu,\ \xi\big\rangle = \big\langle \mu,\ ad_\eta\,\xi\big\rangle\,,$$

*which defines the **coadjoint action** of* $\mathfrak{g}$ *on* $\mathfrak{g}^*$. *For matrices, the coadjoint action may also be expressed as a commutator,*
$\langle \mathrm{ad}^*_\eta\,\mu,\ \xi\rangle = \mathrm{tr}(\mu^T\,,\,[\eta,\xi]) = \mathrm{tr}(\mu^T,(\eta\xi-\xi\eta)) = \mathrm{tr}((\mu^T\eta-\eta^T\mu)\xi) = \mathrm{tr}((\eta^T\mu-\mu^T\eta)^T\xi) = \langle(\eta^T\mu-\mu^T\eta),\ \xi\rangle = \langle[\eta^T,\ \mu],\ \xi\rangle.$

4.9 A momentum map that generalises from $SO(3)$ to other Lie groups.

For any Lie algebra $\mathfrak{g}$, the cotangent lift momentum map satisfies (recall the commutator for the hat map!)

$$\Big\{ J^\xi ,\, J^\eta \Big\} = \pm J^{[\xi ,\, \eta]} ,$$

where $[\xi,\, \eta] = -[\eta,\, \xi]$ is the Lie bracket between $\xi,\, \eta \in \mathfrak{g}$, which we also denote as $[\xi,\, \eta] =: \mathrm{ad}_\xi \eta$. (We'll discuss the $(\pm)$ sign later.)

The corresponding ***Lie–Poisson bracket*** is

$$\Big\{ F(J),\, H(J) \Big\} := \pm \left\langle J \,,\, \left[\frac{\partial H}{\partial J} ,\, \frac{\partial F}{\partial J} \right] \right\rangle_{\mathfrak{g}^* \times \mathfrak{g}} =: \mp \left\langle J \,, \mathrm{ad}_{\partial H/\partial J} \frac{\partial F}{\partial J} \right\rangle_{\mathfrak{g}^* \times \mathfrak{g}} =: \mp \left\langle \mathrm{ad}^*_{\partial H/\partial J} J \,,\, \frac{\partial F}{\partial J} \right\rangle_{\mathfrak{g}^* \times \mathfrak{g}} .$$

Consequently, for Lie-Poisson systems, the dynamics of the cotangent lift momentum map is governed by

$$\frac{dJ}{dt} = \Big\{ J,\, H(J) \Big\} = \mp \, \mathrm{ad}^*_{\partial H/\partial J} J \,.$$

This generalises the angular momentum map Exercise for $SO(3)$ to arbitrary Lie groups and their Lie algebras.

The proof follows by a direct calculation using the Lie-Poisson bracket:

$$\Big\{ J^\xi ,\, J^\eta \Big\} = \Big\{ \big\langle J \,,\, \xi \big\rangle ,\, \big\langle J \,,\, \eta \big\rangle \Big\} = \pm \big\langle J \,,\, [\xi \,,\, \eta] \big\rangle = \pm J^{[\xi \,,\, \eta]}$$

where we have used $\xi = \xi^j e_j$, $\eta = \eta^k e_k$ and $[e_j,\, e_k] = c_{jk}{}^i e_i$ to compute

$$[\xi, \eta] = [\xi^j e_j,\, \eta^k e_k] = \xi^j [e_j,\, e_k] \eta^k = \xi^j \eta^k c_{jk}{}^i e_i = [\xi, \eta]^i e_i \,.$$

Hence, for functions of the ***momentum map*** J we now have the result that

$$\Big\{ J_k,\, J_l \Big\} = \pm c_{kl}{}^m J_m \quad \text{and} \quad \Big\{ F(J),\, H(J) \Big\} = \mp \left\langle J \,, \mathrm{ad}_{\partial H/\partial J} \frac{\partial F}{\partial J} \right\rangle_{\mathfrak{g}^* \times \mathfrak{g}} \quad \text{so} \quad \frac{dJ}{dt} = \Big\{ J,\, H(J) \Big\} = \mp \, \mathrm{ad}^*_{\partial H/\partial J} \, J \,.$$

Thus, the momentum map $J(q,p) : T^*M \to \mathfrak{g}^*$ is Poisson, which means that $\{ F \circ J \,, H \circ J \} = \{ F,\, H \} \circ J$.

The Lagrangian counterpart of Lie–Poisson theory is ***Euler–Poincaré*** theory, from Poincaré [1901], which we will study later (next term).

4.10 Hamilton-Pontryagin principle

Theorem 10 (Hamilton–Pontryagin principle)**.** *The Euler–Poincaré equation*

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} \tag{15}$$

on the dual Lie algebra $\mathfrak{g}^*$ *is equivalent to the following variational principle,*

$$\delta S(\xi, g, \dot g) = \delta \int_a^b l(\xi, g, \dot g)\, dt = 0, \tag{16}$$

for a constrained left-invariant action integral

$$\begin{aligned} S(\xi, g, \dot g) &= \int_a^b l(\xi, g, \dot g)\, dt \\ &= \int_a^b \Big[l(\xi) + \langle\, \mu\,,\, (g^{-1}\dot g - \xi)\,\rangle \Big]\, dt\,. \end{aligned} \tag{17}$$

Proof. The variations of S in formula (17) are given by

$$\delta S = \int_a^b \Big\langle\, \frac{\delta l}{\delta \xi} - \mu\,,\, \delta\xi \,\Big\rangle + \Big\langle\, \delta\mu\,,\, (g^{-1}\dot g - \xi)\,\Big\rangle + \Big\langle\, \mu\,,\, \delta(g^{-1}\dot g)\,\Big\rangle\, dt\,.$$

Substituting $\delta(g^{-1}\dot g) = \dot\eta + \mathrm{ad}_\xi\, \eta$ obtained from $\delta(\dot g) = (\delta g)^{\cdot}$ with $\eta := g^{-1}\delta g$ into the last term produces

$$\begin{aligned} \int_a^b \Big\langle\, \mu\,,\, \delta(g^{-1}\dot g)\,\Big\rangle\, dt &= \int_a^b \Big\langle\, \mu\,,\, \dot\eta + \mathrm{ad}_\xi\, \eta \,\Big\rangle\, dt \\ &= \int_a^b \Big\langle\, -\dot\mu + \mathrm{ad}^*_\xi\, \mu\,,\, \eta \,\Big\rangle\, dt + \Big\langle\, \mu\,,\, \eta \,\Big\rangle\Big|_a^b\,, \end{aligned}$$

where $\eta = g^{-1}\delta g$ vanishes at the endpoints in time. Thus, stationarity $\delta S = 0$ of the Hamilton–Pontryagin variational principle yields the following set of equations:

$$\frac{\delta l}{\delta \xi} = \mu\,, \quad g^{-1}\dot{g} = \xi\,, \quad \dot{\mu} = \mathrm{ad}^*_\xi\,\mu\,. \tag{18}$$

□

Legendre transformation. After the Legendre transformation to

$$h(\mu) = \langle\, \mu\,,\, \xi\,\rangle - \ell(\xi) \tag{19}$$

we have the differential relations

$$dh = \left\langle\, \frac{\partial h}{\partial \mu}\,,\, d\mu \,\right\rangle = \langle\, d\mu\,,\, \xi\,\rangle + \left\langle\, \mu - \frac{\partial l}{\partial \xi}\,,\, \delta\xi \,\right\rangle \tag{20}$$

so that $\partial h/\partial\mu = \xi$, which leads to the Hamiltonian formulation of the Hamilton–Pontryagin equations (21)

$$\dot{\mu} = \mathrm{ad}^*_{\partial h/\partial\mu}\,\mu\,, \quad \frac{dF}{dt} = \left\langle \mathrm{ad}^*_{\partial h/\partial\mu}\,\mu\,,\, \frac{\partial F}{\partial \mu} \right\rangle = \left\langle \mu\,,\, \mathrm{ad}_{\partial h/\partial\mu}\,\frac{\partial F}{\partial \mu} \right\rangle = -\left\langle \mu\,,\, \left[\frac{\partial F}{\partial \mu}\,,\, \frac{\partial H}{\partial \mu}\right] \right\rangle =: \Big\{\, F\,,\, H \Big\}\,. \tag{21}$$

Exercise. Recalculate the Hamilton-Pontryagin variational principle and derive its associated Lie-Poisson bracket for a constrained *right-invariant* action integral

Answer.

$$S(\xi, g, \dot{g}) = \int_a^b \Big[l(\xi) + \langle\, \mu\,,\, (\dot{g}g^{-1} - \xi)\,\rangle \Big]\, dt\,.$$

$$\frac{\delta l}{\delta \xi} = \mu\,, \quad \dot{g}g^{-1} = \xi\,, \quad \dot{\mu} = -\,\mathrm{ad}^*_\xi\,\mu\,.$$

★

▲

We will need the right-invariant Euler-Poincaré and Lie-Poisson equations when we study fluid dynamics.

5 Transformation Theory

motion	linearisation	differential, d
motion equation	infinitesimal transformation	differential k-form
vector field	pull-back	wedge product, $\wedge$
diffeomorphism	push-forward	Lie derivative, $\pounds_Q$
flow	Jacobian matrix	product rule
fixed point	directional derivative	fluid dynamics
equilibrium	commutator	other flows

5.1 Motions, pull-backs, push-forwards, commutators & differentials

- A ***motion*** is defined as a smooth curve $q(t) \in M$ parameterised by $t \in \mathbb{R}$ that solves the ***motion equation***, which is a system of differential equations

$$\dot{q}(t) = \frac{dq}{dt} = f(q) \in TM\,, \tag{22}$$

or in components

$$\dot{q}^i(t) = \frac{dq^i}{dt} = f^i(q) \quad i = 1, 2, \ldots, n\,, \tag{23}$$

- The map $f : q \in M \to f(q) \in T_q M$ is a ***vector field***.

  According to standard theorems about differential equations that are not proven in this course, the solution, or integral curve, $q(t)$ exists, provided f is sufficiently smooth, which will always be assumed to hold.

- Vector fields can also be defined as ***differential operators*** that act on functions, as

$$\frac{d}{dt}G(q) = \dot{q}^i(t)\frac{\partial G}{\partial q^i} = f^i(q)\frac{\partial G}{\partial q^i} \quad i = 1, 2, \ldots, n, \quad \text{(sum on repeated indices)} \tag{24}$$

  for any smooth function $G(q) : M \to \mathbb{R}$.

- To indicate the dependence of the solution of its initial condition $q(0) = q_0$, we write the motion as a smooth transformation

$$q(t) = \phi_t(q_0)\,.$$

  Because the vector field f is independent of time t, for any fixed value of t we may regard ϕ_t as mapping from M into itself that satisfies the composition law

$$\phi_t \circ \phi_s = \phi_{t+s}$$

  and

$$\phi_0 = \mathrm{Id}\,.$$

  Setting $s = -t$ shows that ϕ_t has a smooth inverse. A smooth mapping that has a smooth inverse is called a ***diffeomorphism***. Geometric mechanics deals with diffeomorphisms.

- The smooth mapping $\phi_t : \mathbb{R} \times M \to M$ that determines the solution $\phi_t \circ q_0 = q(t) \in M$ of the motion equation (22) with initial condition $q(0) = q_0$ is called the ***flow*** of the vector field Q.

  A point $q^\star \in M$ at which $f(q^\star) = 0$ is called a ***fixed point*** of the flow ϕ_t, or an ***equilibrium***.

