2 M3-4-5A16 Assessed Problems # 2: Do 4 out of 5 problems

Exercise 2.1 (The fish: a quadratically nonlinear oscillator)

Consider the Hamiltonian dynamics on a symplectic manifold of a system comprising two real degrees of freedom, with real phase space variables (x, y, θ, z), symplectic form

$$\omega = dx \wedge dy + d\theta \wedge dz$$

and Hamiltonian

$$H = \tfrac{1}{2}y^2 + x\left(\tfrac{1}{3}x^2 - z\right) - \tfrac{2}{3}z^{3/2}$$

(A) Write the canonical Poisson bracket for this system.

$\boxed{\boxed{\textit{Answer}}}$

$$\{F\,,\,H\} = H_y F_x - H_x F_y + H_z F_\theta - H_\theta F_z$$

▲

(B) Write Hamilton's canonical equations for this system. Explain how to keep $z \geq 0$, so that H and θ remain real.

$\boxed{\boxed{\textit{Answer}}}$

Hamilton's canonical equations for this system are

$$\begin{aligned}\dot{x} &= \{x, H\} = H_y = y\,,\\ \dot{y} &= \{y, H\} = -\,H_x = -(x^2 - z)\,,\end{aligned}$$

and

$$\begin{aligned}\dot{\theta} &= \{\theta, H\} = H_z = -(x + \sqrt{z}\,)\,,\\ \dot{z} &= \{z, H\} = -\,H_\theta = 0\,.\end{aligned}$$

For H and θ to remain real, one need only choose the initial value of the constant of motion $z \geq 0$. ▲

(C) At what values of x, y and H does the system have stationary points in the (x, y) plane?

$\boxed{\boxed{\textit{Answer}}}$

The system has (x, y) stationary points when its time derivatives vanish: at $y = 0$, $x = \pm\sqrt{z}$ and $H = -\frac{4}{3}z^{3/2}$. ▲

(D) Propose a strategy for solving these equations. In what order should they be solved?

$\boxed{\boxed{\textit{Answer}}}$

Since z is a constant of motion, the equation for its conjugate variable $\theta(t)$ decouples from the others and may be solved as a quadrature *after* first solving for $x(t)$ and $y(t)$ on a level set of z. ▲

(E) Identify the constants of motion of this system and explain why they are conserved.

Answer

There are two constants of motion:
(i) The Hamiltonian H for the canonical equations is conserved, because the Poisson bracket in $\dot{H} = \{H, H\}$ is antisymmetric.
(ii) The momentum z conjugate to θ is conserved, because $H_\theta = 0$. ▲

(F) *Compute the associated Hamiltonian vector field X_H and show that it satisfies*

$$X_H \lrcorner\, \omega = dH$$

Answer

$$\begin{aligned} X_H = \{\,\cdot\,,\, H\} &= H_y\partial_x - H_x\partial_y + H_z\partial_\theta - H_\theta\partial_z \\ &= y\partial_x - (x^2 - z)\partial_y - (x + \sqrt{z})\partial_\theta , \end{aligned}$$

so that

$$X_H \lrcorner\, \omega = ydy + (x^2 - z)dx - (x + \sqrt{z}\,)dz = dH$$

▲

(G) *Write the Poisson bracket that expresses the Hamiltonian vector field X_H as a divergenceless vector field in $\mathbb{R}^3$ with coordinates $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Explain why this Poisson bracket satisfies the Jacobi identity.*

Answer

Write the evolution equations for $\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$ as

$$\begin{aligned} \dot{\mathbf{x}} = \{\mathbf{x}, H\} = \nabla H \times \nabla z &= (H_y, -H_x, 0)^T \\ &= (y, z - x^2, 0)^T \\ &= (\dot{x}, \dot{y}, \dot{z})^T . \end{aligned}$$

Hence, for any smooth function $F(\mathbf{x})$,

$$\{F, H\} = \nabla z \cdot \nabla F \times \nabla H = F_x H_y - H_x F_y \,.$$

This is the canonical Poisson bracket for one degree of freedom, which is known to satisfy the Jacobi identity. ▲

(H) *Identify the Casimir function for this $\mathbb{R}^3$ bracket. Show explicitly that it satisfies the definition of a Casimir function.*

Answer

Substituting $F = \Phi(z)$ for a smooth function Φ into the bracket expression yields

$$\{\Phi(z), H\} = \nabla z \cdot \nabla\Phi(z) \times \nabla H = \nabla H \cdot \nabla z \times \nabla\Phi(z) = 0,$$

for all H. This proves that $F = \Phi(z)$ is a Casimir function for any smooth Φ. ▲

(I) *Sketch a graph of the intersections of the level surfaces in $\mathbb{R}^3$ of the Hamiltonian and Casimir function. Show the directions of flow along these intersections. Identify the locations and types of any relative equilibria at the tangent points of these surfaces.*

Answer

The sketch should show a saddle-node fish shape pointing rightward in the (x, y) plane with elliptic equilibrium at $(x, y) = (\sqrt{z}, 0)$, hyperbolic equilibrium at $(x, y) = (-\sqrt{z}, 0)$ and directions of flow with sign($\dot{x}$)=sign(y). The fish shape is sketched in Figure 1 for $z = 1$. ▲

