1 M3-4-5A16 Assessed Problems # 1: Do 4 out of 5 problems

Exercise 1.1 (Poisson brackets for the Hopf map)

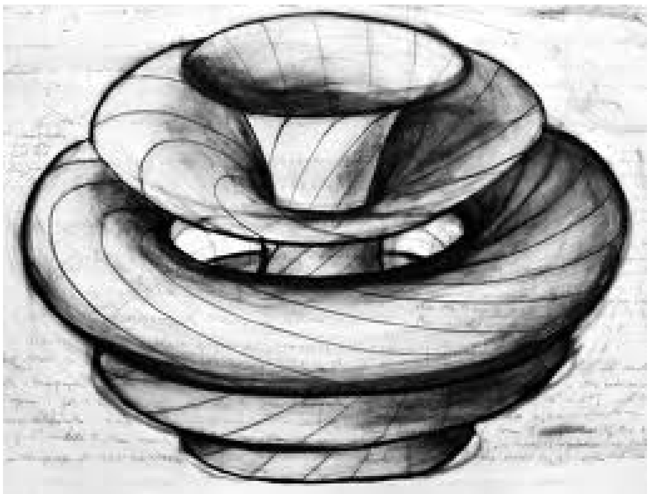

Figure 1: The Hopf map.

In coordinates $(a_1, a_2) \in \mathbb{C}^2$*, the Hopf map* $\mathbb{C}^2/S^1 \to S^3 \to S^2$ *is obtained by transforming to the four quadratic* S^1*-invariant quantities*

$$(a_1, a_2) \to Q_{jk} = a_j a_k^* \,, \quad \textit{with} \quad j, k = 1, 2\,.$$

Let the $\mathbb{C}^2$ *coordinates be expressed as*

$$a_j = q_j + ip_j$$

in terms of canonically conjugate variables satisfying the fundamental Poisson brackets

$$\{q_k, p_m\} = \delta_{km} \quad \textit{with} \quad k, m = 1, 2.$$

(A) Compute the Poisson brackets $\{a_j, a_k^*\}$ *for* $j, k = 1, 2$.

Answer The $\mathbb{C}^2$ coordinates $a_j = q_j + ip_j$ satisfy the Poisson bracket

$$\{a_j\,,\,a_k^*\} = -2i\,\delta_{jk}\,, \quad \text{for } j, k = 1, 2\,.$$

Likewise

$$da_j \wedge da_j^* = -2i\, dq_j \wedge dp_j$$

▲

(B) Is the transformation $(q, p) \to (a, a^*)$ *canonical? Explain why or why not.*

Hint: *a map* $(q, p) \to (Q, P)$ *whose Poisson bracket is* $\{Q, P\} = c\{q, p\}$ *with a constant factor* c *is still regarded as being canonical.*

Answer The transformation $(q, p) \mapsto (a, a^*)$ is indeed canonical. The constant $(-2i)$ is inessential for Hamiltonian dynamics, because it can be absorbed into the definition of time. ▲

(C) Compute the Poisson brackets among Q_{jk}*, with* $j, k = 1, 2$.

Answer The quadratic S^1 invariants on $\mathbb{C}^2$ given by $Q_{jk} = a_j a_k^*$ satisfy the Poisson bracket relations,

$$\{Q_{jk}\,,\,Q_{lm}\} \quad = \quad 2i\,(\delta_{kl}Q_{jm} - \delta_{jm}Q_{kl})\,, \quad j,k,l,m = 1,2\,.$$

Thus, they do close among themselves, but they do not satisfy canonical Poisson bracket relations.

▲

(D) *Make the linear change of variables,*

$$X_0 = Q_{11} + Q_{22}\,, \quad X_1 + iX_2 = 2Q_{12}\,, \quad X_3 = Q_{11} - Q_{22}\,,$$

and compute the Poisson brackets among (X_0, X_1, X_2, X_3).

Answer The quadratic S^1 invariants (X_0, X_1, X_2, X_3) given by

$$X_0 = Q_{11} + Q_{22}\,, \quad X_1 + iX_2 = 2Q_{12}\,, \quad X_3 = Q_{11} - Q_{22}\,,$$

may be expressed in terms of the a_j, $j = 1,2$ as

$$X_0 = |a_1|^2 + |a_2|^2\,, \quad X_1 + iX_2 = 2a_1a_2^*\,, \quad X_3 = |a_1|^2 - |a_2|^2\,.$$

These satisfy the Poisson bracket relations,

$$\{X_0\,,\,X_k\} = 0\,, \quad \{X_j\,,\,X_k\} = -4\epsilon_{jkl}X_l$$

▲

(E) *Express the Poisson bracket* $\{F(\mathbf{X}), H(\mathbf{X})\}$ *in vector form among functions* F *and* H *of* $\mathbf{X} = (X_1, X_2, X_3)$.