  Vice versa, the vector field f is called the ***infinitesimal transformation*** of the mapping ϕ_t, since

$$\frac{d}{dt}\bigg|_{t=0}(\phi_t \circ q_0) = f(q)\,.$$

That is, $f(q)$ is the ***linearisation*** of the flow map ϕ_t at the point $q \in M$.

More generally, the ***directional derivative*** of the function h along the vector field f is given by the action of a differential operator, as

$$\frac{d}{dt}\bigg|_{t=0} h \circ \phi_t = \left[\frac{\partial h}{\partial \phi_t} \frac{d}{dt}(\phi_t \circ q_0) \right]_{t=0} = \frac{\partial h}{\partial q^i}\dot{q}^i = \frac{\partial h}{\partial q^i} f^i(q) =: Qh \,.$$

- Under a smooth change of variables $q = c(r)$ the vector field Q in the expression Qh transforms as

$$Q = f^i(q)\frac{\partial}{\partial q^i} \quad \mapsto \quad R = g^j(r)\frac{\partial}{\partial r^j} \quad \text{with} \quad g^j(r)\frac{\partial c^i}{\partial r^j} = f^i(c(r)) \quad \text{or} \quad g = c_r^{-1} f \circ c \,, \tag{25}$$

where c_r is the ***Jacobian matrix*** of the transformation. That is, since $h(q)$ is a function of q,

$$(Qh) \circ c = R(h \circ c) \,.$$

We express the transformation between the vector fields as $R = c^* Q$ and write this relation as

$$(Qh) \circ c =: c^* Q(h \circ c) \,. \tag{26}$$

The expression c^*Q is called the ***pull-back*** of the vector field Q by the map c. Two vector fields are equivalent under a map c, if one is the pull-back of the other, and fixed points are mapped into fixed points.

The inverse of the pull-back is called the ***push-forward***. It is the pull-back by the inverse map.

- The ***commutator***

$$QR - RQ =: \big[Q,\, R\big]$$

of two vector fields Q and R defines another vector field. Indeed, if

$$Q = f^i(q)\frac{\partial}{\partial q^i} \quad \text{and} \quad R = g^j(q)\frac{\partial}{\partial q^j}$$

then

$$\big[Q,\,R\big] = \left(f^i(q)\frac{\partial g^j(q)}{\partial q^i} - g^i(q)\frac{\partial f^j(q)}{\partial q^i} \right)\frac{\partial}{\partial q^j}$$

because the second-order derivative terms cancel. By the pull-back relation (26) we have

$$c^*\big[Q,\,R\big] = \big[c^*Q,\,c^*R\big] \tag{27}$$

under a change of variables defined by a smooth map, c. This means the definition of the vector field commutator is independent of the choice of coordinates. As we shall see, the ***tangent*** to the relation $c_t^*\big[Q,\,R\big] = \big[c_t^*Q,\,c_t^*R\big]$ at the identity $t = 0$ is the ***Jacobi condition*** for the vector fields to form an algebra, by taking

$$\frac{d}{dt}\bigg|_{t=0} c_t^*\big[Q,\,R\big] = \frac{d}{dt}\bigg|_{t=0}\big[c_t^*Q,\,c_t^*R\big],$$

and using the product rule.

- The ***differential*** of a smooth function $f : M \to M$ is defined as

$$df = \frac{\partial f}{\partial q^i}dq^i\,.$$

- Under a smooth change of variables $s = \phi \circ q = \phi(q)$ the differential of the composition of functions $d(f \circ \phi)$ transforms according to the chain rule as

$$df = \frac{\partial f}{\partial q^i}dq^i\,, \quad d(f\circ\phi) = \frac{\partial f}{\partial \phi^j(q)}\frac{\partial \phi^j}{\partial q^i}dq^i = \frac{\partial f}{\partial s^j}ds^j \quad \Longrightarrow \quad d(f\circ\phi) = (df)\circ\phi \tag{28}$$

That is, the differential d commutes with the pull-back ϕ^* of a smooth transformation ϕ,

$$d(\phi^* f) = \phi^* df\,. \tag{29}$$

In a moment, this pull-back formula will give us the rule for transforming differential forms of any order.

5.2 Wedge products

- Differential k-forms on an n-dimensional manifold are defined in terms of the differential d and the antisymmetric ***wedge product*** ($\wedge$) satisfying

$$dq^i \wedge dq^j = -\, dq^j \wedge dq^i \,, \quad \text{for} \quad i, j = 1, 2, \dots, n \tag{30}$$

By using wedge product, any k-form $\alpha \in \Lambda^k$ on M may be written locally at a point $q \in M$ in the differential basis dq^j as

$$\alpha_m = \alpha_{i_1 \dots i_k}(m) dq^{i_1} \wedge \cdots \wedge dq^{i_k} \in \Lambda^k \,, \quad i_1 < i_2 < \cdots < i_k \,, \tag{31}$$

where the sum over repeated indices is ordered, so that it must be taken over all i_j satisfying $i_1 < i_2 < \cdots < i_k$. Roughly speaking differential forms Λ^k are objects that can be integrated. As we shall see, vector fields also act on differential forms in interesting ways.

- Pull-backs of other differential forms may be built up from their basis elements, the dq^{i_k}. By equation (29),

Theorem 11 (Pull-back of a wedge product)**.** *The pull-back of a wedge product of two differential forms is the wedge product of their pull-backs:*

$$\phi_t^*(\alpha \wedge \beta) = \phi_t^* \alpha \wedge \phi_t^* \beta \,. \tag{32}$$

5.3 Lie derivatives

Definition 12 (Lie derivative of a differential k-form)**.** *The **Lie derivative** of a differential k-form Λ^k by a vector field $Q \in \mathfrak{X}$ is defined by linearising its flow ϕ_t around the identity $t = 0$,*

$$\pounds_Q \Lambda^k = \frac{d}{dt}\bigg|_{t=0} \phi_t^* \Lambda^k \quad \textit{maps} \quad \pounds_Q \Lambda^k \mapsto \Lambda^k \,.$$

Hence, by equation (32), the Lie derivative satisfies the product rule for the wedge product.

Corollary 13 (Product rule for the Lie derivative of a wedge product)**.**

$$\pounds_Q(\alpha \wedge \beta) = \pounds_Q\alpha \wedge \beta + \alpha \wedge \pounds_Q\beta\,. \tag{33}$$

- Pullbacks of vector fields lead to Lie derivative expressions, too.

Definition 14 (Lie derivative of a vector field)**.** *The* ***Lie derivative*** *of a vector field* $Y \in \mathfrak{X}$ *by another vector field* $X \in \mathfrak{X}$ *is defined by linearising the flow* ϕ_t *of* X *around the identity* $t = 0$*,*

$$\pounds_X Y = \frac{d}{dt}\Big|_{t=0} \phi_t^* Y \quad maps \quad \pounds_X \in \mathfrak{X} \mapsto \mathfrak{X}\,.$$

Theorem 15. *The Lie derivative* $\pounds_X Y$ *of a vector field* Y *by a vector field* X *satisfies*

$$\pounds_X Y = \frac{d}{dt}\Big|_{t=0} \phi_t^* Y = [X,\, Y]\,, \tag{34}$$

where $[X,\, Y] = XY - YX$ *is the commutator of the vector fields* X *and* Y*.*

Proof. Denote the vector fields in components as

$$X = X^i(q)\frac{\partial}{\partial q^i} = \frac{d}{dt}\Big|_{t=0} \phi_t^* \quad \text{and} \quad Y = Y^j(q)\frac{\partial}{\partial q^j}\,.$$

Then, by the pull-back relation (26) a direct computation yields, on using the matrix identity $dM^{-1} = -\,M^{-1}dM\,M^{-1}$,

$$
\begin{aligned}
£_X Y &= \frac{d}{dt}\bigg|_{t=0} \phi_t^* Y = \frac{d}{dt}\bigg|_{t=0} \left(Y^j(\phi_t q) \frac{\partial}{\partial(\phi_t q)^j} \right) \\
&= \frac{d}{dt}\bigg|_{t=0} \left(Y^j(\phi_t q) \left[\frac{\partial(\phi_t q)}{\partial q}^{-1} \right]_j^k \frac{\partial}{\partial q^k} \right) \\
&= \left(X^j \frac{\partial Y^k}{\partial q^j} - Y^j \frac{\partial X^k}{\partial q^j} \right) \frac{\partial}{\partial q^k} \\
&= [X,\, Y]\,.
\end{aligned}
$$

□

Corollary 16. *The Lie derivative of the relation (27) for the pull-back of the commutator* $c_t^*\big[Y,\, Z\big] = \big[c_t^* Y,\, c_t^* Z\big]$ *yields the* ***Jacobi condition****, required for the vector fields to form an algebra.*

Proof. By the product rule and the definition of the Lie bracket (34) we have

$$
\frac{d}{dt}\bigg|_{t=0} \phi_t^* \big[Y,\, Z\big] = \big[X,\, \big[Y,\, Z\big]\big] = \big[[X,Y],\, Z\big] + \big[Y,\, [X,Z]\big] = \frac{d}{dt}\bigg|_{t=0} \big[\phi_t^* Y,\, \phi_t^* Z\big]
$$

This is the ***Jacobi identity*** for vector fields. □

Use the hat map and the relation $R_t(\mathbf{x} \times \mathbf{y}) = R_t\mathbf{x} \times R_t\mathbf{y}$ to show that the same argument gives the Jacobi identity for the cross product of vectors in $\mathbb{R}^3$, when ϕ_t^* is a rotation.

5.4 Contraction

Definition 17 (Contraction)**.** *In exterior calculus, the operation of* ***contraction*** *denoted as* $\lrcorner$ *introduces a pairing between vector fields and differential forms. Contraction is also called* ***substitution*** *of a vector field into a differential form. For basis elements in phase space, contraction defines* ***duality relations****,*

$$\partial_q \lrcorner\, dq = 1 = \partial_p \lrcorner\, dp\,, \quad \textit{and} \quad \partial_q \lrcorner\, dp = 0 = \partial_p \lrcorner\, dq\,, \tag{35}$$

so that differential forms are linear functions of vector fields. A ***Hamiltonian vector field****,*

$$X_H = \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p} = H_p\partial_q - H_q\partial_p = \{\,\cdot\,,\,H\,\}\,, \tag{36}$$

satisfies the intriguing linear functional relations with the basis elements in phase space,

$$X_H \lrcorner\, dq = H_p \quad \textit{and} \quad X_H \lrcorner\, dp = -\,H_q\,. \tag{37}$$

Definition 18 (Contraction rules with higher forms)**.** *The rule for contraction or substitution of a vector field into a differential form is to sum the substitutions of X_H over the permutations of the factors in the differential form that bring the corresponding dual basis element into its leftmost position. For example, substitution of the Hamiltonian vector field X_H into the symplectic form $\omega = dq \wedge dp$ yields*

$$X_H \lrcorner\, \omega = X_H \lrcorner\, (dq \wedge dp) = (X_H \lrcorner\, dq)\, dp - (X_H \lrcorner\, dp)\, dq\,.$$

In this example, $X_H \lrcorner\, dq = H_p$ and $X_H \lrcorner\, dp = -\,H_q$, so

$$X_H \lrcorner\, \omega = H_p dp + H_q dq = dH\,,$$

which follows from the duality relations (35).

This calculation has proved the following.

