## 2 M3-4-5A16 Assessed Problems # 2: Do 4 out of 5 problems

**Exercise 2.1 (The fish: a quadratically nonlinear oscillator)**

*Consider the Hamiltonian dynamics on a symplectic manifold of a system comprising two real degrees of freedom, with real phase space variables $(x, y, \theta, z)$, symplectic form*

$$\omega = dx \wedge dy + d\theta \wedge dz$$

*and Hamiltonian*

$$H = \tfrac{1}{2}y^2 + x\left(\tfrac{1}{3}x^2 - z\right) - \tfrac{2}{3}z^{3/2}$$

*(A) Write the canonical Poisson bracket for this system.*

*(B) At what values of $x$, $y$ and $H$ does the system have stationary points in the $(x, y)$ plane?*

*(C) Propose a strategy for solving these equations. In what order should they be solved?*

*(D) Identify the constants of motion of this system and explain why they are conserved.*

*(E) Compute the associated Hamiltonian vector field $X_H$ and show that it satisfies*

$$X_H \,\lrcorner\, \omega = dH$$

*(F) Write the Poisson bracket that expresses the Hamiltonian vector field $X_H$ as a divergenceless vector field in $\mathbb{R}^3$ with coordinates $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Explain why this Poisson bracket satisfies the Jacobi identity.*

*(G) Identify the Casimir function for this $\mathbb{R}^3$ bracket. Show explicitly that it satisfies the definition of a Casimir function.*

*(H) Sketch a graph of the intersections of the level surfaces in $\mathbb{R}^3$ of the Hamiltonian and Casimir function. Show the directions of flow along these intersections. Identify the locations and types of any relative equilibria at the tangent points of these surfaces.*

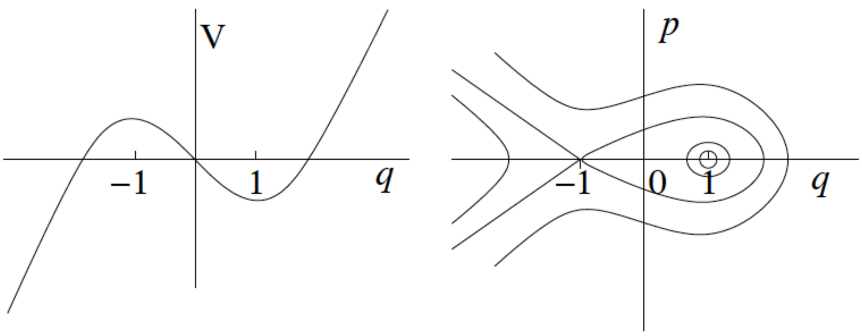


Figure 1: Phase plane for the saddle-node fish shape arising from the intersections of the level surfaces in $\mathbb{R}^3$ of the Hamiltonian and Casimir function.

*(I) Linearise around the relative equilibria on a level set of the Casimir $(z)$ and compute its eigenvalues.*

*(J) If you found a hyperbolic equilibrium point in the previous part connected to itself by a homoclinic orbit, then reduce the equation for the homoclinic orbit to an indefinite integral expression.*

**Exercise 2.2 (3D Volterra system)**

Consider the dynamical system in $(x_1, x_2, x_3) \in \mathbb{R}^3$,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_1x_2 \\ x_2x_3 - x_1x_2 \\ -x_2x_3 \end{bmatrix} = x_2 \begin{bmatrix} x_1 \\ x_3 - x_1 \\ -x_3 \end{bmatrix} \tag{1}$$

This is a 3D version of the Volterra [1931] model of competition among species, which for more species is given by

$$\dot{x}_n = x_n(x_{n+1} - x_{n-1}), \quad n = 1, 2, \dots, N, \quad \text{with} \quad x_0 = 0 = x_{N+1}.$$

The LV system is also widely applicable in physics and chemistry.

(A) Find two conservation laws for the system (1).

(B) Write the system (1) in terms of the cross product of the gradients of the two conserved quantities.

(C) Find the equilibrium states of the system and determine their stability.

(D) Write the system (1) in two Hamiltonian forms that use its two conserved quantities as Hamiltonians.