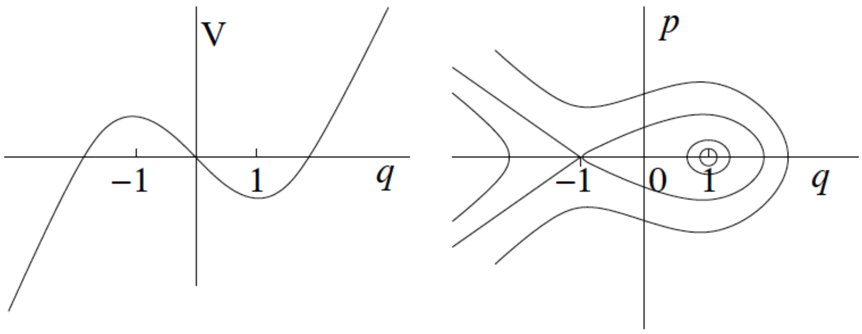

Figure 1: Phase plane for the saddle-node fish shape arising from the intersections of the level surfaces in $\mathbb{R}^3$ of the Hamiltonian and Casimir function.

(J) Linearise around the relative equilibria on a level set of the Casimir (z) and compute its eigenvalues.

$\boxed{\textit{Answer}}$

On a level surface of z the (x,y) coordinates satisfy $\dot{x}=y$ and $\dot{y}=z-x^2$. Linearising around $(x_e,y_e)=(\pm\sqrt{z},0)$ yields with $(x,y)=(x_e+\phi_1(t),y_e+\phi_2(t))$

$$\begin{bmatrix}\dot{\phi}_1\\ \dot{\phi}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2x_e & 0\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}.$$

Its characteristic equation,

$$\det\begin{bmatrix}\lambda & -1\\ 2x_e & \lambda\end{bmatrix}=\lambda^2+2x_e=0,$$

yields $\lambda^2=-2x_e=\mp 2\sqrt{z}$.
Hence, the eigenvalues are,
$\lambda=\pm i\sqrt{2}z^{1/4}$ at the elliptic equilibrium $(x_e,y_e)=(\sqrt{z},0)$, and
$\lambda=\pm\sqrt{2}z^{1/4}$ at the hyperbolic equilibrium $(x_e,y_e)=(-\sqrt{z},0)$. ▲

(K) If you found a hyperbolic equilibrium point in the previous part connected to itself by a homoclinic orbit, then reduce the equation for the homoclinic orbit to an indefinite integral expression.

$\boxed{\textit{Answer}}$

On the homoclinic orbit the Hamiltonian vanishes, so that

$$H=\tfrac{1}{2}y^2+x\left(\tfrac{1}{3}x^2-z\right)-\tfrac{2}{3}z^{3/2}=0.$$

Using $y=\dot{x}$, rearranging and integrating implies the indefinite integral expression, or "quadrature",

$$\int\frac{dx}{\sqrt{2z^{3/2}-x^3+3zx}}=\sqrt{\tfrac{2}{3}}\int dt\,.$$

After some work this integrates to

$$\frac{x(t)+\sqrt{z}}{3\sqrt{z}}=\operatorname{sech}^2\left(\frac{z^{1/4}t}{\sqrt{2}}\right).$$

From this equation, one may also compute the evolution of $\theta(t)$ on the homoclinic orbit by integrating the θ-equation,

$$\frac{d\theta}{dt}=-(x(t)+\sqrt{z}\,).$$

▲

Exercise 2.2 (3D Volterra system)

Consider the dynamical system in $(x_1, x_2, x_3) \in \mathbb{R}^3$,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ x_2 x_3 - x_1 x_2 \\ -x_2 x_3 \end{bmatrix} = x_2 \begin{bmatrix} x_1 \\ x_3 - x_1 \\ -x_3 \end{bmatrix} \tag{1}$$

This is a 3D version of the Volterra [1931] model of competition among species, which for more species is given by

$$\dot{x}_n = x_n(x_{n+1} - x_{n-1}), \quad n = 1, 2, \dots, N, \quad \text{with} \quad x_0 = 0 = x_{N+1}.$$

(A) Find two conservation laws for the system (1).

Answer

The flow of the vector field $(\dot{x}_1, \dot{x}_2, \dot{x}_3) \in T\mathbb{R}^3$ preserves the sum $H = x_1 + x_2 + x_3$ and the product $C = x_1 x_3$.

▲

(B) Write the system (1) in terms of the cross product of the gradients of the two conserved quantities.

Answer

Taking x_2 times the cross product of the gradients of the two conserved quantities for this system yields

$$\dot{\mathbf{x}} = x_2 \nabla H \times \nabla C = x_2 \widehat{\nabla H} \nabla C = -\, x_2 \widehat{\nabla C} \nabla H$$

▲

(C) Find the equilibrium states of the system and determine their stability.

Answer

From the factor of x_2 and the geometry of the intersections of H and C, one sees that the LV system has a plane of hyperbolic fixed points at $x_2 = 0$ and a line of hyperbolic fixed points along the x_2-axis at $x_1 = 0 = x_3$. ▲

(D) Write the system (1) in two Hamiltonian forms that use its two conserved quantities as Hamiltonians.