Answer The Poisson bracket $\{F(\mathbf{X}), H(\mathbf{X})\}$ is given in vector form as

$$\{F(\mathbf{X}), H(\mathbf{X})\} = -\,4\mathbf{X}\cdot\frac{\partial F}{\partial \mathbf{X}}\times\frac{\partial H}{\partial \mathbf{X}}\,.$$

It's the same as the Poisson bracket for the rigid body.

▲

(F) *Show that the quadratic invariants* (X_0, X_1, X_2, X_3) *themselves satisfy a quadratic relation. How is this relevant to the Hopf map?*

Answer The quadratic invariants (X_0, X_1, X_2, X_3) satisfy the quadratic relation

$$X_0^2(\mathbf{X}) = X_1^2 + X_2^2 + X_3^2 = |\mathbf{X}|^2\,.$$

This relation is relevant. It completes the Hopf map, because level sets of X_0 are spheres $S^2 \in S^3$. Moreover, it is relevant to the Poisson bracket in vector form above, which may be written using this relation as

$$\{F(\mathbf{X}), H(\mathbf{X})\} = -\,\frac{1}{2}\frac{\partial X_0^2}{\partial \mathbf{X}}\cdot\frac{\partial F}{\partial \mathbf{X}}\times\frac{\partial H}{\partial \mathbf{X}}\,.$$

▲

Exercise 1.2 (Motion on a sphere)

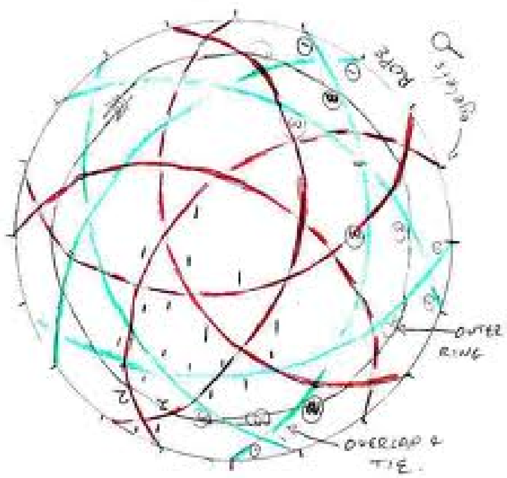

Figure 2: Motion on a sphere.

Motion on a sphere: Part 1, the constraint

Consider Hamilton's principle for the following constrained Lagrangian on $T\mathbb{R}^3$,

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 - \frac{\mu}{2}\left(1 - \|\mathbf{q}\|^2\right) .$$

*Here the quantity μ is called a **Lagrange multiplier** and must be determined as part of the solution.*

Provide a geometric mechanics description of the dynamical system governed by this Lagrangian. In particular, compute the following for it.

1. *Fibre derivative*
2. *Euler-Lagrange equations*
3. *Hamiltonian and canonical equations*
4. *Discussion of solutions. In particular, show that the solutions really do describe motion on a sphere. Do the initial conditions matter?*

$\boxed{\textit{Answer}}$

(i) **Fibre derivative**

This constrained Lagrangian is plainly hyperregular, because

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}}$$

which of course is an isomorphism.

(ii) **Euler-Lagrange equations**

$$\ddot{\mathbf{q}} = \mu \mathbf{q}$$

The motion must preserve $1 - \|\mathbf{q}\|^2 = 0$ and its time derivative $\mathbf{q} \cdot \dot{\mathbf{q}} = 0$ for the initial condition $\mathbf{q}_0 \cdot \mathbf{v}_0 = 0$. Hence one requires

$$\frac{d}{dt}(\mathbf{q} \cdot \dot{\mathbf{q}}) = |\dot{\mathbf{q}}|^2 + \mathbf{q} \cdot \ddot{\mathbf{q}} = 0 \quad \text{by which} \quad \ddot{\mathbf{q}} = \mu \mathbf{q} \quad \text{implies} \quad \mu = -\|\dot{\mathbf{q}}\|^2 / \|\mathbf{q}\|^2 \, .$$

Hence, the Euler-Lagrange equation for this constrained Lagrangian is

$$\ddot{\mathbf{q}} = -\,\frac{\|\dot{\mathbf{q}}\|^2}{\|\mathbf{q}\|^2}\,\mathbf{q}$$

The right side of this equation is the *centripetal force*, required for keeping the motion on the sphere, given by the constraint $1 - \|\mathbf{q}\|^2 = 0$.

