2 M3-4-5A16 Assessed Problems # 2

Exercise 2.1 The Planar Circular Restricted Three-Body Problem (PCR3BP). Consider a comet (or space craft) moving in the plane of Jupiter's orbit in our solar system. Its orbit is mostly heliocentric, but it suffers perturbations due primarily to Jupiter's gravitational field. The effects of the other planets may be neglected.

The two main bodies, such as the Sun and Jupiter, are assigned masses $m_S = 1-\mu$ and $m_J = \mu$, so the total mass $m_S + m_J = 1$. Jupiter's orbit may be taken to be circular. Then the two main bodies rotate in the plane counterclockwise about their common centre of mass, with essentially constant angular velocity, which may be normalized to unity. The third body (the comet or the spacecraft) has a small mass that does not affect the motion, and is free to move in the plane. Let $\mathbf{x} = (x, y) \in \mathbb{R}^2$ be the position of the comet in the plane (Note that these are the position coordinates relative to the positions of the Sun and Jupiter in the rotating frame, not relative to an inertial frame).

The equations of motion of the comet in the rotating frame of Jupiter's orbit are then given by

$$\begin{aligned} \ddot{\mathbf{x}} + 2\widehat{\mathbf{z}} \times \dot{\mathbf{x}} &= \nabla\Phi(\mathbf{x}) \\ \text{with}\quad \Phi(\mathbf{x}) &= \frac{1}{2}|\mathbf{x}|^2 + \frac{\mu}{r_1} + \frac{1-\mu}{r_2}\,, \end{aligned} \tag{1}$$

where and r_1 and r_2 are the planar distances of the comet from the two main bodies.

(a) What is the 2nd term called in the first line of system (1)?

Answer. The 2nd term in the first line of system (1) is the Coriolis force due to moving in a rotating frame. ▲

(b) Explain each of the three terms in $\Phi(\mathbf{x})$.

Answer. The 1st term in $\Phi(\mathbf{x})$ is the potential for the (repulsive) centrifugal force. The 2nd and 3rd terms are potentials for the gravitational attraction of the comet to the two main bodies. ▲

(c) Find a first integral of the motion equations (1) and give its interpretation. This is called the Jacobi integral.

Answer. The Jacobi integral is also the energy

$$E = \frac{1}{2}|\dot{\mathbf{x}}|^2 - \Phi(\mathbf{x})$$

▲

(d) Derive equations (1) from Hamilton's principle.

Answer.

$$0 = \delta S = \delta \int_a^b L(\mathbf{x}, \dot{\mathbf{x}})\,dt = \delta \int_a^b \frac{1}{2}|\dot{\mathbf{x}}|^2 + \dot{\mathbf{x}}\cdot\mathbf{x}\times\widehat{\mathbf{z}} + \Phi(\mathbf{x})\,dt$$

▲

(e) Derive the Hamiltonian for which equations (1) are a canonical system on $T^*\mathbb{R}^2 \simeq \mathbb{R}^4$.

Answer. First, the canonical momentum is $\mathbf{p} := \frac{\partial L}{\partial \dot{\mathbf{x}}} = \dot{\mathbf{x}} + \mathbf{x}\times\widehat{\mathbf{z}}.$. Then the Legendre transform yields

$$H(\mathbf{x}, \mathbf{p}) = \mathbf{p}\cdot\dot{\mathbf{x}} - L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}|\mathbf{p} - \mathbf{x}\times\widehat{\mathbf{z}}|^2 - \Phi(\mathbf{x})$$

for which equations (1) follow as canonical Hamiltonian equations. ▲

(f) Identify the equilibrium solutions of equations (1), named after Lagrange as L_1, L_2, etc. For a hint, see `http://en.wikipedia.org/wiki/Lagrangian_point`. What balances of forces are producing these equilibrium points? (Remember, we are in a rotating frame.)