(E) Verify that this system may be written as $\frac{dL}{dt} = [L, B]$ for the $4 \times 4$ matrices

$$L := \begin{bmatrix} x_1 & 0 & \sqrt{x_1x_2} & 0 \\ 0 & x_1 + x_2 & 0 & \sqrt{x_2x_3} \\ \sqrt{x_1x_2} & 0 & x_2 + x_3 & 0 \\ 0 & \sqrt{x_2x_3} & 0 & x_3 \end{bmatrix} \quad B := \frac{1}{2} \begin{bmatrix} 0 & 0 & -\sqrt{x_1x_2} & 0 \\ 0 & 0 & 0 & -\sqrt{x_2x_3} \\ \sqrt{x_1x_2} & 0 & 0 & 0 \\ 0 & \sqrt{x_2x_3} & 0 & 0 \end{bmatrix}$$

(F) Explain how the conservation laws found earlier are related to the matrices $L$ and $B$.

(G) Give the geometrical interpretation of the formula $\frac{dL}{dt} = [L, B]$ with $4 \times 4$ matrices $L$ and $B$.

(H) Write the system (1) as a double matrix commutator, $\frac{dL}{dt} = [L, [L, N]]$. In particular, find $N$ explicitly.

(I) Give the geometrical interpretation of the formula $\frac{dL}{dt} = [L, [L, N]]$.

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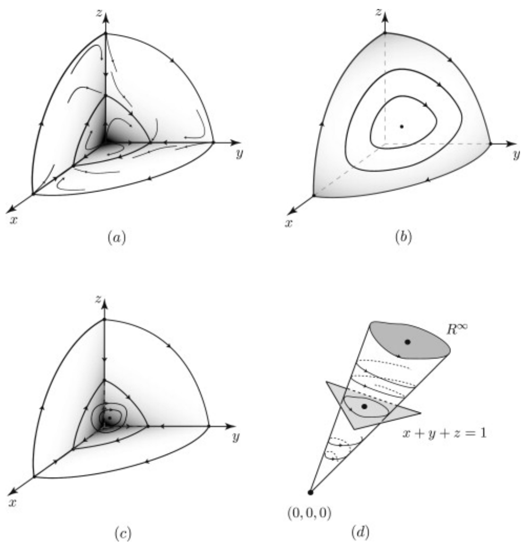


Figure 2: These are sketches of the global dynamics of the 3D May-Leonard system addressed in Problem 2.3 on the positive octant for $\kappa + \lambda = -2$ and $-1 < \kappa < 0$. Courtesy of Ref [1] in Problem 2.3.

**Exercise 2.3 (The 3D May-Leonard system)**

Consider the ***3D May-Leonard system*** governed by the equations:

$$
\begin{aligned}
\dot{x} &= -\,x(x+\kappa y+\lambda z)\,,\\
\dot{y} &= -\,y(y+\kappa z+\lambda x)\,,\\
\dot{z} &= -\,z(z+\kappa x+\lambda y)\,,
\end{aligned}
\tag{2}
$$

for two real constants $\kappa$ and $\lambda$. One notices its cyclic symmetry in $(x, y, z)$. This system describes nonlinear aspects of competition among three species [ML1975].

(A) Show that the system (2) preserves volume in $\mathbb{R}^3$ when $\kappa + \lambda = -\,2$.

(B) For the volume-preserving case $\kappa = -1 = \lambda$, system (2) becomes

$$
\begin{aligned}
\dot{x} &= -\,x(x-y-z)\,,\\
\dot{y} &= -\,y(y-z-x)\,,\\
\dot{z} &= -\,z(z-x-y)\,.
\end{aligned}
\tag{3}
$$

(i) Transform to quadratic variables

$$p_1 = yz\,,\quad p_2 = zx\,,\quad p_3 = xy$$

and find the equations for $\dot{p}_1, \dot{p}_2, \dot{p}_3$ implied by the ML equations (3).

(ii) Show that the equations for $\dot{p}_1, \dot{p}_2, \dot{p}_3$ imply three (linearly dependent) constants of motion.