Answer

(i) When the Hamiltonian is taken as $C = x_1 x_3$ then its Hamiltonian structure is linear and thus Lie-Poisson

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ x_2 x_3 - x_1 x_2 \\ -x_2 x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ 0 \\ x_1 \end{bmatrix} = x_2 \widehat{\nabla H} \nabla C, \tag{2}$$

with an entire plane of critical points at $x_2 = 0$. This system is *Lie-Poisson* for a Lie algebra with structure constants given by

$$c_{12}^2 = -1\,, \quad c_{31}^2 = -1\,, \quad c_{23}^2 = -1\,.$$

(ii) When the Hamiltonian is taken as $H = x_1 + x_2 + x_3$ then its Hamiltonian matrix is found to be *quadratic*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_1x_2 \\ x_2x_3 - x_1x_2 \\ -x_2x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 & x_1 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -\, x_2 \widehat{\nabla C} \nabla H \qquad (3)$$

The Poisson commutation relations associated with this Hamiltonian matrix are quadratic, not linear,

$$\{x_1, x_2\} = x_1x_2 \,, \quad \{x_1, x_3\} = 0 \,, \quad \{x_2, x_3\} = x_2x_3 \,.$$

▲

(E) Verify that this system may be written as $\frac{dL}{dt} = [L, B]$ for the 4×4 matrices

$$L := \begin{bmatrix} x_1 & 0 & \sqrt{x_1x_2} & 0 \\ 0 & x_1 + x_2 & 0 & \sqrt{x_2x_3} \\ \sqrt{x_1x_2} & 0 & x_2 + x_3 & 0 \\ 0 & \sqrt{x_2x_3} & 0 & x_3 \end{bmatrix} \quad B := \frac{1}{2} \begin{bmatrix} 0 & 0 & -\sqrt{x_1x_2} & 0 \\ 0 & 0 & 0 & -\sqrt{x_2x_3} \\ \sqrt{x_1x_2} & 0 & 0 & 0 \\ 0 & \sqrt{x_2x_3} & 0 & 0 \end{bmatrix}$$

(F) Explain how the conservation laws found earlier are related to the matrices L and B.

Answer $\mathrm{tr}L = 2H = 2(x_1 + x_2 + x_3)$ and $\det L = C^2 = (x_1x_3)^2$. ▲

(G) Give the geometrical interpretation of the formula $\frac{dL}{dt} = [L, B]$ with 4×4 matrices L and B.

Answer The Lax pair relation implies that

$$L(t) = O^{-1}(t) L(0) O(t), \quad \text{where} \quad B(t) = O^{-1}\dot{O}(t) \in \mathfrak{so}(4) \quad \text{and} \quad O(t) \in SO(4).$$

Thus, $O(t)L(t)O^{-1}(t) = L(0)$ is conserved. That is, the flow generates a similarity transformation that preserves the spectrum of the 4×4 symmetric matrix $L(0)$. This matrix does not have 4 independent eigenvalues, though, because the system has only 2 conserved quantities. ▲

(H) Write the system (1) as a double matrix commutator, $\frac{dL}{dt} = [L, [L, N]]$. In particular, find N explicitly.

Answer

$$N := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

▲

(I) Give the geometrical interpretation of the formula $\frac{dL}{dt} = [L, [L, N]]$.

Answer The matrix $L(t)$ evolves in a gradient flow that preserves its spectrum until it becomes a multiple of the diagonal matrix N. Thus, the dynamics of L become asymptotically steady as L diagonalizes. Being proportional to the matrix N, the eigenvalues in the diagonalised form of L are doubly degenerate. ▲

Exercise 2.3 (The 3D May-Leonard system)

Consider the ***3D May-Leonard system*** governed by the equations:

$$
\begin{aligned}
\dot{x} &= -\,x(x+\kappa y+\lambda z)\,,\\
\dot{y} &= -\,y(y+\kappa z+\lambda x)\,,\\
\dot{z} &= -\,z(z+\kappa x+\lambda y)\,,
\end{aligned}
\tag{4}
$$

for two real constants κ and λ. One notices its cyclic symmetry in (x, y, z). This system describes nonlinear aspects of competition among three species [ML1975].

(A) Show that the system (4) preserves volume in $\mathbb{R}^3$ when $\kappa+\lambda=-\,2$.

(B) For the volume-preserving case $\kappa=-1=\lambda$, system (4) becomes

$$
\begin{aligned}
\dot{x} &= -\,x(x-y-z)\,,\\
\dot{y} &= -\,y(y-z-x)\,,\\
\dot{z} &= -\,z(z-x-y)\,.
\end{aligned}
\tag{5}
$$

(i) Transform to quadratic variables

$$p_1=yz\,,\quad p_2=zx\,,\quad p_3=xy$$

and find the equations for $\dot{p}_1, \dot{p}_2, \dot{p}_3$ implied by the ML equations (5).

(ii) Show that the equations for $\dot{p}_1, \dot{p}_2, \dot{p}_3$ imply three (linearly dependent) constants of motion.

(iii) Motivated by your result, find two real *quadratic* functions C and H for which the system (5) may be written in Nambu vector form as a cross product of their gradients in $\mathbb{R}^3$, i.e.,

$$\dot{\mathbf{x}}=\nabla C\times\nabla H=\widehat{\mathsf{C}}\,\nabla H=-\,\widehat{\mathsf{H}}\,\nabla C\quad\text{with}\quad \mathbf{x}=(x,y,z)^T\quad\text{and}\quad \widehat{\mathsf{C}}=\nabla C\times=-\,\widehat{\mathsf{C}}^T. \tag{6}$$

In index notation for vector components, $i, j, k = 1, 2, 3$, the first of these would be

$$\dot{x}_i=\widehat{C}_{ik}H_{,\,k}\quad\text{with}\quad \widehat{C}_{ik}=-\,\epsilon_{ikj}C_{,\,j}=-\,\widehat{C}_{ki}\,,$$

by the hat map.