(iii) **Hamiltonian and canonical equations**
The corresponding Hamiltonian is obtained by the Legendre transformation as,

$$H(\mathbf{q},\mathbf{p}) = \frac{1}{2}\|\mathbf{p}\|^2 + \frac{\mu}{2}\left(1 - \|\mathbf{q}\|^2\right)\,, \tag{1}$$

in which the variable $\mathbf{p}$ is the momentum canonically conjugate to the position $\mathbf{q}$. The canonical equations are

$$\begin{aligned}\dot{\mathbf{q}} &= \{\mathbf{q}, H\} = \frac{\partial H}{\partial \mathbf{p}} = \mathbf{p}\,,\\ \dot{\mathbf{p}} &= \{\mathbf{p}, H\} = -\,\frac{\partial H}{\partial \mathbf{q}} = \mu\mathbf{q}\,,\end{aligned}$$

and we must re-determine $\mu = -\,\|\dot{\mathbf{q}}\|^2/\|\mathbf{q}\|^2$ as before.

(iv) The energy corresponding to this Hamiltonian, when evaluated on the constraint is just the kinetic energy

$$E = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 + \frac{\mu}{2}\left(1 - \|\mathbf{q}\|^2\right) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2$$

and it is conserved, as may be checked from the equation of motion, which implies

$$\frac{d}{dt}(\log(\|\dot{\mathbf{q}}\|^2\|\mathbf{q}\|^2) = 0$$

and the result follows since $\|\mathbf{q}\|^2 = 1$ on the constraint.

The motion describes a great circle on S^2, since

$$\frac{d}{dt}(\mathbf{q}\times\dot{\mathbf{q}}) = \mathbf{q}\times\ddot{\mathbf{q}} = 0$$

that is,

$$\mathbf{q}\times\dot{\mathbf{q}}(t) = \mathbf{q}_0\times\mathbf{v}_0,$$

which says that $\dot{\mathbf{q}}(t)$ is always in the plane perpendicular to $\mathbf{q}_0\times\mathbf{v}_0$ and moving on the sphere if initially $\mathbf{q}_0\cdot\mathbf{v}_0 = 0$. Thus, since $\mathbf{q}(t)$ stays on the sphere in a plane passing through the origin, it is a great circle, which is a geodesic on S^2.

▲

Motion on a sphere: Part 2, the penalty

Provide the same kind of geometric mechanics description of the dynamical system governed by the Lagrangian

$$L_\epsilon(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 - \frac{1}{4\epsilon}(1 - \|\mathbf{q}\|^2)^2$$

for a particle with coordinates $\mathbf{q} \in \mathbb{R}^3$ *and constants* $\epsilon > 0$. *For this, let* $\gamma_\epsilon(t)$ *be the curve in* $\mathbb{R}^3$ *obtained by solving the Euler-Lagrange equations for* L_ϵ *with the initial conditions* $\mathbf{q}_0 = \gamma_\epsilon(0)$, $\mathbf{v}_0 = \dot{\gamma}_\epsilon(0)$. *Show that*

$$\lim_{\epsilon \to 0} \gamma_\epsilon(t) = \gamma_0(t)$$

traverses a great circle on the two-sphere S^2, *provided that* $\mathbf{q}_0$ *has unit length and that* $\mathbf{q}_0 \cdot \mathbf{v}_0 = 0$.

Hint: *go through the same first three steps as in the constrained case. Then use the conserved energy to show that*

$$1 - \|\gamma_\epsilon(t)\|^2 \to 0 \quad \textit{as} \quad \epsilon \to 0.$$

Answer

(i) **Fibre derivative**

The fibre derivative gives a linear relation

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}}$$

This means the Lagrangian is hyperregular.

(ii) **Euler-Lagrange equations**

$$\ddot{\mathbf{q}} = \frac{1}{\epsilon}(1 - \|\mathbf{q}\|^2)\mathbf{q}\,. \tag{2}$$

Obviously this equation can only make sense if the ratio $(1 - \|\mathbf{q}\|^2)/\epsilon$ converges as $\epsilon \to 0$. The conserved energy will help us deal with this issue in a moment.