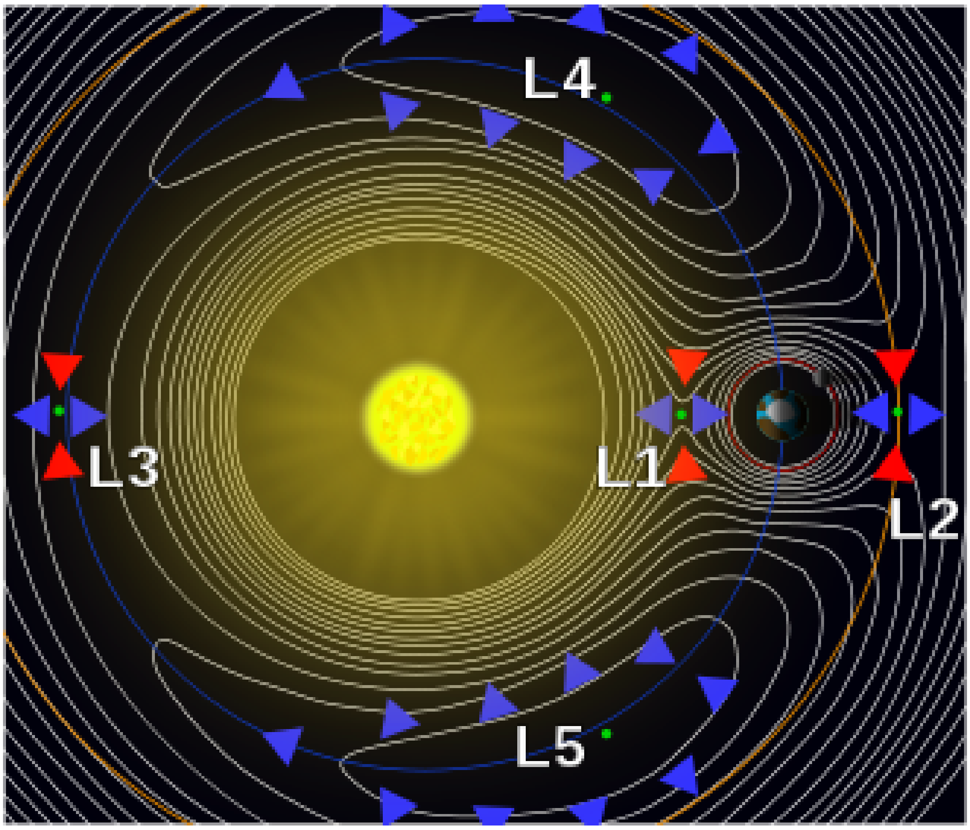

Figure 1: This Figure shows level sets of the Jacobi integral found in part (c).

$\boxed{\text{Answer}}$. The system (1) has five equilibrium points, 3 collinear ones on the x-axis, called L_1, L_2, L_3 and two equilateral points called L_4, L_5. (The latter two were found by Lagrange and are called Lagrange points.) See Figure 2. ▲

(g) Hill's region on the plane in defined as the manifold

$$M(\mu, E) = \{(x, y) | E + \Phi(x, y) \geq 0\} .$$

Show that the boundary of Hill's region $M(\mu, C)$ is a zero velocity curve. Why is this important? Sketch orbits along level sets of Jacobi integral that are just below that of L_2. Hint: For this case, notice that Hill's region contains a "neck" about L_1 and L_2 which open a *passage* for Hill's region to extend from the interior to the exterior of Jupiter's orbit. Take another look at Figure 1.

Answer.

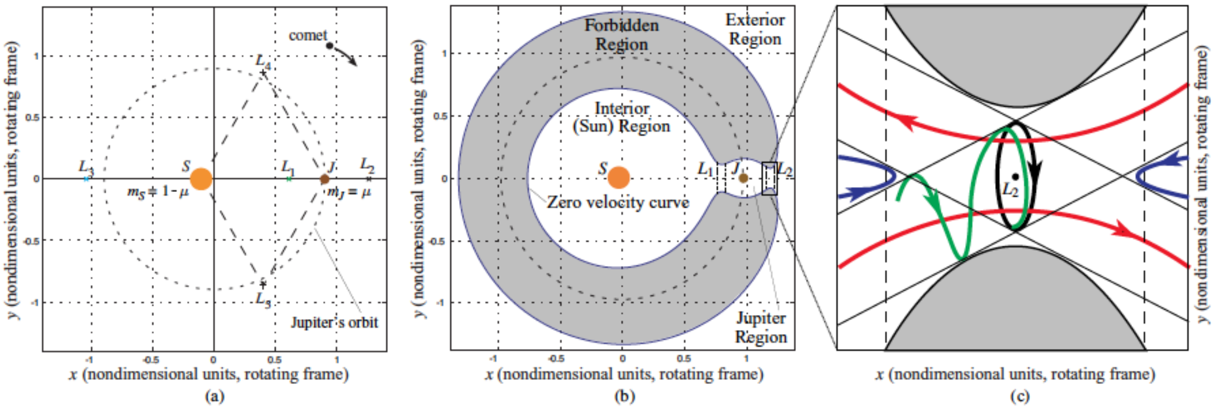

Figure 2: This Figure shows (a) equilibrium points of the PCR3BP as viewed in the rotating frame. (b) Hill's region (schematic, the region in white), which contains a "neck" about L_1 and L_2. (c) The flow in the region near L_2, showing a periodic orbit, a typical asymptotic orbit, two transit orbits and two non-transit orbits. A similar figure holds for the region around L_1.