(iv) Write the 3×3 matrix $\widehat{\mathsf{C}}$ explicitly for $C=y(z-x)$.

(v) Explain what non-uniqueness of this representation of the solutions arises because of the linear dependence among the three constants of motion.

(vi) Show that the system (5) is canonically Hamiltonian on level sets of C and H, by deriving the canonical Poisson brackets.

$\boxed{\textbf{Answer}}$

(A) For the dynamical system (5) in $\mathbb{R}^3$ to preserve volume, it must have zero divergence as a vector field,

$$\partial_x(\dot{x})+\partial_y(\dot{y})+\partial_z(\dot{z})=0\,,$$

in $\mathbb{R}^3$. This occurs when $\kappa+\lambda=-2$, as may be easily checked.

(B) (i) Transforming to the quadratic variables yields

$$\dot{p}_1=\dot{p}_2=\dot{p}_3=2xyz=2\sqrt{p_1p_2p_3}$$

(ii) These equations imply the three linearly dependent constants of motion

$$C_1 = p_2 - p_3 = x(z-y)\,, \quad C_2 = p_3 - p_1 = y(x-z)\,, \quad C_3 = p_1 - p_2 = z(y-x)$$

where $C_1 + C_2 + C_3 = 0$.

(iii) The system of motion equations (5) for $\kappa = -1 = \lambda$ may be written in $\mathbb{R}^3$ vector form (non-uniquely) as a cross product of gradients of these constants of the motion. For example,

$$\dot{\mathbf{x}} = \nabla C_1 \times \nabla C_3 = -\,\nabla C_1 \times \nabla C_2\,.$$

This expression means that the motion in $\mathbb{R}^3$ takes place along the intersections of level sets of the two constants of motion for the system C_1 and C_2, or along a linear combination of them since we have three linearly dependent constants of the ML motion. The system (5) is quadratic and the cross product of the gradients of any two of the quadratic functions C_1, C_2 and C_3 is also quadratic.

In what follows, we will simplify the notation by removing subscripts and make it suggestive of Hamiltonian dynamics by calling one of the constants of the motion H. Namely, for definiteness, we chose

$$-\,C_2 = y(z-x) = C \quad \text{and} \quad C_1 = x(z-y) = H\,,$$

and write

$$\dot{\mathbf{x}} = \nabla C \times \nabla H\,,$$

so the motion takes place along the intersection of their level sets.

The level sets of the quadratic functions C and H intersect in four planes through the origin. Their gradients are

$$\nabla C = (-y, z-x, y) \quad \text{and} \quad \nabla H = (z-y, -x, x)\,.$$

(iv) Consequently, they may be written in Hamiltonian form in two compatible ways:

$$\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} &= -\begin{bmatrix} 0 & y & x-z \\ -\,y & 0 & -\,y \\ z-x & y & 0 \end{bmatrix} \begin{bmatrix} z-y \\ -x \\ x \end{bmatrix} = \widehat{\mathsf{C}}\nabla H \\
&= \begin{bmatrix} 0 & y & x-z \\ -\,y & 0 & -\,y \\ z-x & y & 0 \end{bmatrix} \begin{bmatrix} -y \\ z-x \\ y \end{bmatrix} = -\,\widehat{\mathsf{H}}\nabla C = \begin{bmatrix} -\,x(x-y-z) \\ -\,y(y-z-x) \\ -\,z(z-x-y) \end{bmatrix}
\end{aligned}$$

(v) The representation of the motion equations is non-unique: Other solutions for the quadratic constants of motion comprise the linear combinations,

$$\dot{\mathbf{x}} = \nabla(\alpha C + \beta H) \times \nabla(\gamma C + \epsilon H) = (\alpha\epsilon - \beta\gamma)\nabla C \times \nabla H \quad \text{with} \quad \alpha\epsilon - \beta\gamma = 1.$$

And of course any functions of C and H are also constants of motion. As these parameters change subject the $SL(2, \mathbb{R}$ condition $\alpha\epsilon - \beta\gamma = 1$, the level sets can change shape drastically, but their intersections remain invariant.

(vi) To be Hamiltonian, the system (5) should be expressible in Poisson bracket form. In the present case, the motion in $\mathbb{R}^3$ summons the Nambu bracket,

$$\frac{dF(\mathbf{x})}{dt} = -\nabla C \cdot \nabla F \times \nabla H = \{F, H\}_C\,.$$

Writing this as

$$\{F, H\}_C\, dx \wedge dy \wedge dz = -\,dC \wedge dF \wedge dH$$

reveals that on a level surface of C or H the Poisson bracket would be canonical. The canonical brackets are found by using either one of the constants of motion to eliminate one of the original variables.

▲

References

[1] Llibre, J. and Valls, C. Polynomial, rational and analytic first integrals for a family of 3-dimensional Lotka-Volterra systems, Z. Angew. Math. Phys., 62: 761–777 (2011).

[2] May, R.M., Leonard, W.J.: Nonlinear aspects of competition between three species. SIAM J. Appl. Math. 29: 243– 256 (1975).