(iii) **Hamiltonian and canonical equations**

The corresponding Hamiltonian is obtained by the Legendre transformation as,

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\|\mathbf{p}\|^2 + \frac{1}{4\epsilon}(1 - \|\mathbf{q}\|^2)^2\,, \tag{3}$$

in which the variable $\mathbf{p}$ is the momentum canonically conjugate to the position $\mathbf{q}$. The canonical equations are

$$\begin{aligned}
\dot{\mathbf{q}} &= \{\mathbf{q}, H\} = \frac{\partial H}{\partial \mathbf{p}} = \mathbf{p}\,, \\
\dot{\mathbf{p}} &= \{\mathbf{p}, H\} = -\frac{\partial H}{\partial \mathbf{q}} = \frac{1}{\epsilon}(1 - \|\mathbf{q}\|^2)\mathbf{q}\,,
\end{aligned}$$

and the Hamiltonian H is conserved.

(iv) **Discussion of motion and convergence in the limit as** $\epsilon \to 0$

The energy E_ϵ for the Lagrangian L_ϵ is conserved. In terms of the solution curve $\gamma_\epsilon(t) \in \mathbb{R}^3$, this energy is expressed as

$$\frac{1}{2}\|\dot{\gamma}_\epsilon(t)\|^2 + \frac{1}{4\epsilon}\left(1 - \|\gamma_\epsilon(t)\|^2\right)^2 = \frac{1}{2}\|\mathbf{v}_0\|^2.$$

In particular, this says that $\|\dot{\gamma}_\epsilon(t)\| \le \|\mathbf{v}_0\|$ for all t. Also, from the above equation we conclude that

$$\epsilon\|\dot{\gamma}_\epsilon(t)\|^2 + \frac{1}{2}\left(1 - \|\gamma_\epsilon(t)\|^2\right)^2 = \epsilon\|\mathbf{v}_0\|^2\,.$$

Therefore, $\|\dot{\gamma}_\epsilon(t)\|^2 \leq \|\mathbf{v}_0\|^2$ is bounded by its initial value, so taking the limit as $\epsilon \to 0$ gives $\epsilon\|\dot{\gamma}_\epsilon(t)\|^2 \to 0$. Consequently, the expression for energy gives

$$1 - \|\gamma_\epsilon(t)\|^2 \to 0 \quad \text{as} \quad \epsilon \to 0,$$

that is, $\|\lim_{\epsilon\to 0}\gamma_\epsilon(t)\| = 1$. Thus, $\lim_{\epsilon\to 0}\gamma_\epsilon(t) = \gamma_0(t)$ exists at any fixed time t and lies on the unit sphere S^2. It remains to check that Lagrange's equations continue to make sense in the limit as $\epsilon \to 0$.

Inserting the conserved energy into Lagrange's equations for this problem yields

$$\ddot{\gamma}_\epsilon = \frac{1}{\epsilon}(1 - \|\gamma_\epsilon\|^2)\gamma_\epsilon = \left(\sqrt{\|\mathbf{v}_0\|^2 - \|\dot{\gamma}_\epsilon\|^2}\right)\gamma_\epsilon =: F(\gamma_\epsilon, \dot{\gamma}_\epsilon).$$

Since the equation $\ddot{\gamma}_\epsilon = F(\gamma_\epsilon, \dot{\gamma}_\epsilon)$ is smooth in ϵ as $\epsilon \to 0$, continuous dependence of solutions on parameters (a general fact from theory of ordinary differential equations) shows that γ_ϵ does indeed converge as $\epsilon \to 0$. Observe that

$$\frac{d}{dt}(\gamma_\epsilon \times \dot{\gamma}_\epsilon) = \gamma_\epsilon \times \ddot{\gamma}_\epsilon = 0$$

that is,

$$\gamma_\epsilon(t) \times \dot{\gamma}_\epsilon(t) = \gamma_\epsilon(0) \times \dot{\gamma}_\epsilon(0) = \mathbf{q}_0 \times \mathbf{v}_0.$$

Thus,

$$\gamma_0(t) \times \dot{\gamma}_0(t) = \mathbf{q}_0 \times \mathbf{v}_0,$$

which says that $\dot{\gamma}_0(t)$ is always in the plane perpendicular to $\mathbf{q}_0 \times \mathbf{v}_0$. Thus, since $\gamma_0(t)$ stays on the sphere in a plane passing through the origin, it is a great circle, which is a geodesic on S^2.