This figure was taken from "Dynamical Systems, the Three-Body Problem and Space Mission Design", by Wang Sang Koon, Martin W. Lo, Jerrold E. Marsden, Shane D. Ross, *International Conference on Differential Equations, Berlin, 1999*, Edited by B. Fiedler, K. Gröger and J. Sprekels, World Scientific, 2000, 1167–1181.

BTW, Because the boundary of Hill's region $M(\mu, C)$ is a zero velocity curve, the comet can move only within this region in the (x, y)-plane.

▲

(h) What does the existence of the "neck" in Hill's region mean for escape orbits?

Answer. Existence of the "neck" in Hill's region for orbits whose energy is just below that of L_2 means that these orbits are energetically permitted to make a transit through the neck region from the interior region (inside Jupiter's orbit) to the exterior region (outside Jupiter's orbit) passing through the Hill's region near Jupiter. The same situation occurs for a space craft to be able to leave the Earth-Moon system.

To learn more, see the online book `http://www.cds.caltech.edu/~marsden/volume/missiondesign/KoLoMaRo_DMissionBook_2011-04-25.pdf`.

▲

□

Exercise 2.2 Recall the formula for the kinetic energy of a rigid body:

$$K(\mathcal{R}, \dot{\mathcal{R}}) = \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{x}_0)\left|\dot{\mathcal{R}}\mathbf{x}_0\right|^2 \mathrm{d}^3x_0$$

Here $\mathcal{R}$ is an orthogonal matrix (i.e. $\mathcal{R}\mathcal{R}^T = \mathbb{I}$), $\mathbf{x}_0 \in \mathcal{B} \subset \mathbb{R}^3$ and the motion in space is given by $\mathbf{x}(t) = \mathcal{R}(t)\mathbf{x}_0$ with $\mathcal{R}(0) = Id$.

(a) Show that $\mathcal{R}\mathbf{v}^{(1)} \cdot \mathcal{R}\mathbf{v}^{(2)} = \mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)}$ for two arbitrary vectors $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ in $\mathbb{R}^3$.

(b) Show that $\widehat{\Omega} = \mathcal{R}^{-1}\dot{\mathcal{R}}$ is antisymmetric. Hint: Take the time derivative of $\mathcal{R}^T\mathcal{R}(t) = \mathcal{R}^{-1}\mathcal{R} = Id$.

(c) For $\widehat{\Lambda} = \mathcal{R}^{-1}\delta\mathcal{R}$ and $\delta\mathcal{R}(t) = \frac{\partial}{\partial\epsilon}\mathcal{R}(t,\epsilon)|_{\epsilon=0}$ show that

$$\delta\widehat{\Omega} = \dot{\widehat{\Lambda}} + \left[\widehat{\Omega}, \widehat{\Lambda}\right]$$

(d) Use the inverse of the hat map to show that

$$K = \frac{1}{2}\mathbb{I}\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}$$

where $\mathbb{I}$ is the moment of inertia $\mathbb{I}_{ij} = \int_{\mathcal{B}} \rho(\mathbf{x}_0)\left(|\mathbf{x}_0|^2\delta_{ij} - x_{0\,i}\,x_{0\,j}\right)\mathrm{d}^3x_0$.

(e) Show that $\mathbb{I}$ is a symmetric covariant tensor by writing out its basis and changing variables.

(f) Show that

$$K = \frac{1}{2}\langle \mathbb{J}_0\widehat{\Omega}, \widehat{\Omega}\rangle$$

where $\mathbb{J}_0 = \int_{\mathcal{B}} \rho(\mathbf{x}_0)\,\mathbf{x}_0\mathbf{x}_0^T\,\mathrm{d}^3x_0$.