Exercise 2.4 (Charged particle moving on a sphere S^2 in a magnetic field (CPSB))

Recall that the Lagrangian for a charged particle of mass m in a magnetic field $\mathbf{B} = \mathrm{curl}\mathbf{A}$ in three Euclidean spatial dimensions $(\mathbf{q}, \dot{\mathbf{q}}) \in T_{\mathbf{q}}\mathbb{R}^3$ is

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}\dot{\mathbf{q}}\cdot\dot{\mathbf{q}} + \frac{e}{c}\dot{\mathbf{q}}\cdot\mathbf{A}(\mathbf{q}),$$

for constants m, e, c and prescribed function $\mathbf{A}(\mathbf{q})$. For a constant magnetic field in the $\hat{\mathbf{z}}$-direction, the vector potential is given by $\mathbf{A}(\mathbf{q}) = \frac{B_0}{2}\hat{\mathbf{z}}\times\mathbf{q}$. In this case, the Lagrangian is given by

$$L(q, \dot{q}) = \frac{m}{2}\dot{\mathbf{q}}\cdot\dot{\mathbf{q}} + \frac{e}{c}\,\dot{\mathbf{q}}\cdot\frac{B_0}{2}\hat{\mathbf{z}}\times\mathbf{q}\,.$$

The motion may be restricted to stay on a sphere by passing from a spatially fixed frame into a frame moving with a rotation $O(t) \in SO(3)$ that follows the particle. Then $\mathbf{q}(t) = O(t)\mathbf{q}_0$ and $\dot{\mathbf{q}}(t) = \dot{O}(t)\mathbf{q}_0$. In the moving frame the previous Lagrangian becomes

$$\begin{aligned}
L(O(t), \dot{O}(t)) &= \frac{m}{2}\dot{O}(t)\mathbf{q}_0\cdot\dot{O}(t)\mathbf{q}_0 + \frac{eB_0}{2c}\dot{O}(t)\mathbf{q}_0\cdot\hat{\mathbf{z}}\times O(t)\mathbf{q}_0\\
&= \frac{m}{2}\dot{O}(t)\mathbf{q}_0\cdot OO^{-1}\dot{O}(t)\mathbf{q}_0 + \frac{eB_0}{2c}\dot{O}(t)\mathbf{q}_0\cdot O(O^{-1}\hat{\mathbf{z}}\times\mathbf{q}_0)\\
&= \frac{1}{2}(\boldsymbol{\Omega}\times\mathbf{q}_0)\cdot(\boldsymbol{\Omega}\times\mathbf{q}_0) + \frac{eB_0}{2c}(\boldsymbol{\Omega}\times\mathbf{q}_0)\cdot(\boldsymbol{\Gamma}\times\mathbf{q}_0)\\
&= \frac{1}{2}\boldsymbol{\Omega}\cdot\mathcal{I}\boldsymbol{\Omega} + \frac{eB_0}{2c}\boldsymbol{\Omega}\cdot\mathcal{I}\boldsymbol{\Gamma} =: \ell(\boldsymbol{\Omega}, \boldsymbol{\Gamma})\,.
\end{aligned}$$

The new notation is $O^{-1}\dot{O}(t) = \boldsymbol{\Omega}\times$ with $\boldsymbol{\Omega} \in \mathbb{R}^3$ and $\boldsymbol{\Gamma} = O^{-1}\hat{\mathbf{z}} \in \mathbb{R}^3$.

(A) Derive the CPSB equations using Hamilton's principle with the Lagrangian $\ell(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$.

*Verify that the following equations hold (**Show your work!**)*

$$\frac{d}{dt}(\mathcal{I}\boldsymbol{\Omega}) + \boldsymbol{\Omega}\times\mathcal{I}\boldsymbol{\Omega} + \frac{eB_0}{2\,c}\Big(\underbrace{\boldsymbol{\Omega}\times\mathcal{I}\boldsymbol{\Gamma} + \mathcal{I}(\boldsymbol{\Gamma}\times\boldsymbol{\Omega}) + \boldsymbol{\Gamma}\times\mathcal{I}\boldsymbol{\Omega}}_{\textit{Magnetic torque}}\Big) = 0 \quad \textit{and} \quad \dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}\times\boldsymbol{\Omega}\,.$$

Answer

The CPSB motion equation follows from the Euler-Poincaré action principle $\delta S_{\mathrm{red}} = 0$ for a ***reduced action***,

$$S_{\mathrm{red}} = \int_a^b \ell(\boldsymbol{\Omega}, \boldsymbol{\Gamma})\,dt\,,$$

where variations of vectors $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ are restricted to be of the form

$$\delta\boldsymbol{\Omega} = \dot{\boldsymbol{\Sigma}} + \boldsymbol{\Omega}\times\boldsymbol{\Sigma}\,, \quad \delta\boldsymbol{\Gamma} = \boldsymbol{\Gamma}\times\boldsymbol{\Sigma}\,,$$

arising from variations of the corresponding definitions $O^{-1}\dot{O} = \widehat{\Omega}(t) = \boldsymbol{\Omega}\times$ and $\boldsymbol{\Gamma} = O^{-1}(t)\hat{\mathbf{z}}$ in which $O^{-1}\delta O = \widehat{\Sigma}(t) = \boldsymbol{\Sigma}\times$ is a curve in $\mathbb{R}^3$ that vanishes at the endpoints in time. For this reduced action, the Euler-Poincaré motion equation is

$$\frac{d}{dt}\frac{\partial \ell}{\partial \boldsymbol{\Omega}} + \boldsymbol{\Omega}\times\frac{\partial \ell}{\partial \boldsymbol{\Omega}} + \boldsymbol{\Gamma}\times\frac{\partial \ell}{\partial \boldsymbol{\Gamma}} = 0\,.$$