▲

Exercise 1.3 (The free particle in $\mathbb{H}^2$: #1)

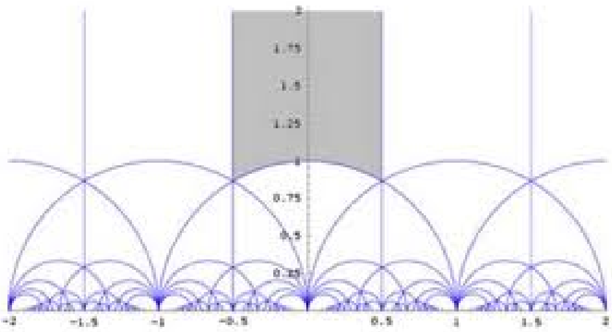

Figure 3: Geodesics on the Lobachevsky half-plane.

In Appendix I of Arnold's book, Mathematical Methods of Classical Mechanics*, page 303, we read.*

> *EXAMPLE. We consider the upper half-plane $y > 0$ of the plane of complex numbers $z = x + iy$ with the metric*
> $$ds^2 = \frac{dx^2 + dy^2}{y^2}\,.$$
> *It is easy to compute that the geodesics of this two-dimensional riemannian manifold are circles and straight lines perpendicular to the x-axis. Linear fractional transformations with real coefficients*
> $$z \to \frac{az+b}{cz+d} \tag{4}$$
> *are isometric transformations of our manifold ($\mathbb{H}^2$), which is called the* Lobachevsky plane.[1]

Consider a free particle of mass m moving on $\mathbb{H}^2$. Its Lagrangian is the kinetic energy corresponding to the Lobachevsky metric Namely,

$$L = \frac{m}{2}\left(\frac{\dot{x}^2 + \dot{y}^2}{y^2}\right). \tag{5}$$

(A) (1) Write the fibre derivatives of the Lagrangian (5) and

(2) compute its Euler-Lagrange equations.

These equations represent geodesic motion on $\mathbb{H}^2$.

(3) Evaluate the Christoffel symbols.

Answer

Fibre derivatives:

$$\frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{y^2} =: p_x \quad \text{and} \quad \frac{\partial L}{\partial \dot{y}} = \frac{m\dot{y}}{y^2} =: p_y$$

Euler-Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y}$ yield, respectively:

$$\frac{d}{dt}\left(\frac{\dot{x}}{y^2}\right) = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\dot{y}}{y^2}\right) = -\,\frac{\dot{x}^2 + \dot{y}^2}{y^3} \tag{6}$$

Expanding these equations yield the Christoffel symbols for the geodesic motion,

$$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0 \quad \text{and} \quad \ddot{y} + \frac{1}{y}\dot{x}^2 - \frac{1}{y}\dot{y}^2 = 0 \quad \Longleftrightarrow \quad \Gamma^1_{12} = -\,\frac{2}{y}\,, \quad \Gamma^2_{11} = \frac{1}{y}\,, \quad \Gamma^2_{22} = -\,\frac{1}{y}\,.$$

▲

[1] These isometric transformations of $\mathbb{H}^2$ have deep significance in physics. They correspond to the most general Lorentz transformation of space-time.

(B) **Hint:** *The Lagrangian in (5) is invariant under the group of linear fractional transformations with real coefficients. These have an $SL(2,\mathbb{R})$ matrix representation*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d} \tag{7}$$

(1) Show that the quantities

$$u = \frac{\dot{x}}{y} \quad \text{and} \quad v = \frac{\dot{y}}{y} \tag{8}$$

are invariant under a subgroup of these symmetry transformations.

(2) Specify this subgroup in terms of the representation (7).

Answer

The quantities (8) are invariant under a subgroup of translations and scalings.

$$\begin{aligned} &T_\tau : (x,y) \mapsto (x+\tau, y) \qquad \text{Flow of } X_T = \partial_x, \quad (\delta x, \delta y) = (1,0), \quad [X_T, X_S] = X_T. \\ &S_\sigma : (x,y) \mapsto (e^\sigma x, e^\sigma y) \qquad \text{Flow of } X_S = x\partial_x + y\partial_y, \quad (\delta x, \delta y) = (x,y). \end{aligned}$$

These transformations are translations T along the x axis and scalings S centered at $(x,y) = (0,0)$. They are represented by elements of (7) as

$$T = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

That is, the transformations T and S are isometries of the metric $ds^2 = (dx^2 + dy^2)/y^2$ on $\mathbb{H}^2$ with $T:\ a = 1 = d, c = 0, b \neq 0$ and $S: a \neq 0, b = 0 = c,\ d = 1$.