Theorem 19 (Hamiltonian vector field)**.** *The Hamiltonian vector field* $X_H = \{\,\cdot\,,\,H\,\}$ *satisfies*

$$X_H \,\lrcorner\, \omega \;=\; dH \quad \textit{with} \quad \omega = dq \wedge dp\,. \tag{38}$$

Remark 20.
The purely geometric nature of relation (38) argues for it to be taken as the definition of a Hamiltonian vector field.

Lemma 21. $d^2 = 0$ *for smooth phase-space functions.*

Proof. For any smooth phase-space function $H(q,p)$, one computes

$$dH = H_q dq + H_p dp$$

and taking the second exterior derivative yields

$$\begin{aligned} d^2 H &= H_{qp}\, dp \wedge dq + H_{pq}\, dq \wedge dp \\ &= (H_{pq} - H_{qp})\, dq \wedge dp = 0\,. \end{aligned}$$

□

Relation (38) also implies the following.

Corollary 22 (Poincaré's theorem)**.** *The flow of* X_H *preserves the exact two-form* ω *for any Hamiltonian* H*.*

Proof. Preservation of ω may be verified first by a formal calculation using (38). Along

$$X_H = (dq/dt, dp/dt) = (\dot q, \dot p) = (H_p, -H_q)\,,$$

for a solution of Hamilton's equations, we have

$$\begin{aligned}
\pounds_{X_H}\omega &= \pounds_{X_H}(dq\wedge dp)\\
&= \frac{d}{dt}\Big|_{t=0} g_t^*(dq\wedge dp)\\
&= \frac{d}{dt}\Big|_{t=0} (g_t^*dq\wedge g_t^*dp)\\
&= d\dot{q}\,\wedge\, dp + dq\,\wedge\, d\dot{p}\\
&= dH_p\,\wedge\, dp - dq\,\wedge\, dH_q\\
&= d(H_p\,dp + H_q\,dq)\\
&= d(X_H \,\lrcorner\, \omega)\\
&= d(dH) = 0\,.
\end{aligned}$$

The first two steps use the product rule for Lie derivatives of differential forms

$$\begin{aligned}
\pounds_{X_H}(dq\wedge dp) &= \frac{d}{dt}\Big|_{t=0} g_t^*(dq\wedge dp) = \frac{d}{dt}\Big|_{t=0}(g_t^*dq\wedge g_t^*dp)\\
&= \Big[\frac{d}{dt}g_t^*dq\wedge g_t^*dp + g_t^*dq\wedge \frac{d}{dt}g_t^*dp\Big]_{t=0}\\
&= \pounds_{X_H}dq\wedge dp + dq\wedge \pounds_{X_H}dp
\end{aligned} \tag{39}$$

and the third-to-the-last and last steps use the property of the exterior derivative d that $d^2 = 0$ for continuous forms. The latter is due to the equality of cross derivatives $H_{pq} = H_{qp}$ and antisymmetry of the wedge product $dq\,\wedge\, dp = -\,dp\,\wedge\, dq$. □

Definition 23 (Symplectic flow)**.** *A flow is **symplectic** if it preserves the phase-space area or symplectic two-form,* $\omega = dq\,\wedge\, dp$.

According to this definition, Corollary 22 may be simply re-stated as

Corollary 24 (Poincaré's theorem). *The flow of a Hamiltonian vector field is symplectic.*

Definition 25 (Canonical transformations). *A smooth invertible map g of the phase space T^*M is called a* ***canonical transformation*** *if it preserves the canonical symplectic form ω on T^*M, i.e., $g^*\omega = \omega$, where $g^*\omega$ denotes the pull-back of ω under the map g.*

Remark 26 (Criterion for a canonical transformation).
Suppose in original coordinates (p, q) the symplectic form is expressed as $\omega = dq \wedge dp$. A transformation $g : T^*M \mapsto T^*M$ written as $(Q, P) = (Q(p, q), P(p, q)$ is canonical if the direct computation shows that $dQ \wedge dP = g^*(dq \wedge dp) = c\, dq \wedge dp$, up to a constant factor c. (Such a constant factor c is unimportant, since it may be absorbed into the units of time in Hamilton's canonical equations.)

Remark 27.
By Corollary 24 (Poincaré's Theorem), the Hamiltonian phase flow g_t is a one-parameter group of canonical transformations.

Theorem 28 (Preservation of Hamiltonian form). *Canonical transformations preserve the Hamiltonian form.*

Proof. The coordinate-free relation $X_H \,\lrcorner\, \omega = dH$ with $\omega = dq \wedge dp$ keeps its form if

$$dQ \wedge dP = g^*(dq \wedge dp) = c\, dq \wedge dp\,,$$

up to the constant factor c. Hence, Hamilton's equations re-emerge in canonical form in the new coordinates, up to a rescaling by c which may be absorbed into the units of time. □

5.5 Summary of differential-form operations

Besides the wedge product, three basic operations are commonly applied to differential forms. These are contraction, exterior derivative and Lie derivative.

- Contraction $\lrcorner$ with a vector field X lowers the degree:

$$X \lrcorner \Lambda^k \mapsto \Lambda^{k-1}\,.$$

- Exterior derivative d raises the degree:

$$d\Lambda^k \mapsto \Lambda^{k+1}\,.$$

- Lie derivative $\pounds_X$ by vector field X preserves the degree:

$$\pounds_X\Lambda^k \mapsto \Lambda^k\,, \quad \text{where} \quad \pounds_X\Lambda^k = \frac{d}{dt}\bigg|_{t=0}\phi_t^*\Lambda^k\,,$$

  in which ϕ_t is the flow of the vector field X. In analogy with fluids one may write $\pounds_X\Lambda^k = \frac{d}{dt}\Lambda^k$ along $\frac{dx}{dt} = X$.

- Lie derivative $\pounds_X$ satisfies ***Cartan's formula***: (The proof is a direct calculation.)

$$\pounds_X\alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha) \quad \text{for} \quad \alpha = \alpha_{i_1\ldots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k} \in \Lambda^k\,.$$

Remark 29.
Note also that the Lie derivative commutes with the exterior derivative. That is,

$$d(\pounds_X\alpha) = \pounds_X d\alpha\,, \quad \text{for} \quad \alpha \in \Lambda^k(M) \quad \text{and} \quad X \in \mathfrak{X}(M)\,.$$

5.6 Examples of contraction, or interior product

Definition 30 (Contraction, or interior product)**.** *Let $\alpha \in \Lambda^k$ be a k-form on a manifold M,*

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k \,, \quad \textit{with} \quad i_1 < i_2 < \cdots < i_k \,,$$

and let $X = X^j \partial_j$ be a vector field. The contraction or interior product $X \,\lrcorner\, \alpha$ of a vector field X with a k-form α is defined by

$$X \,\lrcorner\, \alpha = X^j \alpha_{j i_2 \dots i_k} dx^{i_2} \wedge \cdots \wedge dx^{i_k} \,. \tag{40}$$

Note that

$$\begin{aligned} X \,\lrcorner\, (Y \,\lrcorner\, \alpha) &= X^l Y^m \alpha_{m l i_3 \dots i_k} dx^{i_3} \wedge \cdots \wedge dx^{i_k} \\ &= -\,Y \,\lrcorner\, (X \,\lrcorner\, \alpha) \,, \end{aligned}$$

by antisymmetry of $\alpha_{m l i_3 \dots i_k}$, particularly in its first two indices.

Remark 31 (Examples of contraction)**.**

1. A mnemonic device for keeping track of signs in contraction or substitution of a vector field into a differential form is to sum the substitutions of $X = X^j \partial_j$ over the permutations that bring the corresponding dual basis element into the leftmost position in the k-form α. For example, in two dimensions, contraction of the vector field $X = X^j \partial_j = X^1 \partial_1 + X^2 \partial_2$ into the two-form $\alpha = \alpha_{jk} dx^j \wedge dx^k$ with $\alpha_{21} = -\,\alpha_{12}$ yields

$$X \,\lrcorner\, \alpha = X^j \alpha_{j i_2} dx^{i_2} = X^1 \alpha_{12} dx^2 + X^2 \alpha_{21} dx^1 \,.$$

   Likewise, in three dimensions, contraction of the vector field $X = X^1 \partial_1 + X^2 \partial_2 + X^3 \partial_3$ into the three-form

$\alpha = \alpha_{123} dx^1 \wedge dx^2 \wedge dx^3$ with $\alpha_{213} = -\alpha_{123}$, etc. yields

$$\begin{aligned} X \lrcorner \alpha &= X^1 \alpha_{123} dx^2 \wedge dx^3 + \text{cyclic permutations} \\ &= X^j \alpha_{j i_2 i_3} dx^{i_2} \wedge dx^{i_3} \quad \text{with} \quad i_2 < i_3 \,. \end{aligned}$$

2. The rule for contraction of a vector field with a differential form develops from the relation

$$\partial_j \lrcorner dx^k = \delta^k_j \,,$$

in the coordinate basis $e_j = \partial_j := \partial / \partial x^j$ and its dual basis $e^k = dx^k$. Contraction of a vector field with a one-form yields the dot product, or inner product, between a covariant vector and a contravariant vector is given by

$$X^j \partial_j \lrcorner v_k dx^k = v_k \delta^k_j X^j = v_j X^j \,,$$

or, in vector notation,

$$X \lrcorner \mathbf{v} \cdot d\mathbf{x} = \mathbf{v} \cdot \mathbf{X} \,.$$

This is the ***dot product of vectors*** $\mathbf{v}$ and $\mathbf{X}$.

3. By the linearity of its definition (40), contraction of a vector field X with a differential k-form α satisfies

$$(hX) \lrcorner \alpha = h(X \lrcorner \alpha) = X \lrcorner h\alpha \,.$$

Our previous calculations for two-forms and three-forms provide the following additional expressions for contraction of a vector field with a differential form, which may be written in vector notation as:

$$
\begin{aligned}
X \lrcorner\, \mathbf{B}\cdot d\mathbf{S} &= -\,\mathbf{X}\times\mathbf{B}\cdot d\mathbf{x}\,,\\
X \lrcorner\, d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S}\,,\\
d(X \lrcorner\, d^{\,3}x) &= d(\mathbf{X}\cdot d\mathbf{S}) = (\operatorname{div}\mathbf{X})\, d^{\,3}x\,.
\end{aligned}
$$

Remark 32 (Physical examples of contraction)**.**
The first of these contraction relations represents the Lorentz, or Coriolis force, when $\mathbf{X}$ is particle velocity and $\mathbf{B}$ is either magnetic field, or rotation rate, respectively. The second contraction relation is the flux of the vector $\mathbf{X}$ through a surface element. The third is the exterior derivative of the second, thereby yielding the divergence of the vector $\mathbf{X}$.