In this case, we have

$$S_{\rm red} = \int_a^b \ell(\boldsymbol{\Omega},\boldsymbol{\Gamma})\,dt = \int_a^b \frac{1}{2}\boldsymbol{\Omega}\cdot\mathcal{I}\,\boldsymbol{\Omega} + \frac{eB_0}{2c}\boldsymbol{\Gamma}\cdot\mathcal{I}\,\boldsymbol{\Omega}\,dt\,,$$

and one computes the derivatives

$$\boldsymbol{\Pi} = \frac{\partial \ell}{\partial \boldsymbol{\Omega}} = \mathcal{I}\,\boldsymbol{\Omega} + \frac{eB_0}{2c}\mathcal{I}\,\boldsymbol{\Gamma} \quad\text{and}\quad \frac{\partial \ell}{\partial \boldsymbol{\Gamma}} = \frac{eB_0}{2c}\mathcal{I}\,\boldsymbol{\Omega}\,.$$

Thus, the Euler-Poincaré motion equation becomes

$$\frac{d}{dt}\left(\mathcal{I}\,\boldsymbol{\Omega} + \frac{eB_0}{2c}\mathcal{I}\,\boldsymbol{\Gamma}\right) + \boldsymbol{\Omega}\times\left(\mathcal{I}\,\boldsymbol{\Omega} + \frac{eB_0}{2c}\mathcal{I}\,\boldsymbol{\Gamma}\right) + \frac{eB_0}{2c}\boldsymbol{\Gamma}\times\mathcal{I}\,\boldsymbol{\Omega} = 0\,,$$

or

$$\dot{\boldsymbol{\Pi}} - \boldsymbol{\Pi}\times\boldsymbol{\Omega} + \boldsymbol{\Gamma}\times\frac{eB_0}{2c}\mathcal{I}\,\boldsymbol{\Omega} = 0\,.$$

Upon using $\dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}\times\boldsymbol{\Omega}$ we find

$$\frac{d}{dt}\left(\mathcal{I}\,\boldsymbol{\Omega}\right) + \boldsymbol{\Omega}\times\mathcal{I}\,\boldsymbol{\Omega} + \frac{eB_0}{2\,c}\Big(\underbrace{\mathcal{I}(\boldsymbol{\Gamma}\times\boldsymbol{\Omega}) + \boldsymbol{\Omega}\times\mathcal{I}\,\boldsymbol{\Gamma} + \boldsymbol{\Gamma}\times\mathcal{I}\,\boldsymbol{\Omega}}_{\text{Magnetic torque}}\Big) = 0\,.$$

Then, as expected, kinetic energy is conserved,

$$\frac{d}{dt}\left(\frac{1}{2}\boldsymbol{\Omega}\cdot\mathcal{I}\,\boldsymbol{\Omega}\right) = 0.$$

[Geometric meaning of the magnetic torque]We define superscript $\flat$ (called flat, in music) as the metric adjoint with respect to $\mathcal{I}$. In particular, we denote

$$\mathcal{I}(\boldsymbol{\Omega}\times\boldsymbol{\Gamma}) = (\mathrm{ad}_\Omega\Gamma)^\flat$$

We also define $\mathrm{ad}^\dagger$ (called ad-dagger) by the expression

$$(\mathrm{ad}^\dagger_\Omega\,\Gamma)^\flat := \mathrm{ad}^*_\Omega(\Gamma^\flat) = -\,\boldsymbol{\Omega}\times\mathcal{I}\,\boldsymbol{\Gamma} \tag{7}$$

for any $\Omega,\Gamma\in\mathfrak{so}(3)\simeq\mathbb{R}^3$. In terms of $\flat$ and $\mathrm{ad}^\dagger$, the magnetic torque is given by

$$-\Big(\underbrace{\mathrm{ad}_\Omega\,\Gamma}_{\text{Skew}} + \underbrace{\mathrm{ad}^\dagger_\Omega\,\Gamma + \mathrm{ad}^\dagger_\Gamma\,\Omega}_{\text{Symmetric}}\Big)^\flat \tag{8}$$

The term in brackets also appears in the covariant derivative of one vector field in a Lie algebra by another one. ▲

(B) Find two constants of the motion for the CHTE equations.

$\boxed{\textit{Answer}}$

Since $\boldsymbol{\Gamma}$ is a unit vector, we have conservation of $|\boldsymbol{\Gamma}|^2$. In fact, this is clear from its evolution equation

$$\dot{\boldsymbol{\Gamma}} = \frac{d}{dt}O^{-1}(t)\hat{\mathbf{z}} = -\,O^{-1}\dot{O}\,\boldsymbol{\Gamma} = -\,\widehat{\Omega}\,\boldsymbol{\Gamma} = -\,\boldsymbol{\Omega}\times\boldsymbol{\Gamma}$$

However, this is trivial, so it doesn't count.

We also have conservation of the total kinetic energy,

$$E = \frac{1}{2}\boldsymbol{\Omega}\cdot\mathcal{I}\,\boldsymbol{\Omega}$$

which we expected because we know the Lorentz force does no work. However, we will also verify it directly and by Legendre transformation to the energy Hamiltonian.