(iii) Motivated by your result, find two real *quadratic* functions $C$ and $H$ for which the system (3) may be written in Nambu vector form as a cross product of their gradients in $\mathbb{R}^3$, i.e.,

$$\dot{\mathbf{x}} = \nabla C\times\nabla H = \widehat{\mathsf{C}}\,\nabla H = -\,\widehat{\mathsf{H}}\,\nabla C \quad\text{with}\quad \mathbf{x} = (x,y,z)^T \quad\text{and}\quad \widehat{\mathsf{C}} = \nabla C\times = -\,\widehat{\mathsf{C}}^T. \tag{4}$$

In index notation for vector components, $i, j, k = 1, 2, 3$, the first of these would be

$$\dot{x}_i = \widehat{C}_{ik}H_{,k} \quad\text{with}\quad \widehat{C}_{ik} = -\,\epsilon_{ikj}C_{,j} = -\,\widehat{C}_{ki}\,,$$

by the hat map.

(iv) Write the $3\times 3$ matrix $\widehat{\mathsf{C}}$ explicitly for $C = y(z-x)$.

(v) Explain what non-uniqueness of this representation of the solutions arises because of the linear dependence among the three constants of motion.

(vi) Show that the system (3) is canonically Hamiltonian on level sets of $C$ and $H$, by deriving the canonical Poisson brackets.

# References


[1] Blé, G., V. Castellanos, J. Llibre and I. Quilantán, Integrability and global dynamics of the May–Leonard model, Nonlinear Analysis: Real World Applications 14 (2013) 280–293

[2] Llibre, J. and Valls, C. Polynomial, rational and analytic first integrals for a family of 3-dimensional Lotka-Volterra systems, Z. Angew. Math. Phys., 62: 761–777 (2011).

[3] May, R.M., Leonard, W.J.: Nonlinear aspects of competition between three species. SIAM J. Appl. Math. 29: 243– 256 (1975).

**Exercise 2.4 (Charged particle moving on a sphere $S^2$ in a magnetic field (CPSB) )**

*Recall that the Lagrangian for a charged particle of mass $m$ in a magnetic field $\mathbf{B} = \mathrm{curl}\mathbf{A}$ in three Euclidean spatial dimensions $(\mathbf{q}, \dot{\mathbf{q}}) \in T_{\mathbf{q}}\mathbb{R}^3$ is*

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}\dot{\mathbf{q}}\cdot\dot{\mathbf{q}} + \frac{e}{c}\dot{\mathbf{q}}\cdot\mathbf{A}(\mathbf{q}),$$

*for constants $m$, $e$, $c$ and prescribed function $\mathbf{A}(\mathbf{q})$. For a constant magnetic field in the $\hat{\mathbf{z}}$-direction, the vector potential is given by $\mathbf{A}(\mathbf{q}) = \frac{B_0}{2}\hat{\mathbf{z}}\times\mathbf{q}$. In this case, the Lagrangian is given by*

$$L(q, \dot{q}) = \frac{m}{2}\dot{\mathbf{q}}\cdot\dot{\mathbf{q}} + \frac{e}{c}\,\dot{\mathbf{q}}\cdot\frac{B_0}{2}\hat{\mathbf{z}}\times\mathbf{q}\,.$$

*The motion may be restricted to stay on a sphere by passing from a spatially fixed frame into a frame moving with a rotation $O(t)\in SO(3)$ that follows the particle. Then $\mathbf{q}(t) = O(t)\mathbf{q}_0$ and $\dot{\mathbf{q}}(t) = \dot{O}(t)\mathbf{q}_0$. In the moving frame the previous Lagrangian becomes*

$$\begin{aligned}
L(O(t), \dot{O}(t)) &= \frac{m}{2}\dot{O}(t)\mathbf{q}_0\cdot\dot{O}(t)\mathbf{q}_0 + \frac{eB_0}{2c}\dot{O}(t)\mathbf{q}_0\cdot\hat{\mathbf{z}}\times O(t)\mathbf{q}_0\\
&= \frac{m}{2}\dot{O}(t)\mathbf{q}_0\cdot OO^{-1}\dot{O}(t)\mathbf{q}_0 + \frac{eB_0}{2c}\dot{O}(t)\mathbf{q}_0\cdot O(O^{-1}\hat{\mathbf{z}}\times\mathbf{q}_0)\\
&= \frac{1}{2}(\boldsymbol{\Omega}\times\mathbf{q}_0)\cdot(\boldsymbol{\Omega}\times\mathbf{q}_0) + \frac{eB_0}{2c}(\boldsymbol{\Omega}\times\mathbf{q}_0)\cdot(\boldsymbol{\Gamma}\times\mathbf{q}_0)\\
&= \frac{1}{2}\boldsymbol{\Omega}\cdot\mathcal{I}\,\boldsymbol{\Omega} + \frac{eB_0}{2c}\boldsymbol{\Omega}\cdot\mathcal{I}\,\boldsymbol{\Gamma} =: \ell(\boldsymbol{\Omega}, \boldsymbol{\Gamma})\,.
\end{aligned}$$