Exercise. Show that

$$X \lrcorner\, (X \lrcorner\, \mathbf{B}\cdot d\mathbf{S}) = 0$$

and

$$(X \lrcorner\, \mathbf{B}\cdot d\mathbf{S}) \wedge \mathbf{B}\cdot d\mathbf{S} = 0\,,$$

for any vector field X and two-form $\mathbf{B}\cdot d\mathbf{S}$. ★

Proposition 33 (Contracting through wedge product)**.** *Let α be a k-form and β be a one-form on a manifold M and let $X = X^j\partial_j$ be a vector field. Then the contraction of X through the wedge product $\alpha\wedge\beta$ satisfies*

$$X \lrcorner\, (\alpha\wedge\beta) = (X \lrcorner\, \alpha)\wedge\beta + (-1)^k\alpha\wedge(X \lrcorner\, \beta)\,. \tag{41}$$

Proof. The proof is a straightforward calculation using the definition of contraction. The exponent k in the factor $(-1)^k$ counts the number of exchanges needed to get the one-form β to the left most position through the k-form α. □

Proposition 34. *[Contraction is natural under pull-back]*
That is,

$$\phi^*(X(m) \,\lrcorner\, \alpha) = X(\phi(m)) \,\lrcorner\, \phi^*\alpha = \phi^* X \,\lrcorner\, \phi^*\alpha \,. \tag{42}$$

Proof. Direct verification using the relation between pull-back of forms and push-forward of vector fields. Note the implication, $\pounds_X(Y \,\lrcorner\, \alpha) = [X\,,\,Y] \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X \alpha)$. □

Definition 35 (Alternative notations for contraction)**.** *Besides the hook notation with* $\lrcorner$, *one also finds in the literature the following two alternative notations for contraction of a vector field* X *with* k*-form* $\alpha \in \Lambda^k$ *on a manifold* M*:*

$$X \,\lrcorner\, \alpha = i_X \alpha = \alpha(X, \underbrace{\,\cdot\,,\cdot\,,\ldots,\,\cdot\,}_{k-1\ slots}) \in \Lambda^{k-1} \,. \tag{43}$$

In the last alternative, one leaves a dot $(\,\cdot\,)$ *in each remaining slot of the form that results after contraction. For example, contraction of the Hamiltonian vector field* $X_H = \{\,\cdot\,,\,H\}$ *with the symplectic two-form* $\omega \in \Lambda^2$ *produces the one-form*

$$X_H \,\lrcorner\, \omega = \omega(X_H,\,\cdot\,) = -\,\omega(\,\cdot\,,\,X_H) = dH \,.$$

In this alternative notation, the proof of formula (42) in Proposition 34 may be written, as follows.

Proof. Since forms are multilinear maps to the real numbers, one may define the pull back of a k-form, α, by

$$\phi^*\alpha(X_1, X_2, \ldots) := \alpha(\phi_* X, \phi_* X_2, \ldots) \,.$$

Therefore, we are able to use the following proof.

$$
\begin{aligned}
\phi^* X \,\lrcorner\, \phi^*\alpha(X_1, X_2, ...) &= \phi^*\alpha(\phi^* X, X_1, X_2, ...) \\
&= \alpha(\phi_*\phi^* X, \phi_* X_1, \phi_* X_2, ...) \\
&= \alpha(X, \phi_* X_1, \phi_* X_2, ...) \\
&= (X \,\lrcorner\, \alpha)(\phi_* X_1, \phi_* X_2, ...) \\
&= \phi^*(X \,\lrcorner\, \alpha)(X_1, X_2, ...)
\end{aligned}
$$

Now, if we allow $X_1, X_2, \dots$ to be arbitrary, then formula (42) in Proposition 34 follows. □

Proposition 36 (Hamiltonian vector field definitions)**.** *The two definitions of Hamiltonian vector field* X_H

$$
dH = X_H \,\lrcorner\, \omega \quad \textit{and} \quad X_H = \{\,\cdot\,,\, H\}
$$

are equivalent.

Proof. The symplectic Poisson bracket satisfies $\{F, H\} = \omega(X_F, X_H)$, because

$$
\omega(X_F, X_H) := X_H \,\lrcorner\, X_F \,\lrcorner\, \omega = X_H \,\lrcorner\, dF = -X_F \,\lrcorner\, dH = \{F,\, H\}\,.
$$

□

Remark 37.
The relation $\{F,\, H\} = \omega(X_F, X_H)$ means that the Hamiltonian vector field defined via the symplectic form coincides exactly with the Hamiltonian vector field defined using the Poisson bracket.

5.7 Exercises in exterior calculus operations

Vector notation for differential basis elements One denotes differential basis elements dx^i and $dS_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$, for $i, j, k = 1, 2, 3$ in vector notation as

$$
\begin{aligned}
d\mathbf{x} &:= (dx^1, dx^2, dx^3)\,, \\
d\mathbf{S} &= (dS_1, dS_2, dS_3) \\
&:= (dx^2 \wedge dx^3,\; dx^3 \wedge dx^1,\; dx^1 \wedge dx^2)\,, \\
dS_i &:= \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k\,, \\
d^{\,3}x &= dVol := dx^1 \wedge dx^2 \wedge dx^3 \\
&= \frac{1}{6}\epsilon_{ijk}dx^i \wedge dx^j \wedge dx^k\,.
\end{aligned}
$$

Exercise. (Vector calculus operations) Show that contraction $\,\lrcorner\, : \mathfrak{X} \times \Lambda^k \to \Lambda^{k-1}$ of the vector field $X = X^j\partial_j =: \mathbf{X}\cdot\nabla$ with the differential basis elements $d\mathbf{x}$, $d\mathbf{S}$ and $d^{\,3}x$ recovers the following familiar operations among vectors:

$$
\begin{aligned}
X \,\lrcorner\, d\mathbf{x} &= \mathbf{X}\,, \\
X \,\lrcorner\, d\mathbf{S} &= \mathbf{X} \times d\mathbf{x}\,, \\
(\text{or,}\quad X \,\lrcorner\, dS_i &= \epsilon_{ijk}X^j dx^k) \\
Y \,\lrcorner\, X \,\lrcorner\, d\mathbf{S} &= \mathbf{X} \times \mathbf{Y}\,, \\
X \,\lrcorner\, d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S} = X^k dS_k\,, \\
Y \,\lrcorner\, X \,\lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y} \cdot d\mathbf{x} = \epsilon_{ijk}X^iY^j dx^k\,, \\
Z \,\lrcorner\, Y \,\lrcorner\, X \,\lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y} \cdot \mathbf{Z}\,.
\end{aligned}
$$

★

Exercise. (Exterior derivatives in vector notation) Show that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation:

$$
\begin{aligned}
df &= f_{,j}\,dx^j =: \nabla f \cdot d\mathbf{x}\,, \\
0 = d^2 f &= f_{,jk}\,dx^k \wedge dx^j\,, \\
df \wedge dg &= f_{,j}\,dx^j \wedge g_{,k}\,dx^k \\
&=: (\nabla f \times \nabla g)\cdot d\mathbf{S}\,, \\
df \wedge dg \wedge dh &= f_{,j}\,dx^j \wedge g_{,k}\,dx^k \wedge h_{,l}\,dx^l \\
&=: (\nabla f \cdot \nabla g \times \nabla h)\,d^{\,3}x\,.
\end{aligned}
$$

★

Exercise. (Vector calculus formulas) Show that the exterior derivative yields the following vector calculus formulas:

$$
\begin{aligned}
df &= \nabla f \cdot d\mathbf{x}\,, \\
d(\mathbf{v}\cdot d\mathbf{x}) &= (\operatorname{curl}\mathbf{v})\cdot d\mathbf{S}\,, \\
d(\mathbf{A}\cdot d\mathbf{S}) &= (\operatorname{div}\mathbf{A})\,d^{\,3}x\,.
\end{aligned}
$$

The compatibility condition $d^2 = 0$ is written for these forms as

$$
\begin{aligned}
0 = d^2 f = d(\nabla f\cdot d\mathbf{x}) &= (\operatorname{curl}\operatorname{grad} f)\cdot d\mathbf{S}\,, \\
0 = d^2(\mathbf{v}\cdot d\mathbf{x}) = d\big((\operatorname{curl}\mathbf{v})\cdot d\mathbf{S}\big) &= (\operatorname{div}\operatorname{curl}\mathbf{v})\,d^{\,3}x\,.
\end{aligned}
$$

The product rule is written for these forms as

$$
\begin{aligned}
d\big(f(\mathbf{A}\cdot d\mathbf{x})\big) &= df \wedge \mathbf{A}\cdot d\mathbf{x} + f\operatorname{curl}\mathbf{A}\cdot d\mathbf{S} \\
&= \big(\nabla f \times \mathbf{A} + f\operatorname{curl}\mathbf{A}\big)\cdot d\mathbf{S} \\
&= \operatorname{curl}(f\mathbf{A})\cdot d\mathbf{S}\,,
\end{aligned}
$$

$$
\begin{aligned}
d\big((\mathbf{A}\cdot d\mathbf{x})\wedge(\mathbf{B}\cdot d\mathbf{x})\big) &= (\mathrm{curl}\,\mathbf{A})\cdot d\mathbf{S}\wedge\mathbf{B}\cdot d\mathbf{x} - \mathbf{A}\cdot d\mathbf{x}\wedge(\mathrm{curl}\,\mathbf{B})\cdot d\mathbf{S}\\
&= \big(\mathbf{B}\cdot\mathrm{curl}\,\mathbf{A} - \mathbf{A}\cdot\mathrm{curl}\,\mathbf{B}\big)d^{\,3}x\\
&= d\big((\mathbf{A}\times\mathbf{B})\cdot d\mathbf{S}\big)\\
&= \mathrm{div}(\mathbf{A}\times\mathbf{B})\,d^{\,3}x\,.
\end{aligned}
$$

These calculations yield familiar formulas from vector calculus for quantities curl(grad), div(curl), curl($f\mathbf{A}$) and div($\mathbf{A}\times\mathbf{B}$). ★

5.8 Integral calculus formulas

Exercise. (Integral calculus formulas) Show that the Stokes' theorem for the vector calculus formulas yields the following familiar results in $\mathbb{R}^3$:

- The ***fundamental theorem of calculus***, upon integrating df along a curve in $\mathbb{R}^3$ starting at point a and ending at point b:

$$\int_a^b df \;=\; \int_a^b \nabla f\cdot d\mathbf{x} = f(b) - f(a)\,.$$

- The ***classical Stokes theorem***, for a compact surface S with boundary ∂S:

$$\int_S(\mathrm{curl}\,\mathbf{v})\cdot d\mathbf{S} \;=\; \oint_{\partial S}\mathbf{v}\cdot d\mathbf{x}\,.$$

(For a planar surface $S\in\mathbb{R}^2$, this is ***Green's theorem***.)

- The ***Gauss divergence theorem***, for a compact spatial domain D with boundary ∂D:

$$\int_D(\mathrm{div}\,\mathbf{A})\,d^{\,3}x \;=\; \oint_{\partial D}\mathbf{A}\cdot d\mathbf{S}\,.$$

★

These exercises illustrate the following,

Theorem 38 (Stokes' theorem)**.** *Suppose M is a compact oriented k-dimensional manifold with boundary ∂M and α is a smooth $(k-1)$-form on M. Then*

$$\int_M d\alpha = \oint_{\partial M} \alpha \,.$$

5.9 Summary and an exercise

Summary

The pull-back ϕ_t^* of a smooth flow ϕ_t generated by a smooth vector field X on a smooth manifold M commutes with the exterior derivative d, wedge product $\wedge$ and contraction $\,\lrcorner\,$.