Finally, we conservation of $\boldsymbol{\Gamma}\cdot\boldsymbol{\Pi} = \hat{\mathbf{z}}\cdot O(t)\boldsymbol{\Pi}$, which is the vertical ($\hat{\mathbf{z}}$) component of the spatial angular momentum $\boldsymbol{\pi} = O(t)\boldsymbol{\Pi}$ along the constant magnetic field $\hat{\mathbf{z}}B_0$. ▲

(C) *Derive the CPSB Hamiltonian $h(\boldsymbol{\Pi},\boldsymbol{\Gamma})$ and its variational derivatives, by Legendre-transforming $l(\boldsymbol{\Omega},\boldsymbol{\Gamma})$, the reduced Lagrangian for CPSB.*

Answer

With fibre derivative

$$\boldsymbol{\Pi} = \frac{\partial \ell}{\partial \boldsymbol{\Omega}} = \mathcal{I}\,\boldsymbol{\Omega} + \frac{eB_0}{2c}\mathcal{I}\,\boldsymbol{\Gamma}\,,$$

the Hamiltonian is computed from the Legendre transform as

$$h(\boldsymbol{\Pi},\boldsymbol{\Gamma}) = \boldsymbol{\Pi}\cdot\boldsymbol{\Omega} - \ell(\boldsymbol{\Omega},\boldsymbol{\Gamma}) = \frac{1}{2}\boldsymbol{\Omega}\cdot\mathcal{I}\boldsymbol{\Omega} = \frac{1}{2}\left(\boldsymbol{\Pi} - \frac{eB_0}{2c}\mathcal{I}\,\boldsymbol{\Gamma}\right)\cdot\mathcal{I}^{-1}\left(\boldsymbol{\Pi} - \frac{eB_0}{2c}\mathcal{I}\,\boldsymbol{\Gamma}\right)$$

and one computes the derivatives

$$\frac{\partial h}{\partial \boldsymbol{\Pi}} = \boldsymbol{\Omega} \quad\text{and}\quad \frac{\partial h}{\partial \boldsymbol{\Gamma}} = -\frac{eB_0}{2c}\mathcal{I}\mathcal{I}^{-1}\left(\boldsymbol{\Pi} - \frac{eB_0}{2c}\mathcal{I}\,\boldsymbol{\Gamma}\right) = -\frac{eB_0}{2c}\mathcal{I}\,\boldsymbol{\Omega}\,.$$

▲

(D) *Write the CPSB equations in Lie–Poisson bracket matrix form.*

Answer

The Euler-Poincaré motion equation and the auxiliary equation for $\boldsymbol{\Gamma}$ are given by

$$\dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi}\times\boldsymbol{\Omega} - \boldsymbol{\Gamma}\times\frac{eB_0}{2c}\mathcal{I}\,\boldsymbol{\Omega} \quad\text{and}\quad \dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}\times\boldsymbol{\Omega}\,.$$

These equations may be written in Hamiltonian form as

$$\begin{bmatrix}\dot{\boldsymbol{\Pi}}\\ \dot{\boldsymbol{\Gamma}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Pi}\times & \boldsymbol{\Gamma}\times\\ \boldsymbol{\Gamma}\times & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{\Omega}\\ -\frac{eB_0}{2c}\mathcal{I}\,\boldsymbol{\Omega}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Pi}\times & \boldsymbol{\Gamma}\times\\ \boldsymbol{\Gamma}\times & 0\end{bmatrix}\begin{bmatrix}\partial h/\partial\boldsymbol{\Pi}\\ \partial h/\partial\boldsymbol{\Gamma}\end{bmatrix}$$

Hence, Lie–Poisson bracket matrix form

$$\frac{d}{dt}f(\boldsymbol{\Pi},\boldsymbol{\Gamma}) = \left[\tfrac{\partial f}{\partial \boldsymbol{\Pi}}\,,\quad \tfrac{\partial f}{d\boldsymbol{\Gamma}}\right]\begin{bmatrix}\dot{\boldsymbol{\Pi}}\\ \dot{\boldsymbol{\Gamma}}\end{bmatrix} = \left[\tfrac{\partial f}{\partial \boldsymbol{\Pi}}\,,\quad \tfrac{\partial f}{d\boldsymbol{\Gamma}}\right]\begin{bmatrix}\boldsymbol{\Pi}\times & \boldsymbol{\Gamma}\times\\ \boldsymbol{\Gamma}\times & 0\end{bmatrix}\begin{bmatrix}\partial h/\partial\boldsymbol{\Pi}\\ \partial h/\partial\boldsymbol{\Gamma}\end{bmatrix} =: \{f,h\}$$

▲

Exercise 2.5 (Anisotropic harmonic oscillator on the sphere S^{n-1})

The motion of a particle of mass m undergoing anisotropic harmonic oscillations in $\mathbb{R}^n$ is governed by Hamilton's principle with the following Lagrangian

$$L(\mathbf{x},\dot{\mathbf{x}})=\frac{m}{2}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}-\frac{1}{2}\mathbf{x}\cdot\mathcal{K}_0\mathbf{x},$$

for $(\mathbf{x},\dot{\mathbf{x}})\in T_{\mathbf{x}}\mathbb{R}^n$ and a constant $n\times n$ symmetric matrix $\mathcal{K}_0$ that determines the spring constant in each direction.

One restricts the motion to stay on the S^{n-1} sphere by setting $\mathbf{x}(t)=O(t)\mathbf{x}_0$ and $\dot{\mathbf{x}}(t)=\dot{O}(t)\mathbf{x}_0$, with $(O,\dot{O})\in T_O SO(n)$.