▲

(C) (1) Use the invariant quantities (u,v) in (8) as new variables in Hamilton's principle.

Hint: *the transformed Lagrangian is*

$$\ell(u,v) = \frac{m}{2}(u^2 + v^2)\,.$$

(2) Find the corresponding conserved Noether quantities.

Answer

(1) The translations T along the x axis and scalings S centered at $(x,y) = (0,0)$ leave invariant the quantities

$$u = \frac{\dot{x}}{y} \quad \text{and} \quad v = \frac{\dot{y}}{y}\,,$$

in terms of which the Lagrangian L in (5) reduces to

$$\ell(u,v) = \frac{m}{2}(u^2 + v^2)\,.$$

The reduced Hamilton's principle in the variables u and v yields,

$$\begin{aligned} 0 = \delta S = \delta \int_a^b \ell(u,v)\,dt &= \int_a^b m(u\delta u + v\delta v)\,dt \\ &= m \int_a^b \frac{u}{y}(\delta\dot{x} - u\delta y) + \frac{v}{y}(\delta\dot{y} - v\delta y)\,dt \\ &= -\,m \int_a^b \left(\frac{d}{dt}\frac{u}{y}\right)\delta x + \left(\frac{d}{dt}\frac{v}{y} + \frac{u^2+v^2}{y}\right)\delta y\,dt + m\left[\frac{u}{y}\,\delta x + \frac{v}{y}\,\delta y\right]_a^b \end{aligned}$$

Thus, Hamilton's principle recovers equations (6) in the variables u and v.

(2) Applying Noether's theorem to the endpoint term in these variables yields conservation of

$$C_T = \frac{u}{y}, \text{ for } (\delta x, \delta y) = (1, 0) \text{ translations,}$$

and

$$C_S = \frac{ux}{y} + v \text{ for } (\delta x, \delta y) = (x, y) \text{ scaling.}$$

▲

(D) Transform the Euler-Lagrange equations from x and y to the variables u and v that are invariant under the symmetries of the Lagrangian.

Then:

(1) Show that the resulting system conserves the kinetic energy expressed in these variables.

(2) Discuss its integral curves and critical points in the uv plane.

(3) Show that the u and v equations can be integrated explicitly in terms of sech and tanh.

Hint: *In the uv variables, the Euler-Lagrange equations for the Lagrangian (5) are expressed as*

$$\frac{d}{dt}\frac{u}{y} = 0 \quad \textit{and} \quad \frac{d}{dt}\frac{v}{y} + \frac{u^2 + v^2}{y} = 0\,.$$

Expanding these equations using $u = \dot{x}/y$ and $v = \dot{y}/y$ yields

$$\dot{u} = uv\,, \qquad \dot{v} = -\,u^2 \tag{9}$$

Answer

(1) Equations (9) imply conservation of the kinetic energy

$$\ell(u, v) = \frac{m}{2}(u^2 + v^2) = E$$

(2) The integral curves of the system of equations (9) in the uv plane are either critical points along the axis $u = 0$, or they are heteroclinic connections between these points that are semi-circles around the origin on level sets of the energy E.

The critical points at $u = \dot{x}/y = 0$ are relative equilibria of the system corresponding to vertical motion on the xy plane. Those corresponding to "upward motion" ($\dot{y} > 0$) are unstable and the ones corresponding to "downward motion" ($\dot{y} < 0$) are stable.

(3) The trial solutions $u = \tanh$ and $v = \text{sech}$ quickly converge to the exact solutions of the uv system.

▲

(E) (1) Legendre transform the Lagrangian (5) to the Hamiltonian side, obtain the canonical equations and

(2) derive the Poisson brackets for the variables u and v. **Hint:** $\{yp_x, yp_y\} = yp_x$.