(g) Denote the identity matrix by $\mathbf{I}$ and show that

$$\mathbb{I} = \mathrm{Tr}(\mathbb{J}_0)\mathbf{I} - \mathbb{J}_0$$

(h) Define $\widehat{\omega} = \dot{\mathcal{R}}\mathcal{R}^{-1}$ and show that

$$K = \frac{1}{2}\langle \mathbb{J}\widehat{\omega}, \widehat{\omega}\rangle =: K(\widehat{\omega}, \mathbb{J})$$

where $\mathbb{J} = \mathcal{R}\mathbb{J}_0\mathcal{R}^{-1}$.

(i) Define $\widehat{\Gamma} = (\delta\mathcal{R})\mathcal{R}^{-1}$, with $\delta\mathcal{R}(t) = \frac{\partial}{\partial\epsilon}\mathcal{R}(t,\epsilon)|_{\epsilon=0}$ and verify that

$$\delta\mathbb{J} = \widehat{\Gamma}\mathbb{J} - \mathbb{J}\widehat{\Gamma} =: \left[\widehat{\Gamma}, \mathbb{J}\right]$$

(j) Show that

$$\delta\widehat{\omega} = \dot{\widehat{\Gamma}} + \left[\widehat{\Gamma}, \widehat{\omega}\right]$$

(k) Use the formulas above to write the Euler-Poincaré equation arising from the variation

$$0 = \delta S = \delta\int_{t_1}^{t_2} K(\widehat{\omega}, \mathbb{J})\,\mathrm{d}t$$

(l) Use the formulas above to write Euler-Poincaré equation arising from the variation

$$0 = \delta S = \delta\int_{t_1}^{t_2} K\,dt = \delta\int_{t_1}^{t_2} \frac{1}{2}\mathbb{I}\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}\,\mathrm{d}t$$

Compare the results of your variational formulas and Euler-Poincaré equations in the previous part. What is the difference between them?

(m) Perform the Legendre transformations and write the Hamiltonian formulations of the previous two parts.

□

Exercise 2.3 The standard theory of uniaxial molecules in liquid crystals endows each molecule with a director $\mathbf{n}$, so that $|\mathbf{n}|^2 \equiv 1$. In the presence of a (constant) external field $\mathbf{E}$, the Lagrangian of a fixed rotating molecule is written as follows:

$$L(\mathbf{n}, \dot{\mathbf{n}}) = \frac{1}{2}|\dot{\mathbf{n}}|^2 - \frac{1}{2}(\mathbf{n} \cdot \mathbf{E})^2$$

where all physical constants have been dropped for simplicity. Now, laboratory measurements ensure that the director $\mathbf{n}$ undergoes purely rotational dynamics, i.e.

$$\mathbf{n}(t) = \chi(t)\mathbf{n}_0 \qquad \text{where} \qquad \chi(t) \in SO(3)\,.$$

(a) Write the above Lagrangian as a function on the tangent bundle $TSO(3)$, i.e. $L : TSO(3) \to \mathbb{R}$, and determine whether the result is left or right invariant under rotations. What happens in the case $\mathbf{E} = 0$?

$\boxed{\text{Answer}}$. Under rotations from the right, the spatial vectors transform as $\mathbf{E} \to \mathbf{E}R$ and $\mathbf{n}_0 \to \mathbf{n}_0 R$

$$\mathbf{n} \cdot \mathbf{E} = \chi(t)\mathbf{n}_0 \cdot \mathbf{E} \to \chi(t)\mathbf{n}_0 R \cdot \mathbf{E}R = \mathrm{tr}\left((\chi(t)\mathbf{n}_0 R)^T \mathbf{E}R \right) = \mathrm{tr}\left(R^T \left(\chi(t)\mathbf{n}_0\right)^T \mathbf{E}R \right) = \mathbf{n} \cdot \mathbf{E}$$

Thus, the potential energy is right invariant, but not left invariant. When $\mathbf{E} = 0$ the kinetic energy remains, which is both left and right invariant. ▲

(b) Show that the above Lagrangian can be written in the form

$$L = \frac{1}{2}|\boldsymbol{\omega} \times \mathbf{n}|^2 - \frac{1}{2}(\mathbf{n} \cdot \mathbf{E})^2 =: l(\boldsymbol{\omega}, \mathbf{n})$$

where $\widehat{\omega} = \dot{\chi}\chi^{-1}$ is the *spatial* angular velocity.