That is, for k-forms α, $\beta \in \Lambda^k(M)$, and $m \in M$, the pull-back ϕ_t^* satisfies

$$\begin{aligned} d(\phi_t^*\alpha) &= \phi_t^* d\alpha \,, \\ \phi_t^*(\alpha \wedge \beta) &= \phi_t^*\alpha \wedge \phi_t^*\beta \,, \\ \phi_t^*(X \lrcorner\, \alpha) &= \phi_t^* X \lrcorner\, \phi_t^*\alpha \,. \end{aligned}$$

In addition, the Lie derivative $\pounds_X \alpha$ of a k-form $\alpha \in \Lambda^k(M)$ by the vector field X tangent to the flow ϕ_t on M is defined either dynamically or geometrically (by Cartan's formula) as

$$\pounds_X \alpha \;=\; \frac{d}{dt}\bigg|_{t=0} (\phi_t^*\alpha) = X \lrcorner\, d\alpha + d(X \lrcorner\, \alpha), \tag{44}$$

in which the last is Cartan's geometric formula in (44) for the Lie derivative.

Exercise.

(a) Verify the formula $[X\,,\,Y]\,\lrcorner\,\alpha = \pounds_X(Y\,\lrcorner\,\alpha) - Y\,\lrcorner\,(\pounds_X\alpha)$.

(b) Use (b) to verify $\pounds_{[X\,,\,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$.

(c) Use (c) to verify the Jacobi identity.

(d) Use (c) to verify that the divergence-free vector fields are closed under commutation.

(e) For a top-form α show that

$$[X\,,\,Y]\,\lrcorner\,\alpha = d\big(X\,\lrcorner\,(Y\,\lrcorner\,\alpha)\big)\,. \tag{45}$$

(f) Write the equivalent of equation (45) as a formula in vector calculus.

★

Answer.

(a) The required formula follows immediately from the product rule in (39) for the dynamical definition of the Lie derivative. Since pull-back commutes with contraction, insertion of a vector field into a k-form transforms under the flow ϕ_t of a smooth vector field Y as

$$\phi_t^*(Y\,\lrcorner\,\alpha) = \phi_t^*Y\,\lrcorner\,\phi_t^*\alpha\,.$$

A direct computation using the dynamical definition of the Lie derivative $\pounds_Y\alpha = \frac{d}{dt}|_{t=0}(\phi_t^*\alpha)$, then yields

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\phi_t^*\big(Y\,\lrcorner\,\alpha\big) &= \Big(\frac{d}{dt}\Big|_{t=0}\phi_t^*Y\Big)\,\lrcorner\,\alpha \\ &\quad + Y\,\lrcorner\,\Big(\frac{d}{dt}\Big|_{t=0}\phi_t^*\alpha\Big)\,.\end{aligned}$$

Hence, we recognise that the desired formula is the ***product rule*** met earlier in equation (39):

$$\pounds_X(Y\,\lrcorner\,\alpha) = (\pounds_XY)\,\lrcorner\,\alpha + Y\,\lrcorner\,(\pounds_X\alpha)\,.$$

(b) Insert $\pounds_X Y = [X\,,\,Y]$ into the product rule formula in part (b). Then

$$[X\,,\,Y] \,\lrcorner\, \alpha = \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X \alpha).$$

Now use Cartan's formula in (44)

$$\pounds_X \alpha \;=\; \frac{d}{dt}\Big|_{t=0}(\phi_t^* \alpha) = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha),$$

to compute the required result, as

$$\begin{aligned}
\pounds_{[X\,,\,Y]}\alpha &= d([X\,,\,Y] \,\lrcorner\, \alpha) + [X\,,\,Y] \,\lrcorner\, d\alpha \\
&= d\big(\,\pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X \alpha)\big) \\
&\quad + \pounds_X(Y \,\lrcorner\, d\alpha) - Y \,\lrcorner\, (\pounds_X d\alpha) \\
&= \pounds_X d(Y \,\lrcorner\, \alpha) - d(Y \,\lrcorner\, (\pounds_X \alpha) \\
&\quad + \pounds_X(Y \,\lrcorner\, d\alpha) - Y \,\lrcorner\, d(\pounds_X \alpha) \\
&= \pounds_X(\pounds_Y \alpha) - \pounds_Y(\pounds_X \alpha)\,.
\end{aligned}$$

Can you think of an alternative proof based on the dynamical definition of the Lie derivative?

(c) Applying part (b), ($\pounds_{[X\,,\,Y]}\alpha = \pounds_X \pounds_Y \alpha - \pounds_Y \pounds_X \alpha$) to $\alpha = d^3x$ proves that $\pounds_{[X\,,\,Y]} d^3x = 0$; since both $\pounds_Y d^3x = 0 = \pounds_X d^3x$, because, e.g., $\pounds_Y d^3x = (\mathrm{div} Y)\, d^3x$.

(d) As a consequence of part (b),

$$\begin{aligned}
\pounds_{[\,Z\,,[X\,,\,Y]]}\alpha &= \pounds_Z(\pounds_X \pounds_Y - \pounds_Y \pounds_X)\alpha - (\pounds_X \pounds_Y - \pounds_Y \pounds_X)\pounds_Z \alpha \\
&= \pounds_Z \pounds_X \pounds_Y \alpha - \pounds_Z \pounds_Y \pounds_X \alpha - \pounds_X \pounds_Y \pounds_Z \alpha + \pounds_Y \pounds_X \pounds_Z \alpha\,,
\end{aligned}$$

and summing over cyclic permutations verifies that

$$\pounds_{[\,Z\,,[X\,,Y]]}\,\alpha + \pounds_{[\,X\,,[Y\,,Z]]}\,\alpha + \pounds_{[\,Y\,,[Z\,,X]]}\,\alpha = 0\,.$$

This is the ***Jacobi identity for the Lie derivative***.

(e) Substituting the relation $\pounds_X Y = [X,\, Y]$ into the product rule above in part (b) and rearranging yields

$$[X,\, Y] \lrcorner\, \alpha = \pounds_X (Y \lrcorner\, \alpha) - Y \lrcorner\, (\pounds_X \alpha)\,, \tag{46}$$

as required, for an arbitrary k-form α.

From formula (46), we have

$$\begin{aligned}
[X\,,\, Y] \lrcorner\, \alpha &= \pounds_X (Y \lrcorner\, \alpha) - Y \lrcorner\, (\pounds_X \alpha) \\
&= d\big(X \lrcorner\, (Y \lrcorner\, \alpha) + X \lrcorner\, d(Y \lrcorner\, \alpha)\big) - Y \lrcorner\, (\pounds_X \alpha) \\
&= d\big(X \lrcorner\, (Y \lrcorner\, \alpha)\big) + X \lrcorner\, (\pounds_Y \alpha - Y \lrcorner\, d\alpha) - Y \lrcorner\, (\pounds_X \alpha) \\
&= d\big(X \lrcorner\, (Y \lrcorner\, \alpha)\big) + X \lrcorner\, (\pounds_Y \alpha) - Y \lrcorner\, (\pounds_X \alpha) \\
[X\,,\, Y] \lrcorner\, \alpha &= d\big(X \lrcorner\, (Y \lrcorner\, \alpha)\big) + (\operatorname{div} \mathbf{Y}) X \lrcorner\, \alpha - (\operatorname{div} \mathbf{X}) Y \lrcorner\, \alpha.
\end{aligned} \tag{47}$$

The last two steps to obtain (47) follow, because $d\alpha = 0$ and $\pounds_X \alpha = (\operatorname{div} \mathbf{X})\, \alpha$ for a top-form α.

For divergence-free vectors $\mathbf{X}$ and $\mathbf{Y}$, the last result takes the elegant form,

$$[X\,,\, Y] \lrcorner\, \alpha = d\big(X \lrcorner\, (Y \lrcorner\, \alpha)\big), \tag{48}$$

when $\operatorname{div} \mathbf{X} = 0 = \operatorname{div} \mathbf{Y}$.

(f) The vector calculus formula to which equation (47) is equivalent may be found by writing its left and right sides in a coordinate basis, as

$$\begin{aligned}
[X\,,\, Y] \lrcorner\, \alpha &= (\mathbf{X} \cdot \nabla \mathbf{Y} - \mathbf{Y} \cdot \nabla \mathbf{X}) \cdot d\mathbf{S} \\
d\big(X \lrcorner\, (Y \lrcorner\, \alpha)\big) + X \lrcorner\, (\pounds_Y \alpha) - Y \lrcorner\, (\pounds_X \alpha) &= -\operatorname{curl}(\mathbf{X} \times \mathbf{Y}) \cdot d\mathbf{S} + (\operatorname{div} \mathbf{Y})\, \mathbf{X} \cdot d\mathbf{S} - (\operatorname{div} \mathbf{X})\, \mathbf{Y} \cdot d\mathbf{S}
\end{aligned}$$

Thus, equation (47) is equivalent to the vector calculus identity

$$(\mathbf{X} \cdot \nabla \mathbf{Y} - \mathbf{Y} \cdot \nabla \mathbf{X}) = -\operatorname{curl}(\mathbf{X} \times \mathbf{Y}) + (\operatorname{div} \mathbf{Y})\, \mathbf{X} - (\operatorname{div} \mathbf{X})\, \mathbf{Y}\,.$$

This is the fundamental identity of fluid mechanics when $\mathbf{X} = \mathbf{u}$ and $\mathbf{Y} = \boldsymbol{\omega}$. That is,

$$-\operatorname{curl}(\mathbf{u} \times \boldsymbol{\omega}) = \mathbf{u} \cdot \nabla \boldsymbol{\omega} + (\operatorname{div} \mathbf{u})\, \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} - (\operatorname{div} \boldsymbol{\omega})\, \mathbf{u}\,.$$

▲

6 Geometric formulations of ideal fluid dynamics

6.1 Euler's fluid equations

Euler's equations for the incompressible motion of an ideal flow of a fluid of unit density and velocity $\mathbf{u}$ satisfying $\operatorname{div}\mathbf{u}=0$ in a rotating frame with Coriolis parameter $\operatorname{curl}\mathbf{R}=2\boldsymbol{\Omega}$ are given in the form of Newton's law of force by

$$\underbrace{\partial_t\,\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}}_{\text{acceleration}}=\underbrace{\mathbf{u}\times 2\boldsymbol{\Omega}}_{\text{Coriolis}}-\underbrace{\nabla p}_{\text{pressure}}\,. \tag{49}$$

Requiring preservation of the divergence-free (volume-preserving) constraint $\nabla\cdot\mathbf{u}=0$ results in a Poisson equation for pressure p, which may be written in several equivalent forms,

$$\begin{aligned}-\Delta p &= \operatorname{div}\big(\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{u}\times 2\boldsymbol{\Omega}\big)\\ &= u_{i,j}u_{j,i}-\operatorname{div}\big(\mathbf{u}\times 2\boldsymbol{\Omega}\big)\\ &= \operatorname{tr}\mathsf{S}^2-\frac{1}{2}|\operatorname{curl}\mathbf{u}|^2-\operatorname{div}\big(\mathbf{u}\times 2\boldsymbol{\Omega}\big)\,,\end{aligned} \tag{50}$$

where $\mathsf{S}=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T)$ is the ***strain-rate tensor***.