*The new notation is $O^{-1}\dot{O}(t) = \boldsymbol{\Omega}\times$ with $\boldsymbol{\Omega}\in\mathbb{R}^3$ and $\boldsymbol{\Gamma} = O^{-1}\hat{\mathbf{z}}\in\mathbb{R}^3$.*

*(A) Derive the CPSB equations using Hamilton's principle with the Lagrangian $\ell(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$.*

*Verify that the following equations hold (**Show your work!**)*

$$\frac{d}{dt}\left(\mathcal{I}\,\boldsymbol{\Omega}\right) + \boldsymbol{\Omega}\times\mathcal{I}\,\boldsymbol{\Omega} + \frac{eB_0}{2\,c}\Big(\underbrace{\boldsymbol{\Omega}\times\mathcal{I}\,\boldsymbol{\Gamma} + \mathcal{I}(\boldsymbol{\Gamma}\times\boldsymbol{\Omega}) + \boldsymbol{\Gamma}\times\mathcal{I}\,\boldsymbol{\Omega}}_{\textit{Magnetic torque}}\Big) = 0 \quad \textit{and} \quad \dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}\times\boldsymbol{\Omega}\,.$$

*(B) Find two constants of the motion for the CHTE equations.*

*(C) Derive the CPSB Hamiltonian $h(\boldsymbol{\Pi}, \boldsymbol{\Gamma})$ and its variational derivatives, by Legendre-transforming $l(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$, the reduced Lagrangian for CPSB.*

*(D) Write the CPSB equations in Lie–Poisson bracket matrix form.*

**Exercise 2.5 (Anisotropic harmonic oscillator on the sphere $S^{n-1}$)**

*The motion of a particle of mass $m$ undergoing anisotropic harmonic oscillations in $\mathbb{R}^n$ is governed by Hamilton's principle with the following Lagrangian*

$$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{m}{2}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} - \frac{1}{2}\mathbf{x}\cdot\mathcal{K}_0\mathbf{x},$$

*for $(\mathbf{x},\dot{\mathbf{x}}) \in T_{\mathbf{x}}\mathbb{R}^n$ and a constant $n\times n$ symmetric matrix $\mathcal{K}_0$ that determines the spring constant in each direction.*

*One restricts the motion to stay on the $S^{n-1}$ sphere by setting $\mathbf{x}(t) = O(t)\mathbf{x}_0$ and $\dot{\mathbf{x}}(t) = \dot{O}(t)\mathbf{x}_0$, with $(O,\dot{O}) \in T_OSO(n)$.*

*(A) Show that for this type of motion the original Lagrangian becomes*

$$\begin{aligned}\ell(\widehat{\Omega},\mathcal{K}) &= \frac{m}{2}\mathrm{tr}\big((\widehat{\Omega}\mathbf{x}_0)^T(\widehat{\Omega}\mathbf{x}_0)\big) - \frac{1}{2}\mathrm{tr}\big(\mathbf{x}_0\mathbf{x}_0^T\mathcal{K}\big)\\ &= \frac{m}{2}\mathrm{tr}\big(\mathcal{I}\widehat{\Omega}^T\widehat{\Omega}\big) - \frac{1}{2}\mathrm{tr}\big(\mathcal{I}\mathcal{K}\big)\end{aligned}$$

*with*

$$\mathcal{I} = \mathbf{x}_0\mathbf{x}_0^T\,,\quad \widehat{\Omega}(t) = O^{-1}\dot{O}(t) \in \mathfrak{so}(n) \quad\text{and}\quad \mathcal{K}(t) = O^{-1}\mathcal{K}_0 O(t)\,,$$

*where $\mathcal{I}$ and $\mathcal{K}_0$ are $n\times n$ constant symmetric matrices.*