$\boxed{\text{Answer}}$. Using $\mathbf{n}(t) = \chi(t)\mathbf{n}_0$ leads to

$$|\dot{\mathbf{n}}|^2 = |\dot{\chi}(t)\mathbf{n}_0 \cdot \dot{\chi}(t)\mathbf{n}_0|^2 = \left|\dot{\chi}\chi^{-1}\mathbf{n} \cdot \dot{\chi}\chi^{-1}\mathbf{n}\right|^2 = |\widehat{\omega}\mathbf{n} \cdot \widehat{\omega}\mathbf{n}|^2 = |\boldsymbol{\omega} \times \mathbf{n}|^2$$

▲

(c) In the theory of nematic liquid crystals one introduces the *conformation tensor* $\mathcal{J} = \mathbf{1} - \mathbf{n}\mathbf{n}^T$ as a new dynamical variable.

Verify the evolution relation

$$\mathcal{J}(t) = \chi(t)\mathcal{J}_0\chi(t)^T\,,$$

where $\mathcal{J}_0 = \mathbf{1} - \mathbf{n}_0\mathbf{n}_0^T$, and show that the Lagrangian $l(\boldsymbol{\omega}, \mathbf{n})$ can be written as

$$l = \frac{1}{2}\boldsymbol{\omega} \cdot \mathcal{J}\boldsymbol{\omega} + \frac{1}{2}\mathbf{E} \cdot \mathcal{J}\mathbf{E} =: \ell(\boldsymbol{\omega}, \mathcal{J})$$

(Recall that Lagrangian functions are defined up to additive constants.)

$\boxed{\text{Answer}}$.

$$\begin{aligned}
l(\boldsymbol{\omega}, \mathbf{n}) &= \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{n}) \cdot (\boldsymbol{\omega} \times \mathbf{n}) - \frac{1}{2}(\mathbf{n} \cdot \mathbf{E})(\mathbf{n} \cdot \mathbf{E}) \\
&= \frac{1}{2}\boldsymbol{\omega} \cdot (\mathbf{n} \times (\boldsymbol{\omega} \times \mathbf{n})) + \frac{1}{2}(\mathbf{E} \cdot \mathcal{J}\mathbf{E}) - \frac{1}{2}|\mathbf{E}|^2 \\
&= \frac{1}{2}\boldsymbol{\omega} \cdot \mathcal{J}\boldsymbol{\omega} + \frac{1}{2}\mathbf{E} \cdot \mathcal{J}\mathbf{E} - \frac{1}{2}|\mathbf{E}|^2 =: \ell(\boldsymbol{\omega}, \mathcal{J})
\end{aligned}$$

▲

(d) Verify the following relations

$$\delta\boldsymbol{\omega} = \dot{\boldsymbol{\Lambda}} + \boldsymbol{\Lambda} \times \boldsymbol{\omega}\,, \qquad \delta\mathcal{J} = \big[\,\widehat{\Lambda}, \mathcal{J}\,\big], \qquad \frac{d\mathcal{J}}{dt} = \big[\,\widehat{\omega}, \mathcal{J}\,\big], \qquad \frac{\partial \ell}{\partial \mathcal{J}} = \frac{1}{2}\big(\boldsymbol{\omega}\boldsymbol{\omega}^T - \mathbf{E}\mathbf{E}^T\big)$$

where $\widehat{\Lambda} := \delta\mathcal{R}\mathcal{R}^{-1}$ is the *spatial* angular variation.

Answer. All of these relations follow from straightforward calculations. ▲

(e) Given the property $\mathrm{d}\,\mathrm{Tr}(\mathcal{J})/\mathrm{d}t = \mathrm{Tr}(\mathrm{d}\mathcal{J}/\mathrm{d}t)$, verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}(\mathcal{J}^n) = 0$$

for arbitrary positive, integer values of n. Upon defining $c_n := \mathrm{Tr}(\mathcal{J}^n)$, use this result to explain why $\dot{\Omega}(c_n) = 0$ for any n and any function $\Omega : \mathbb{R} \to \mathbb{R}$.