The Newton's law equation for Euler fluid motion in (49) may be rearranged into an alternative form,

$$\partial_t\,\mathbf{v}-\mathbf{u}\times\boldsymbol{\omega}+\nabla\Big(p+\frac{1}{2}|\mathbf{u}|^2\Big)=0\,, \tag{51}$$

where we denote

$$\mathbf{v}\equiv\mathbf{u}+\mathbf{R}\,,\quad \boldsymbol{\omega}=\operatorname{curl}\mathbf{v}=\operatorname{curl}\mathbf{u}+2\boldsymbol{\Omega}\,, \tag{52}$$

and introduce the ***Lamb vector***,

$$\boldsymbol{\ell}:=-\,\mathbf{u}\times\boldsymbol{\omega}\,, \tag{53}$$

which represents the nonlinearity in Euler's fluid equation (51). The Poisson equation (50) for pressure p may now be expressed in terms of the divergence of the Lamb vector,

$$-\,\Delta\left(p+\frac{1}{2}|\mathbf{u}|^2\right)=\operatorname{div}(-\,\mathbf{u}\times\operatorname{curl}\mathbf{v})=\operatorname{div}\boldsymbol{\ell}\,. \tag{54}$$

Remark 39 (***Boundary conditions).***
Because the velocity $\mathbf{u}$ must be tangent to any fixed boundary, the normal component of the motion equation must vanish. This requirement produces a Neumann condition for pressure given by

$$\partial_n \left(p + \frac{1}{2}|\mathbf{u}|^2 \right) + \hat{\mathbf{n}} \cdot \boldsymbol{\ell} = 0 \,, \tag{55}$$

at a fixed boundary with unit outward normal vector $\hat{\mathbf{n}}$.

Remark 40 (***Helmholtz vorticity dynamics).***
Taking the curl of the Euler fluid equation (51) yields the ***Helmholtz vorticity equation***

$$\partial_t \, \boldsymbol{\omega} - \text{curl}(\mathbf{u} \times \boldsymbol{\omega}) = 0 \,, \tag{56}$$

whose geometrical meaning will emerge in discussing Stokes' Theorem 56 for the vorticity of a rotating fluid.

The rotation terms have now been fully integrated into both the dynamics and the boundary conditions. In this form, the ***Kelvin circulation theorem*** and the ***Stokes vorticity theorem*** will emerge naturally together as geometrical statements.

6.2 Kelvin's circulation theorem

Theorem 41 (***Kelvin's circulation theorem).*** *The Euler equations (49) preserve the circulation integral $I(t)$ defined by*

$$I(t) = \oint_{c(\mathbf{u})} \mathbf{v} \cdot d\mathbf{x} \,, \tag{57}$$

where $c(\mathbf{u})$ is a closed circuit moving with the fluid at velocity $\mathbf{u}$.

Proof. The dynamical definition of the Lie derivative in (44) yields the following for the time rate of change of this circulation integral:

$$
\begin{aligned}
\frac{d}{dt}\oint_{c(\mathbf{u})} \mathbf{v}\cdot d\mathbf{x} &= \oint_{c(\mathbf{u})} \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\mathbf{v}\cdot d\mathbf{x}) \\
&= \oint_{c(\mathbf{u})} \left(\frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial x^j}u^j + v_j\frac{\partial u^j}{\partial \mathbf{x}}\right)\cdot d\mathbf{x} \\
&= -\oint_{c(\mathbf{u})} \nabla\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right)\cdot d\mathbf{x} \\
&= -\oint_{c(\mathbf{u})} d\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right) = 0\,.
\end{aligned}
\tag{58}
$$

The Cartan formula in (44) defines the Lie derivative of the circulation integrand in the equivalent form that we need for the third step and will also use in a moment for Stokes' theorem:

$$
\begin{aligned}
\pounds_{\mathbf{u}}(\mathbf{v}\cdot d\mathbf{x}) &= \big(\mathbf{u}\cdot\nabla\mathbf{v} + v_j\nabla u^j\big)\cdot d\mathbf{x} \\
&= u \,\lrcorner\, d(\mathbf{v}\cdot d\mathbf{x}) + d(u \,\lrcorner\, \mathbf{v}\cdot d\mathbf{x}) \\
&= u \,\lrcorner\, (\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}) + d(\mathbf{u}\cdot\mathbf{v}) \\
&= \big(-\,\mathbf{u}\times\mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\big)\cdot d\mathbf{x}\,.
\end{aligned}
\tag{59}
$$

This identity recasts Euler's equation into the following geometric form:

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\mathbf{v}\cdot d\mathbf{x}) &= (\partial_t\mathbf{v} -\, \mathbf{u}\times\mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v}))\cdot d\mathbf{x} \\
&= -\nabla\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right)\cdot d\mathbf{x} \\
&= -d\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right).
\end{aligned}
\tag{60}
$$

This finishes the last step in the proof (58), because the integral of an exact differential around a closed loop vanishes. □

The exterior derivative of the Euler fluid equation in the form (60) yields Stokes' theorem, after using the commutativity of the exterior and Lie derivatives $[d,\ \pounds_{\mathbf{u}}]=0$,

$$
\begin{aligned}
d\pounds_{\mathbf{u}}(\mathbf{v}\cdot d\mathbf{x}) &= \pounds_{\mathbf{u}}\, d(\mathbf{v}\cdot d\mathbf{x}) \\
&= \pounds_{\mathbf{u}}(\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}) \\
&= -\,\mathrm{curl}\big(\mathbf{u}\times \mathrm{curl}\,\mathbf{v}\big)\cdot d\mathbf{S} \\
&= \big[\mathbf{u}\cdot\nabla \mathrm{curl}\,\mathbf{v} + \mathrm{curl}\,\mathbf{v}(\mathrm{div}\,\mathbf{u}) - (\mathrm{curl}\,\mathbf{v})\cdot\nabla\mathbf{u}\big]\cdot d\mathbf{S}\,, \\
(\text{by } \mathrm{div}\,\mathbf{u}=0) \quad &= \big[\mathbf{u}\cdot\nabla \mathrm{curl}\,\mathbf{v} - (\mathrm{curl}\,\mathbf{v})\cdot\nabla\mathbf{u}\big]\cdot d\mathbf{S} \\
&=: \big[\,u,\ \mathrm{curl}\,v\,\big]\cdot d\mathbf{S}\,,
\end{aligned}
\tag{61}
$$

where $[\,u,\ \mathrm{curl}\,v\,]$ denotes (minus) the ***Jacobi–Lie bracket*** of the vector fields u and $\mathrm{curl}\,v$.

This calculation proves the following.

Theorem 42. *Euler's fluid equations (51) imply that*

$$
\frac{\partial\omega}{\partial t} = -\,[\,u,\ \omega\,]
\tag{62}
$$

where $[\,u,\ \omega\,]$ denotes the Jacobi–Lie bracket of the divergenceless vector fields u and $\omega := \mathrm{curl}\,v$.

The exterior derivative of Euler's equation in its geometric form (60) is equivalent to the curl of its vector form (51). That is,

$$
d\Big(\frac{\partial}{\partial t}+\pounds_{\mathbf{u}}\Big)(\mathbf{v}\cdot d\mathbf{x}) = \Big(\frac{\partial}{\partial t}+\pounds_{\mathbf{u}}\Big)(\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}) = 0\,.
\tag{63}
$$

Hence from the calculation in (61) and the Helmholtz vorticity equation (63) we have

$$
\Big(\frac{\partial}{\partial t}+\pounds_{\mathbf{u}}\Big)(\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}) = \Big(\partial_t\,\boldsymbol{\omega} - \mathrm{curl}(\mathbf{u}\times\boldsymbol{\omega})\Big)\cdot d\mathbf{S} = 0\,,
\tag{64}
$$

in which one denotes $\boldsymbol{\omega} := \mathrm{curl}\,\mathbf{v}$. This Lie-derivative version of the Helmholtz vorticity equation may be used to prove the following form of Stokes' theorem for the Euler equations in a rotating frame.

Theorem 43. ***[Kelvin/Stokes' theorem for vorticity of a rotating fluid]***

$$
\begin{aligned}
\frac{d}{dt}\oint_{c(\mathbf{u})} \mathbf{v}\cdot d\mathbf{x} &= \frac{d}{dt}\iint_{S(\mathbf{u})} \operatorname{curl}\mathbf{v}\cdot d\mathbf{S} \\
&= \iint_{S(\mathbf{u})} \Big(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\Big)(\operatorname{curl}\mathbf{v}\cdot d\mathbf{S}) \\
&= \iint_{S(\mathbf{u})} \Big(\partial_t\,\boldsymbol{\omega} - \operatorname{curl}(\mathbf{u}\times\boldsymbol{\omega})\Big)\cdot d\mathbf{S} = 0\,,
\end{aligned}
\tag{65}
$$

where the surface $S(\mathbf{u})$ is bounded by an arbitrary circuit $\partial S = c(\mathbf{u})$ moving with the fluid.

6.3 Steady solutions: Lamb surfaces

According to Theorem 42, Euler's fluid equations (51) imply that

$$
\frac{\partial\omega}{\partial t} = -\,[\,u,\,\omega\,]\,. \tag{66}
$$

Consequently, the vector fields u, ω in *steady* Euler flows, which satisfy $\partial_t\omega = 0$, also satisfy the condition necessary for the Frobenius theorem to hold – namely, that their Jacobi–Lie bracket vanishes. That is, in smooth steady, or equilibrium, solutions of Euler's fluid equations, the flows of the two divergenceless vector fields u and ω commute with each other and lie on a surface in three dimensions.

A sufficient condition for this commutation relation is that the ***Lamb vector*** $\boldsymbol{\ell} := -\,\mathbf{u}\times\operatorname{curl}\mathbf{v}$ in (53) satisfies

$$
\boldsymbol{\ell} := -\,\mathbf{u}\times\operatorname{curl}\mathbf{v} = \nabla H(\mathbf{x})\,, \tag{67}
$$

for some smooth function $H(\mathbf{x})$. This condition means that the flows of vector fields u and $\operatorname{curl} v$ (which are *steady* flows of the Euler equations) are both confined to the same surface $H(\mathbf{x}) = const$. Such a surface is called a ***Lamb surface***.

The vectors of velocity ($\mathbf{u}$) and total vorticity ($\operatorname{curl}\mathbf{v}$) for a steady Euler flow are both perpendicular to the normal vector to the Lamb surface along $\nabla H(\mathbf{x})$. That is, the Lamb surface is invariant under the flows of both vector fields, *viz*

$$
\pounds_u H = \mathbf{u}\cdot\nabla H = 0 \quad\text{and}\quad \pounds_{\operatorname{curl} v} H = \operatorname{curl}\mathbf{v}\cdot\nabla H = 0\,. \tag{68}
$$

The Lamb surface condition (67) has the following coordinate-free representation.

Theorem 44 (***Lamb surface condition***)**.** *The Lamb surface condition (67) is equivalent to the following double substitution of vector fields into the volume form,*

$$dH = u \lrcorner \operatorname{curl} v \lrcorner d^3x \,. \tag{69}$$

Proof. Recall that the contraction of vector fields with forms yields the following useful formula for the surface element:

$$\nabla \lrcorner d^3x = d\mathbf{S} \,. \tag{70}$$

Then using results from previous exercises in vector calculus operations one finds by direct computation that

$$\begin{aligned} u \lrcorner \operatorname{curl} v \lrcorner d^3x &= u \lrcorner (\operatorname{curl} \mathbf{v} \cdot d\mathbf{S}) \\ &= -\big(\mathbf{u} \times \operatorname{curl} \mathbf{v}\big) \cdot d\mathbf{x} \\ &= \nabla H \cdot d\mathbf{x} \\ &= dH \,. \end{aligned} \tag{71}$$

□

Remark 45.
Formula (71)

$$u \lrcorner (\operatorname{curl} \mathbf{v} \cdot d\mathbf{S}) = dH$$

is to be compared with

$$X_h \lrcorner \omega = dH \,,$$

in the definition of a Hamiltonian vector field in Equation (38) of Theorem 19. Likewise, the stationary case of the Helmholtz vorticity equation (63), namely,

$$\pounds_{\mathbf{u}}(\operatorname{curl} \mathbf{v} \cdot d\mathbf{S}) = 0 \,, \tag{72}$$

is to be compared with the proof of Poincaré's theorem in Corollary 22

$$\pounds_{X_h}\omega = d(X_h \lrcorner \omega) = d^2H = 0 \,.$$

Thus, the two-form $\operatorname{curl} \mathbf{v} \cdot d\mathbf{S}$ plays the same role for stationary Euler fluid flows as the symplectic form $dq \wedge dp$ plays for canonical Hamiltonian flows. We seek the corresponding symplectic coordinates.