Answer

The equations of motion on the Hamiltonian formulation are defined by introducing the momenta:

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{y^2}, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = \frac{m\dot{y}}{y^2},$$

and the Hamiltonian

$$H = \frac{y^2}{2m}\left(p_x^2 + p_y^2\right).$$

One gets

$$\begin{aligned} \dot{x} &= \frac{y^2 p_x}{m} \qquad \dot{p}_x = 0, \\ \dot{y} &= \frac{y^2 p_y}{m} \qquad \dot{p}_y = \frac{-y}{m}\left(p_x^2 + p_y^2\right). \end{aligned} \tag{10}$$

By defining

$$u = y p_x/m, \qquad v = y p_y/m,$$

the Hamiltonian can be written as

$$H = h(u,v) = \frac{1}{2}\left(u^2 + v^2\right),$$

and the equations of motion (10) become, using $\{yp_x, yp_y\} = yp_x$,

$$\dot{u} = uv \qquad \dot{v} = -\,u^2. \tag{11}$$

These equations are Hamiltonian with respect to the Lie-Poisson bracket

$$\{u,v\} = u,$$

and the reduced Hamiltonian $h(u,v)$ in terms of the invariant variables. Namely,

$$\dot{u} = \{u,h\} = uv \quad \dot{v} = \{v,h\} = -\,u^2.$$

▲

Exercise 1.4 (The free particle in $\mathbb{H}^2$: #2)

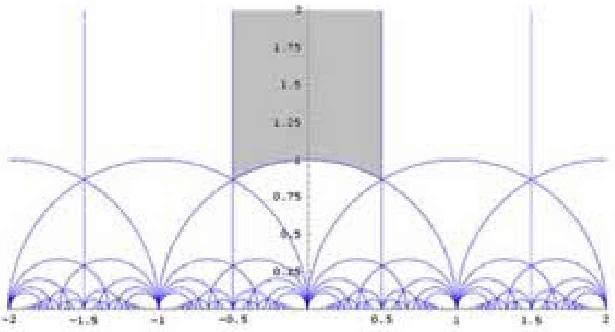

Figure 4: Geodesics on the Lobachevsky half-plane.

Consider the following pair of differential equations for $(u, v) \in \mathbb{R}^2$,

$$\dot{u} = uv\,, \qquad \dot{v} = -\,u^2\,. \tag{12}$$

These equations have discrete symmetries under combined reflection and time reversal, $(u, t) \to (-u, -t)$ *and* $(v, t) \to (-v, -t)$*. (This is called PT symmetry in the* (u, v) *plane.)*

(A) Find 2×2 *real matrices* L *and* B *for which the system (12) may be written as a Lax pair, namely, as*

$$\frac{dL}{dt} = [L, B]\,.$$

$\boxed{\textbf{\textit{Hint:}}}$ *a basis of* 2×2 *real matrices is given by*

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Explain what the Lax pair relation means and determine a constant of the motion from it.

$\boxed{\textbf{\textit{Hint:}}}$ *consider the similarity transformation* $L(t) = O^{-1}(t)L(0)O(t)$.

$\boxed{\textit{Answer}}$

We introduce two linear 2×2 matrices, one symmetric ($L^T = L$) and the other skew-symmetric ($B^T = -B$), as required for the commutator $[L, B]$ to be symmetric:

$$L = \begin{bmatrix} -v & u \\ u & v \end{bmatrix} = u \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - v \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = u\sigma_1 - v\sigma_3,$$

$$B = \frac{1}{2} \begin{bmatrix} 0 & u \\ -u & 0 \end{bmatrix} = \frac{u}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \frac{u}{2}\sigma_2.$$

Both matrices must be linear homogeneous, so that the commutator $[L, B]$ and time derivative $\frac{dL}{dt}$ can match powers using (12). The $\mathfrak{sl}(2, \mathbb{R})$ σ-matrices satisfy

$$[\sigma_1,\, \sigma_2] = 2\sigma_3, \quad [\sigma_2,\, \sigma_3] = 2\sigma_1, \quad \text{and} \quad [\sigma_3,\, \sigma_1] = -\,2\sigma_2.$$

Thus, we find the Lax pair relation,

$$\frac{dL}{dt} = u^2\sigma_3 + uv\,\sigma_1 = [L,\, B] = \left[u\sigma_1 - v\sigma_3,\, \frac{u}{2}\sigma_2\right].$$

What the Lax pair relation means: isospectrality. The Lax pair relation implies that

$$L(t) = O^{-1}(t)L(0)O(t), \quad \text{where} \quad B = O^{-1}\dot{O} \quad \text{and} \quad O(t) \in SO(2).$$

Thus, $OL(t)O^{-1} = L(0)$ is conserved. That is, the flow generates a similarity transformation that conserves the initial spectrum of the 2×2 symmetric matrix $L(0)$. Such a flow is said to be *isospectral*. The traceless matrix $L(t)$ has one independent eigenvalue and the system (12) has only one conserved quantity. The conserved quantity is the determinant, $\det L(t) = \det L(0)$.