Answer. The simplest proof is that

$$\mathrm{tr}\left(\mathcal{J}(t)^2\right) = \mathrm{tr}(\chi(t)\mathcal{J}_0\chi(t)^T\chi(t)\mathcal{J}_0\chi(t)^T) = \mathrm{tr}(\chi(t)\mathcal{J}_0^2\chi(t)^T) = \mathrm{tr}(\mathcal{J}_0^2)$$

Likewise, $\mathrm{tr}\left(\mathcal{J}(t)^n\right) = \mathrm{tr}(\mathcal{J}_0^n) = const.$ ▲

(f) Compute the equations of motion from Hamilton's principle $0 = \delta S = \delta\int_a^b \ell(\boldsymbol{\omega}, \mathcal{J})\,dt$.

Answer.

$$\begin{aligned} 0 = \delta S = \delta\int_a^b \ell(\boldsymbol{\omega}, \mathcal{J})\,dt &= \int_a^b \frac{\partial \ell}{\partial \boldsymbol{\omega}}\cdot\delta\boldsymbol{\omega} + \mathrm{tr}\left(\frac{\partial \ell}{\partial \mathcal{J}}\delta\mathcal{J}\right)dt \\ &= \int_a^b (\mathcal{J}\boldsymbol{\omega})\cdot\delta\boldsymbol{\omega} + \frac{1}{2}\left(\boldsymbol{\omega}\cdot\delta\mathcal{J}\boldsymbol{\omega} + \mathbf{E}\cdot\ \delta\mathcal{J}\mathbf{E}\right)dt \\ &= \int_a^b (\mathcal{J}\boldsymbol{\omega})\cdot(\dot{\boldsymbol{\Lambda}} + \boldsymbol{\Lambda}\times\boldsymbol{\omega}) + \frac{1}{2}\Big(\boldsymbol{\omega}\cdot\big[\,\widehat{\Lambda}, \mathcal{J}\,\big]\boldsymbol{\omega} + \mathbf{E}\cdot\big[\,\widehat{\Lambda}, \mathcal{J}\,\big]\mathbf{E}\Big)\,dt \end{aligned}$$

In a side calculation, we compute

$$\boldsymbol{\omega}\cdot\big[\,\widehat{\Lambda}, \mathcal{J}\,\big]\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\widehat{\Lambda}\mathcal{J}\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\mathcal{J}\widehat{\Lambda}\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\boldsymbol{\Lambda}\times(\mathcal{J}\boldsymbol{\omega}) - (\mathcal{J}\boldsymbol{\omega})\cdot\boldsymbol{\Lambda}\times\boldsymbol{\omega} = -2\boldsymbol{\Lambda}\cdot(\boldsymbol{\omega}\times\mathcal{J}\boldsymbol{\omega})$$

so in δS the quadratic terms in $\boldsymbol{\omega}$ *cancel* each other. Consequently, upon integrating by parts, we find

$$0 = \delta S = -\int_a^b \boldsymbol{\Lambda}\cdot\left(\frac{d}{dt}(\mathcal{J}\boldsymbol{\omega}) + (\mathbf{E}\times\mathcal{J}\mathbf{E})\right)dt + \Big[\boldsymbol{\Lambda}\cdot\mathcal{J}\boldsymbol{\omega}\Big]_a^b$$

Thus, the equations of motion are

$$\frac{d}{dt}(\mathcal{J}\boldsymbol{\omega}) + (\mathbf{E}\times\mathcal{J}\mathbf{E}) = 0 \quad \text{and} \quad \frac{d\mathcal{J}}{dt} = \big[\,\widehat{\omega}, \mathcal{J}\,\big]$$

When $\mathbf{E} = 0$, the motion equation becomes conservation of spatial angular momentum $\mathcal{J}\boldsymbol{\omega}$, as indicated by the Noether theorem coming from the endpoint terms. This is different from the rigid body though, because the moment of inertia, while fixed in the body frame, is changing in the spatial frame in which we are working.

▲

□

Exercise 2.4 (a) Let G be a matrix Lie group and let $h(t) \in G$ be a curve such that $h(0) = e$ and $\dot{h}(0) = \xi$. Then, fix $g \in G$ and $\eta \in \mathfrak{g} = T_eG$ and consider the adjoint operators

$$\mathrm{Ad}_g\,\xi := \frac{d}{dt}\Big|_{t=0} I_g h(t)$$

$$\mathrm{ad}_\xi\,\eta := \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{h(t)}\,\eta$$

where $I_g := ghg^{-1}$ is the conjugation operator. (Fully developed solutions are found in Holm, Schmah, Stoica [2009] *Geometric mechanics and Symmetry*, Oxford).