Definition 46. *The* ***Clebsch representation*** *of the one-form* $\mathbf{v}\cdot d\mathbf{x}$ *is defined by*

$$\mathbf{v}\cdot d\mathbf{x} = -\,\Pi\, d\Xi + d\Psi\,. \tag{73}$$

The functions Ξ*,* Π *and* Ψ *are called* ***Clebsch potentials*** *for the vector* $\mathbf{v}$*.*[4]

In terms of the Clebsch representation (73) of the one-form $\mathbf{v}\cdot d\mathbf{x}$, the total vorticity flux $\operatorname{curl}\mathbf{v}\cdot d\mathbf{S} = d(\mathbf{v}\cdot d\mathbf{x})$ is the exact two-form,

$$\operatorname{curl}\mathbf{v}\cdot d\mathbf{S} = d\Xi\wedge d\Pi\,. \tag{74}$$

This amounts to writing the flow lines of the *vector field* of the total vorticity $\operatorname{curl} v$ as the intersections of level sets of surfaces $\Xi = const$ and $\Pi = const$. In other words,

$$\operatorname{curl}\mathbf{v} = \nabla\Xi\times\nabla\Pi\,, \tag{75}$$

with the assumption that these level sets foliate $\mathbb{R}^3$. That is, one assumes that any point in $\mathbb{R}^3$ along the flow of the total vorticity vector field $\operatorname{curl} v$ may be assigned to a regular intersection of these level sets. The main result of this assumption is the following theorem.

Theorem 47 (***Lamb surfaces are symplectic manifolds*****).** *[ArKh1992, ArKh1998] The steady flow of the vector field* u *satisfying the symmetry relation given by the vanishing of the commutator* $[\,u,\,\operatorname{curl} v\,] = 0$ *on a three-dimensional manifold* $M\in\mathbb{R}^3$ *reduces to incompressible flow on a two-dimensional symplectic manifold whose canonically conjugate coordinates* $(\Xi,\,\Pi)$ *are provided by the total vorticity flux*

$$\operatorname{curl} v \,\lrcorner\, d^{\,3}x = \operatorname{curl}\mathbf{v}\cdot d\mathbf{S} = d\Xi\wedge d\Pi\,.$$

The reduced flow is canonically Hamiltonian on this symplectic manifold. Furthermore, the reduced Hamiltonian is precisely the restriction of the invariant H *onto the reduced phase space.*

Proof. Restricting formula (71) to coordinates on a total vorticity flux surface (74) yields the exterior derivative of the Hamiltonian,

$$\begin{aligned} dH(\Xi,\,\Pi) &= u\,\lrcorner\,(\operatorname{curl}\mathbf{v}\cdot d\mathbf{S}) \\ &= u\,\lrcorner\,(d\Xi\wedge d\Pi) \\ &= (\mathbf{u}\cdot\nabla\Xi)\,d\Pi - (\mathbf{u}\cdot\nabla\Pi)\,d\Xi \\ &=: \frac{d\Xi}{dT}\,d\Pi - \frac{d\Pi}{dT}\,d\Xi \\ &= \frac{\partial H}{\partial\Pi}\,d\Pi + \frac{\partial H}{\partial\Xi}\,d\Xi\,, \end{aligned} \tag{76}$$

[4] The Clebsch representation is another example of a cotangent lift momentum map.

where $T \in \mathbb{R}$ is the time parameter along the flow lines of the steady vector field u, which carries the Lagrangian fluid parcels. On identifying corresponding terms, the steady flow of the fluid velocity $\mathbf{u}$ is found to obey the canonical Hamiltonian equations,

$$(\mathbf{u} \cdot \nabla \Xi) = \pounds_u \Xi \quad =: \quad \frac{d\Xi}{dT} = \frac{\partial H}{\partial \Pi} = \big\{ \Xi,\, H \big\}\,, \tag{77}$$

$$(\mathbf{u} \cdot \nabla \Pi) = \pounds_u \Pi \quad =: \quad \frac{d\Pi}{dT} = -\,\frac{\partial H}{\partial \Xi} = \big\{ \Pi,\, H \big\}\,, \tag{78}$$

where $\{\,\cdot\,,\,\cdot\,\}$ is the canonical Poisson bracket for the symplectic form $d\Xi \wedge d\Pi$. □

Corollary 48. *The vorticity flux $d\Xi \wedge d\Pi$ is invariant under the flow of the velocity vector field u.*

Proof. By (76), one verifies

$$\pounds_u (d\Xi \wedge d\Pi) = d\big(u \,\lrcorner\, (d\Xi \wedge d\Pi)\big) = d^2 H = 0\,.$$

This is the standard computation in the proof of Poincaré's theorem in Corollary 22 for the preservation of a symplectic form by a canonical transformation. Its interpretation here is that the steady Euler flows preserve the total vorticity flux, $\mathrm{curl}\,\mathbf{v} \cdot d\mathbf{S} = d\Xi \wedge d\Pi$. □

6.4 The conserved helicity of ideal incompressible flows

Definition 49 (Helicity)**.** *The helicity $\Lambda[\mathrm{curl}\,\mathbf{v}]$ of a divergence-free vector field $\mathrm{curl}\, v$ that is tangent to the boundary ∂D of a simply connected domain $D \in \mathbb{R}^3$ is defined as*

$$\Lambda[\mathrm{curl}\,\mathbf{v}] = \int_D \mathbf{v} \cdot \mathrm{curl}\,\mathbf{v}\, d^{\,3}x\,, \tag{79}$$

where $\mathbf{v}$ is a divergence-free vector-potential for the field $\mathrm{curl}\,\mathbf{v}$.

Remark 50.
The helicity is unchanged by adding a gradient to the vector $\mathbf{v}$. Thus, $\mathbf{v}$ is not unique and $\mathrm{div}\,\mathbf{v} = 0$ is not a restriction for simply connected domains in $\mathbb{R}^3$, provided $\mathrm{curl}\,\mathbf{v}$ is tangent to the boundary ∂D.

The helicity of a vector field $\mathrm{curl}\, v$ measures the total linking of its field lines, or their relative winding. (For details and mathematical history, see [ArKh1998].) The idea of helicity goes back to Helmholtz and Kelvin in the 19th century. The principal feature of this concept for fluid dynamics is embodied in the following theorem.

Theorem 51 (***Euler flows preserve helicity).*** *When homogeneous or periodic boundary conditions are imposed, Euler's equations for an ideal incompressible fluid flow in a rotating frame with Coriolis parameter* $\operatorname{curl}\mathbf{R} = 2\boldsymbol{\Omega}$ *preserves the helicity*

$$\Lambda[\operatorname{curl}\mathbf{v}] = \int_D \mathbf{v}\cdot\operatorname{curl}\mathbf{v}\, d^{\,3}x\,, \tag{80}$$

with $\mathbf{v} = \mathbf{u} + \mathbf{R}$*, for which* $\mathbf{u}$ *is the divergenceless fluid velocity* $(\operatorname{div}\mathbf{u} = 0)$ *and* $\operatorname{curl}\mathbf{v} = \operatorname{curl}\mathbf{u} + 2\boldsymbol{\Omega}$ *is the total vorticity.*

Proof. Rewrite the geometric form of the Euler equations (60) for rotating incompressible flow with unit mass density in terms of the circulation one-form $v := \mathbf{v}\cdot d\mathbf{x}$ as

$$\big(\partial_t + \pounds_u\big)v = -\,d\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right) =: -\,d\varpi\,, \tag{81}$$

and $\pounds_u\, d^{\,3}x = 0$, where ϖ is an augmented pressure variable,

$$\varpi := p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\,. \tag{82}$$

The fluid velocity vector field is denoted as $u = \mathbf{u}\cdot\nabla$ with $\operatorname{div}\mathbf{u} = 0$. Then the ***helicity density***, defined as

$$v\wedge dv = \mathbf{v}\cdot\operatorname{curl}\mathbf{v}\, d^{\,3}x = \lambda\, d^{\,3}x\,,\quad \text{with}\quad \lambda = \mathbf{v}\cdot\operatorname{curl}\mathbf{v}\,, \tag{83}$$

obeys the dynamics it inherits from the Euler equations,

$$\big(\partial_t + \pounds_u\big)(v\wedge dv) = -d\varpi\wedge dv - v\wedge d^2\varpi = -d(\varpi\, dv)\,, \tag{84}$$

after using $d^2\varpi = 0$ and $d^2 v = 0$. In vector form, this result may be expressed as a conservation law,

$$\big(\partial_t\lambda + \operatorname{div}\lambda\mathbf{u}\big)d^{\,3}x = -\operatorname{div}(\varpi\operatorname{curl}\mathbf{v})\, d^{\,3}x\,. \tag{85}$$

Consequently, the time derivative of the integrated helicity in a domain D obeys

$$\begin{aligned} \frac{d}{dt}\Lambda[\operatorname{curl}\mathbf{v}] &= \int_D \partial_t\lambda\, d^{\,3}x = -\int_D \operatorname{div}(\lambda\mathbf{u} + \varpi\operatorname{curl}\mathbf{v})\, d^{\,3}x \\ &= -\oint_{\partial D}(\lambda\mathbf{u} + \varpi\operatorname{curl}\mathbf{v})\cdot d\mathbf{S}\,, \end{aligned} \tag{86}$$

which vanishes when homogeneous, or periodic, or even Neumann boundary conditions are imposed on the values of $\mathbf{u}$ and $\operatorname{curl}\mathbf{v}$ at the boundary ∂D. □

Remark 52.
This result means the ***helicity integral***

$$\Lambda[\mathrm{curl}\,\mathbf{v}] = \int_D \lambda\, d^{\,3}x$$

is conserved in periodic domains, or in all of $\mathbb{R}^3$ with vanishing boundary conditions at spatial infinity. However, if either the velocity or total vorticity at the boundary possesses a nonzero normal component, then the boundary is a *source* of helicity (that is, it causes winding of field lines of $\mathrm{curl}\,\mathbf{v}$). For a fixed impervious boundary, the normal component of velocity does vanish, but no such condition is imposed on the total vorticity by the physics of fluid flow. Thus, we have the following.