*(B) Derive the variational relations,*

$$\delta\widehat{\Omega} = \frac{d\widehat{\Xi}}{dt} + \big[\,\widehat{\Omega}\,,\,\widehat{\Xi}\,\big] \qquad \delta\mathcal{K} = \big[\,\mathcal{K}\,,\,\widehat{\Xi}\,\big]\,.$$

*(C) Compute the reduced Euler–Lagrange equations for the Lagrangian $\ell(\widehat{\Omega},\mathcal{K})$ by taking matrix variations in its Hamilton's principle $\delta S = 0$ with $S = \int \ell(\widehat{\Omega},\mathcal{K})\,dt$, to find*

$$\delta S = \frac{1}{2}\int_a^b \mathrm{tr}\big(M^T\delta\widehat{\Omega}\,\big)\,dt + \frac{1}{2}\int_a^b \mathrm{tr}\big(\,\Xi\,[\,\mathcal{K}\,,\,\mathcal{I}\,]\big)\,dt\,,$$

*with matrix commutator $[\mathcal{K},\mathcal{I}] := \mathcal{K}\mathcal{I} - \mathcal{I}\mathcal{K}$, variation $\Xi := O^{-1}\delta O \in \mathfrak{so}(n)$ and variational derivative $M := \partial l/\partial\Omega = \mathcal{I}\Omega + \Omega\mathcal{I}$.*

*(D) By integrating by parts, invoking homogeneous endpoint conditions, then rearranging, derive the following formula for the variation,*

$$\delta S = -\frac{1}{2}\int_a^b \mathrm{tr}\bigg(\Big(\frac{dM}{dt} - [\,M\,,\,\Omega\,] - [\,\mathcal{K}\,,\,\mathcal{I}\,]\Big)\,\Xi\bigg)\,dt\,.$$

*This means that Hamilton's principle for $\delta S = 0$ with arbitrary $\Xi$ implies an equation for the evolution of $M$ given by*

$$\frac{dM}{dt} = \big[\,M\,,\,\widehat{\Omega}\,\big] + \big[\,\mathcal{K}(t)\,,\,\mathcal{I}\,\big]\,. \tag{5}$$

*(E) Derive a differential equation for $\mathcal{K}(t)$ from the time derivative of its definition $\mathcal{K}(t) := O^{-1}(t)\mathcal{K}_0O(t)$, as*

$$\frac{d\mathcal{K}}{dt} = \big[\,\mathcal{K}\,,\,\widehat{\Omega}\,\big]\,. \tag{6}$$

*The last two equations constitute a closed dynamical system for $M(t)$ and $\mathcal{K}(t)$, with initial conditions specified by the values of $\widehat{\Omega}(0)$ and $\mathcal{K}(0) = \mathcal{K}_0$ for $O(0) = \mathrm{Id}$ at time $t = 0$.*

(F) *Following Manakov's idea [Man1976], show that these equations may be combined into a commutator of polynomials,*

$$\frac{d}{dt}\big(\mathcal{K} + \lambda M + \lambda^2 \mathcal{I}^2\big) = \big[\,\mathcal{K} + \lambda M + \lambda^2 \mathcal{I}^2\,,\,\widehat{\Omega} + \lambda \mathcal{I}\,\big]\,. \tag{7}$$

(G) *Show that the commutator form (7) implies for every non-negative integer power $K$ that*

$$\frac{d}{dt}(\mathcal{K} + \lambda M + \lambda^2 \mathcal{I}^2)^K = [(\mathcal{K} + \lambda M + \lambda^2 \mathcal{I}^2)^K, (\widehat{\Omega} + \lambda \mathcal{I})]\,.$$

(H) *Show that*

$$\mathrm{tr}(\mathcal{K} + \lambda M + \lambda^2 \mathcal{I}^2)^K = \mathrm{constant}\,, \tag{8}$$

*for each power of $\lambda$. That is, all the coefficients of each power of $\lambda$ are constant in time for the motion of a rigid body in a quadratic field.*

$\boxed{\textit{Answer}}$ Since the commutator is antisymmetric, its trace vanishes and $K$ conservation laws emerge, as

$$\frac{d}{dt}\mathrm{tr}(\mathcal{K} + \lambda M + \lambda^2 \mathcal{I}^2)^K = 0\,,$$

after commuting the trace operation with the time derivative. ▲