(i) Where does ξ live? Why? $\boxed{\text{Answer}}$. By its definition $\xi := \dot{h}(0) \in T_eG$. ▲

(ii) Specialize to $G = GL(n, \mathbb{R})$ and verify

$$\mathrm{Ad}_g\,\xi = g\xi g^{-1}\,, \qquad \mathrm{ad}_\xi\,\eta = \xi\eta - \eta\xi$$

(iii) Specialize to $G = SO(3)$ and change the notation to $g = R$ and $\xi = \widehat{\xi}$ (analogously $\eta = \widehat{\eta}$). Then verify

$$\mathrm{Ad}_R\,\widehat{\xi} = R\widehat{\xi}R^T\,, \qquad \mathrm{ad}_{\widehat{\xi}}\,\widehat{\eta} = \widehat{\xi}\widehat{\eta} - \widehat{\eta}\widehat{\xi}$$

(iv) Use the hat map $\widehat{\xi}\mathbf{v} = \boldsymbol{\xi} \times \mathbf{v}$ to identify the Lie algebras

$$(\mathfrak{so}(3), [\,,\,]) \simeq (\mathbb{R}^3, \times)$$

Show that

$$\mathrm{Ad}_R\,\widehat{\xi} = \widehat{R\boldsymbol{\xi}}\,, \qquad \mathrm{ad}_{\widehat{\xi}}\,\widehat{\eta} = \widehat{\boldsymbol{\xi}\times\boldsymbol{\eta}}$$

which implies

$$\mathrm{Ad}_R\,\boldsymbol{\xi} = R\boldsymbol{\xi}\,, \qquad \mathrm{ad}_{\boldsymbol{\xi}}\,\boldsymbol{\eta} = \boldsymbol{\xi}\times\boldsymbol{\eta}$$

Hint: Apply $\mathrm{Ad}_R\,\widehat{\xi}$ and $\mathrm{ad}_{\widehat{\xi}}\,\widehat{\eta}$ to an arbitrary vector. Then, in the second case, recall the Jacobi identity for cross products.

(b) Consider the dual coadjoint operators

$$\left\langle \mathrm{Ad}^*_g\,\mu, \eta \right\rangle := \langle \mu, \mathrm{Ad}_g\,\eta \rangle\,, \qquad \left\langle \mathrm{ad}^*_\xi\,\mu, \eta \right\rangle := \langle \mu, \mathrm{ad}_\xi\,\eta \rangle$$

(i) Show that when $G = GL(n, \mathbb{R})$, one has

$$\mathrm{Ad}^*_g\,\mu = g^T \mu g^{-T}, \qquad \mathrm{ad}^*_\xi\,\mu = -\left[\mu, \xi^T\right]$$

(ii) Find the coadjoint operators associated to

$$\mathrm{Ad}_R\,\boldsymbol{\xi} = R\boldsymbol{\xi} \qquad \mathrm{ad}_{\boldsymbol{\xi}}\,\boldsymbol{\eta} = \boldsymbol{\xi}\times\boldsymbol{\eta}$$

(c) Write the Euler-Poincaré equations of rigid body dynamics in terms of the skew-symmetric matrix $\widehat{\Omega} = \mathcal{R}^{-1}\dot{\mathcal{R}}$, where $\mathcal{R}(t) \in SO(3)$ is the primitive Lagrangian coordinate. *(See Exercise 2.2).*

(d) Consider a matrix Lie group G and a Lagrangian $L : TG \to \mathbb{R}$. Suppose that L is **right-invariant**. That is, suppose

$$L(g, \dot{g}) = L(gh, \dot{g}h)\,, \quad \forall h \in G\,.$$

Then, upon defining

$$L(g, \dot{g}) = L(gg^{-1}, \dot{g}g^{-1}) =: \ell(\xi)$$

with $\xi = \dot{g}g^{-1}$, consider the Euler-Poincaré variational principle

$$\delta \int_{t_1}^{t_2} \ell(\xi)\, dt = 0\,.$$

(i) Show that $\xi \in \mathfrak{g}$, the matrix Lie algebra of the matrix Lie group G.

(ii) Show that

$$\delta\xi = \dot{\nu} + [\nu, \xi]$$

where $\nu := (\delta g)g^{-1}$.

(iii) Show that this Euler-Poincaré variational principle yields the equations

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} + \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi} = 0$$

(Notice the difference in sign with respect to the Euler-Poincaré equations for left-invariant quantities)

$\square$