Corollary 53. *A flux of total vorticity* $\mathrm{curl}\,\mathbf{v}$ *into the domain is a source of helicity.*

> **Exercise.** Use Cartan's formula in (44) to compute $\pounds_u(v \wedge dv)$ in Equation (84). ★

> **Exercise.** Compute the helicity for the one-form $v = \mathbf{v} \cdot d\mathbf{x}$ in the Clebsch representation (73). What does this mean for the linkage of the vortex lines that admit the Clebsch representation? ★

Theorem 54 (Diffeomorphisms preserve helicity)**.** *The helicity* $\Lambda[\xi]$ *of any divergenceless vector field* ξ *is preserved under the action on* ξ *of any volume-preserving diffeomorphism of the manifold* M *[ArKh1998].*

Remark 55 (Helicity is a topological invariant)**.**
The helicity $\Lambda[\xi]$ is a topological invariant, not a dynamical invariant, because its invariance is independent of which diffeomorphism acts on ξ. This means the invariance of helicity is independent of which Hamiltonian flow produces the diffeomorphism. This is the hallmark of a Casimir function. Although it is defined above with the help of a metric, every volume-preserving diffeomorphism carries a divergenceless vector field ξ into another such field with the same helicity. However, independently of any metric properties, the action of diffeomorphisms does not create or destroy linkages of the characteristic curves of divergenceless vector fields.

6.5 Ertel theorem for potential vorticity

Euler–Boussinesq equations The Euler–Boussinesq equations for the incompressible motion of an ideal flow of a stratified fluid and velocity $\mathbf{u}$ satisfying $\mathrm{div}\,\mathbf{u} = 0$ in a rotating frame with Coriolis parameter $\mathrm{curl}\,\mathbf{R} = 2\boldsymbol{\Omega}$ are given by

$$\underbrace{\partial_t\,\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}}_{\text{acceleration}} = \underbrace{-\,gb\nabla z}_{\text{buoyancy}} + \underbrace{\mathbf{u}\times 2\boldsymbol{\Omega}}_{\text{Coriolis}} - \underbrace{\nabla p}_{\text{pressure}} \tag{87}$$

where $-g\nabla z$ is the constant downward acceleration of gravity and b is the buoyancy, which satisfies the ***advection relation***,

$$\partial_t\, b + \mathbf{u}\cdot\nabla b = 0\,. \tag{88}$$

As for Euler's equations without buoyancy, requiring preservation of the divergence-free (volume-preserving) constraint $\nabla\cdot\mathbf{u} = 0$ results in a Poisson equation for pressure p,

$$-\,\Delta\left(p + \frac{1}{2}|\mathbf{u}|^2\right) = \mathrm{div}(-\,\mathbf{u}\times\mathrm{curl}\,\mathbf{v}) + g\partial_z b\,, \tag{89}$$

which satisfies a Neumann boundary condition because the velocity $\mathbf{u}$ must be tangent to the boundary.

The Newton's law form of the Euler–Boussinesq equations (87) may be rearranged as

$$\partial_t\,\mathbf{v} -\ \mathbf{u}\times\mathrm{curl}\,\mathbf{v} + gb\nabla z + \nabla\left(p + \frac{1}{2}|\mathbf{u}|^2\right) = 0\,, \tag{90}$$

where $\mathbf{v}\equiv\mathbf{u}+\mathbf{R}$ and $\nabla\cdot\mathbf{u} = 0$.

Theorem 56. ***[The Kelvin/Stokes' theorem for vorticity of a stratified, rotating fluid]***

$$\begin{aligned}
\frac{d}{dt}\oint_{c(\mathbf{u})}\mathbf{v}\cdot d\mathbf{x} &= \frac{d}{dt}\iint_{S(\mathbf{u})}\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}\\
&= \iint_{S(\mathbf{u})}\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S})\\
&= \iint_{S(\mathbf{u})}\Big(\partial_t\,\boldsymbol{\omega} - \mathrm{curl}\,(\mathbf{u}\times\boldsymbol{\omega})\Big)\cdot d\mathbf{S}\\
&= \iint_{S(\mathbf{u})}\Big(-\ g\nabla b\times\nabla z\Big)\cdot d\mathbf{S}\,,
\end{aligned} \tag{91}$$

where the surface $S(\mathbf{u})$ is bounded by an arbitrary circuit $\partial S = c(\mathbf{u})$ moving with the fluid. Thus, non-alignment of the gradient of buoyancy ∇b with the vertical ∇z creates circulation. Compare this result with equation (65) in the absence of stratification.

Geometrically, equation (90) may be written as

$$\big(\partial_t + \pounds_u\big)v + gbdz + d\varpi = 0\,, \tag{92}$$

where ϖ is defined in (81). In addition, the buoyancy satisfies

$$\big(\partial_t + \pounds_u\big) b = 0\,, \quad \text{with} \quad \pounds_u\, d^{\,3}x = 0\,. \tag{93}$$

The fluid velocity vector field is denoted as $u = \mathbf{u}\cdot\nabla$ and the circulation one-form as $v = \mathbf{v}\cdot d\mathbf{x}$. The exterior derivatives of the two equations in (92) are written as

$$\big(\partial_t + \pounds_u\big) dv = -g db\wedge dz \quad \text{and} \quad \big(\partial_t + \pounds_u\big) db = 0\,. \tag{94}$$

Consequently, one finds from the product rule for Lie derivatives (39) that

$$\big(\partial_t + \pounds_u\big)(dv\wedge db) = 0 \quad \text{or} \quad \partial_t\, q + \mathbf{u}\cdot\nabla q = 0\,, \tag{95}$$

in which the quantity

$$q = \nabla b\cdot \operatorname{curl}\mathbf{v} \tag{96}$$

is called ***potential vorticity*** and is abbreviated as PV. The potential vorticity is an important diagnostic for many processes in geophysical fluid dynamics. Conservation of PV on fluid parcels is called ***Ertel's theorem***.

Remark 57 (***Ertel's theorem for the vorticity vector field).***
Writing the vorticity vector field $\omega = \boldsymbol{\omega}\cdot\nabla$, we have

$$\big(\partial_t + \pounds_u\big)\omega = \partial_t\omega + [u,\,\omega] = g\nabla z\times\nabla b\cdot\nabla\,.$$

Thus, conservation of the potential vorticity may also be proved by the product rule, as

$$\big(\partial_t + \pounds_u\big) q = \big(\partial_t + \pounds_u\big)(\boldsymbol{\omega}\cdot\nabla b) = \big(\partial_t + \pounds_u\big)(\omega b) = \Big(\big(\partial_t + \pounds_u\big)\omega\Big) b + \omega\big(\partial_t + \pounds_u\big) b = 0\,.$$

Remark 58 (***Material derivative formulation).***
Denoting

$$\frac{D}{Dt} = \partial_t + \pounds_u \quad \text{and} \quad \omega = \boldsymbol{\omega}\cdot\nabla$$

provides an intuitive expression of the Ertel theorem (95) that helps understand it in terms of the time derivative $\frac{D}{Dt}$ following the flow of the fluid particles. Namely, it suggests writing in vector form

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla) = g\nabla z\times\nabla b\cdot\nabla \quad \text{and} \quad \frac{Db}{Dt} = 0\,,$$

so that the product rule for derivatives yields conservation of PV on fluid parcels, as

$$\frac{Dq}{Dt} = \frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla b) = \Big(\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla)\Big)b + (\boldsymbol{\omega}\cdot\nabla)\frac{Db}{Dt} = g\nabla z\times\nabla b\cdot\nabla b + (\boldsymbol{\omega}\cdot\nabla)\frac{Db}{Dt} = 0\,.$$

Remark 59 (***The conserved quantities associated with Ertel's theorem).***
The constancy of the scalar quantities b and q on fluid parcels implies conservation of the spatially integrated quantity,

$$C_\Phi = \int_D \Phi(b,q)\,d^{\,3}x\,, \tag{97}$$

for any smooth function Φ for which the integral exists.

Proof.

$$\begin{aligned}\frac{d}{dt}C_\Phi &= \int_D \Phi_b\partial_t b + \Phi_q\partial_t q\,d^{\,3}x = -\int_D \Phi_b\mathbf{u}\cdot\nabla b + \Phi_q\mathbf{u}\cdot\nabla q\,d^{\,3}x\\ &= -\int_D \mathbf{u}\cdot\nabla\Phi(b,q)\,d^{\,3}x = -\int_D \nabla\cdot\big(\mathbf{u}\,\Phi(b,q)\big)\,d^{\,3}x = -\oint_{\partial D}\Phi(b,q)\,\mathbf{u}\cdot\hat{\mathbf{n}}\,dS = 0\,,\end{aligned}$$

when the normal component of the velocity $\mathbf{u}\cdot\hat{\mathbf{n}}$ vanishes at the boundary ∂D.

□

Remark 60 (***Energy conservation).***
In addition to C_Φ, the Euler–Boussinesq fluid equations (90) also conserve the total energy

$$E = \int_D \frac{1}{2}|\mathbf{u}|^2 + gbz\ d^{\,3}x\,, \tag{98}$$

which is the sum of the kinetic and potential energies.

We do not develop the Hamiltonian formulation of the three-dimensional stratified rotating fluid equations studied here. However, one may imagine that the conserved quantity C_Φ with the arbitrary function Φ would play an important role. For more explanation in the framework of Geometric Mechanics, see [Ho2011GM] and references therein.

These issues will be discussed in Spring term in M3-4-5A34, *Geometry, Mechanics and Symmetry.*

References

[AbMa1978] Abraham, R. and Marsden, J. E. [1978]
Foundations of Mechanics,
2nd ed. Reading, MA: Addison-Wesley.

[ArKh1992] Arnold, V. I. and Khesin, B. A. [1992]
Topological methods in hydrodynamics.
Annu. Rev. Fluid Mech. **24**, 145–166.

[ArKh1998] Arnold, V. I. and Khesin, B. A. [1998]
Topological Methods in Hydrodynamics. New York: Springer.

[Ho2005] Holm, D. D. [2005]
The Euler–Poincaré variational framework for modeling fluid dynamics.
In *Geometric Mechanics and Symmetry: The Peyresq Lectures*,
edited by J. Montaldi and T. Ratiu.
London Mathematical Society Lecture Notes Series 306.
Cambridge: Cambridge University Press.

[Ho2011GM] Holm, D. D. [2011]
Geometric Mechanics I: Dynamics and Symmetry,
Second edition, World Scientific: Imperial College Press, Singapore, .

[Ho2011] Holm, D. D. [2011]
Applications of Poisson geometry to physical problems,
Geometry & Topology Monographs **17**, 221–384.

[Ho2012] Holm, D. D. [2012]
Euler-Poincaré Theory from the Rigid Body to Solitons
6th GMC Summer School Lectures, Miraflores de La Sierra, Spain, 22-26 June 2012
GMC Notes, No. 2. Download at `http://gmcnetwork.org/drupal/?q=notes`

[HoSmSt2009] Holm, D. D., Schmah, T. and Stoica, C. [2009]
Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions,
Oxford University Press.

[Ma1976] Manakov, S. V. [1976]
Note on the integration of Euler's equations for the dynamics of an n-dimensional rigid body.
Funct. Anal. Appl. **10**, 328–329.

[MaRa1994] Marsden, J. E. and Ratiu, T. S. [1994]
Introduction to Mechanics and Symmetry.
Texts in Applied Mathematics, Vol. 75. New York: Springer-Verlag.

[Po1901] H. Poincaré, Sur une forme nouvelle des équations de la méchanique, *C.R. Acad. Sci.* **132** (1901) 369–371.

[RaTuSbSoTe2005] Ratiu, T. S., Tudoran, R., Sbano, L., Sousa Dias, E. and Terra, G. [2005]
A crash course in geometric mechanics.
In *Geometric Mechanics and Symmetry: The Peyresq Lectures*,
edited by J. Montaldi and T. Ratiu. London Mathematical Society Lecture Notes Series 306.
Cambridge: Cambridge University Press.