(A) Show that for this type of motion the original Lagrangian becomes

$$\begin{aligned}\ell(\widehat{\Omega},\mathcal{K})&=\frac{m}{2}\mathrm{tr}\big((\widehat{\Omega}\mathbf{x}_0)^T(\widehat{\Omega}\mathbf{x}_0)\big)-\frac{1}{2}\mathrm{tr}\big(\mathbf{x}_0\mathbf{x}_0^T\mathcal{K}\big)\\&=\frac{m}{2}\mathrm{tr}\big(\mathcal{I}\widehat{\Omega}^T\widehat{\Omega}\big)-\frac{1}{2}\mathrm{tr}\big(\mathcal{I}\mathcal{K}\big)\end{aligned}$$

with

$$\mathcal{I}=\mathbf{x}_0\mathbf{x}_0^T\,,\quad \widehat{\Omega}(t)=O^{-1}\dot{O}(t)\in\mathfrak{so}(n)\quad\textit{and}\quad \mathcal{K}(t)=O^{-1}\mathcal{K}_0O(t)\,,$$

where $\mathcal{I}$ and $\mathcal{K}_0$ are $n\times n$ constant symmetric matrices.

(B) Derive the variational relations,

$$\delta\widehat{\Omega}=\frac{d\widehat{\Xi}}{dt}+\big[\,\widehat{\Omega}\,,\,\widehat{\Xi}\,\big]\qquad \delta\mathcal{K}=\big[\,\mathcal{K}\,,\,\widehat{\Xi}\,\big]\,.$$

(C) Compute the reduced Euler–Lagrange equations for the Lagrangian $\ell(\widehat{\Omega},\mathcal{K})$ by taking matrix variations in its Hamilton's principle $\delta S=0$ with $S=\int\ell(\widehat{\Omega},\mathcal{K})\,dt$, to find (with $m=1$)

$$\delta S=\frac{1}{2}\int_a^b\mathrm{tr}\big(M^T\delta\widehat{\Omega}\,\big)\,dt+\frac{1}{2}\int_a^b\mathrm{tr}\big(\,\Xi\,[\,\mathcal{K}\,,\,\mathcal{I}\,]\big)\,dt\,,$$

with matrix commutator $[\mathcal{K},\mathcal{I}]:=\mathcal{K}\mathcal{I}-\mathcal{I}\mathcal{K}$, variation $\Xi:=O^{-1}\delta O\in\mathfrak{so}(n)$ and variational derivative $M:=\partial l/\partial\Omega=\mathcal{I}\Omega+\Omega\mathcal{I}$.

(D) By integrating by parts, invoking homogeneous endpoint conditions, then rearranging, derive the following formula for the variation,

$$\delta S=-\frac{1}{2}\int_a^b\mathrm{tr}\bigg(\Big(\frac{dM}{dt}-[\,M\,,\,\Omega\,]-[\,\mathcal{K}\,,\,\mathcal{I}\,]\Big)\,\Xi\bigg)\,dt\,.$$

This means that Hamilton's principle for $\delta S=0$ with arbitrary Ξ implies an equation for the evolution of M given by

$$\frac{dM}{dt}=\big[\,M\,,\,\widehat{\Omega}\,\big]+\big[\,\mathcal{K}(t)\,,\,\mathcal{I}\,\big]\,.\tag{9}$$

(E) Derive a differential equation for $\mathcal{K}(t)$ from the time derivative of its definition $\mathcal{K}(t):=O^{-1}(t)\mathcal{K}_0O(t)$, as

$$\frac{d\mathcal{K}}{dt}=\big[\,\mathcal{K}\,,\,\widehat{\Omega}\,\big]\,.\tag{10}$$

The last two equations constitute a closed dynamical system for $M(t)$ and $\mathcal{K}(t)$, with initial conditions specified by the values of $\widehat{\Omega}(0)$ and $\mathcal{K}(0)=\mathcal{K}_0$ for $O(0)=\mathrm{Id}$ at time $t=0$.

(F) *Following Manakov's idea [Man1976], show that these equations may be combined into a commutator of polynomials,*

$$\frac{d}{dt}\big(\mathcal{K}+\lambda M+\lambda^2\mathcal{I}^2\big)=\big[\,\mathcal{K}+\lambda M+\lambda^2\mathcal{I}^2\,,\,\widehat{\Omega}+\lambda\mathcal{I}\,\big]\,. \tag{11}$$

(G) *Show that the commutator form (11) implies for every non-negative integer power K that*

$$\frac{d}{dt}(\mathcal{K}+\lambda M+\lambda^2\mathcal{I}^2)^K=[(\mathcal{K}+\lambda M+\lambda^2\mathcal{I}^2)^K,(\widehat{\Omega}+\lambda\mathcal{I})]\,.$$

(H) *Show that*

$$\mathrm{tr}(\mathcal{K}+\lambda M+\lambda^2\mathcal{I}^2)^K=\text{constant}\,, \tag{12}$$

for each power of λ. That is, all the coefficients of each power of λ are constant in time for the motion of a rigid body in a quadratic field.

$\boxed{\boxed{\textit{Answer}}}$ Since the commutator is antisymmetric, its trace vanishes and K conservation laws emerge, as

$$\frac{d}{dt}\mathrm{tr}(\mathcal{K}+\lambda M+\lambda^2\mathcal{I}^2)^K=0\,,$$

after commuting the trace operation with the time derivative. ▲