▲

(B) Write the system (12) as a double matrix commutator, $\frac{dL}{dt} = [L, [L, N]]$. In particular, find N explicitly and explains what this means for the solutions. **Hint:** *compute $\frac{d}{dt}\,\mathrm{tr}\,(LN)$.*

Answer Substituting $N := \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ into $\frac{dL}{dt} = [L, [L, N]]$ yields $b - a =$, so for example we may set

$$N := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

What this means for the solutions: Gradient flow. The evolution by the double bracket relation $\frac{dL}{dt} = [L, [L, N]]$ is a *gradient flow* that preserves the spectrum of L but decreases the quantity $\mathrm{tr}\,(LN)$ according to

$$\frac{d}{dt}\,\mathrm{tr}\,(LN) = -\,\mathrm{tr}\,([L,\, N]^T[L,\, N])\,,$$

until L becomes diagonal and hence $[L,\, N] \to 0$, because N is diagonal. Thus, the dynamics (12) becomes asymptotically steady as L tends to a diagonal matrix. This means the system (12) must asymptotically approach a stable equilibria that is consistent with its initial conditions and conservation laws. For the present case, substituting the explicit forms of L and N yields

$$\frac{d}{dt}\,\mathrm{tr}\,(LN) = \frac{1}{2}\,\dot{v} = -\,\frac{1}{2}\,u^2 = -\,\mathrm{tr}\,([L,\, N]^T[L,\, N])\,,$$

which holds by (12) and thus checks the previous calculation. In the present case, it will turn out that $\lim_{t\to\infty} u(t) = 0$, which will verify $[L,\, N] \to 0$, as the off-diagonal parts of L will vanish asymptotically. ▲

(C) Find explicit solutions and discuss their motion and asymptotic behaviour:

(1) in time; and

(2) in the (u, v) phase plane. **Hint:** *keep the* tanh *function in mind.*

Answer

Keeping the tanh function in mind and recalling that

$$\frac{d\tanh(ct)}{dt} = c\,\mathrm{sech}^2(ct) \qquad \frac{d\,\mathrm{sech}(ct)}{dt} = -\,c\,\mathrm{sech}(ct)\tanh(ct),$$

we find, for $u(0) = c$ and $v(0) = 0$,

$$v(t) = -c\tanh(ct) \quad \text{and} \quad u(t) = c\,\mathrm{sech}(ct),$$

and of course we check, $2h = u^2 + v^2 = c^2(\tanh^2 + \mathrm{sech}^2) = c^2$.

Motion and asymptotic behaviour.

(a) **In time:** We have $\lim_{t\to\infty}(u(t), v(t)) = (0, -c)$. Consequently, the quantity $u(t)$ falls exponentially with time, from $u(0)$ toward the line of fixed points at $u = 0$, while $u(t)$ goes to a constant equal to $-u(0)$.

(b) **In the (u, v) phase plane:** Since h is conserved, the motion is along a family of semi-circles, each parameterised by its radius $c = \sqrt{2h}$, as

$$u^2 + v^2 = c^2 \quad \text{for} \quad u > 0 \quad \text{and} \quad u < 0\,,$$

lying in the upper and lower (u, v) half planes. These semi-circular motions are mirror images, reflected across the line of fixed points at $u = 0$. The equations of motion are PT-symmetric, so the fixed points along $u = 0$ in the (u, v) plane are stable for $v < 0$, and unstable for $v > 0$.
Thus, the two families of semi-circular motion both connect the line of fixed points at $u = 0$ to itself. One family of semi-circles lies in the upper half (u, v) plane, and the other lies symmetrically placed to complete the circles in the lower half (u, v) plane. The flows along each reflection-symmetric pair of semi-circles pass in the same (negative) v direction, from $v = c$ to $v = -c$.

▲

(D) Explain why the solution behaviour found in the previous part is consistent with the behaviour predicted by the double bracket relation.

Answer

This analysis is consistent with the conclusion from the double-bracket relation $\frac{dL}{dt} = [L, [L, N]\,]$ that the dynamics of L-matrix

$$L = \begin{bmatrix} -v & u \\ u & v \end{bmatrix}$$

asymptotically becomes steady. In fact, since $\lim_{t\to\infty} u = 0$ and $\lim_{t\to\infty} v(t) = -c$, the L-matrix asymptotically diagonalises! ▲

Exercise 1.5 (Nambu Poisson brackets on $\mathbb{R}^3$)

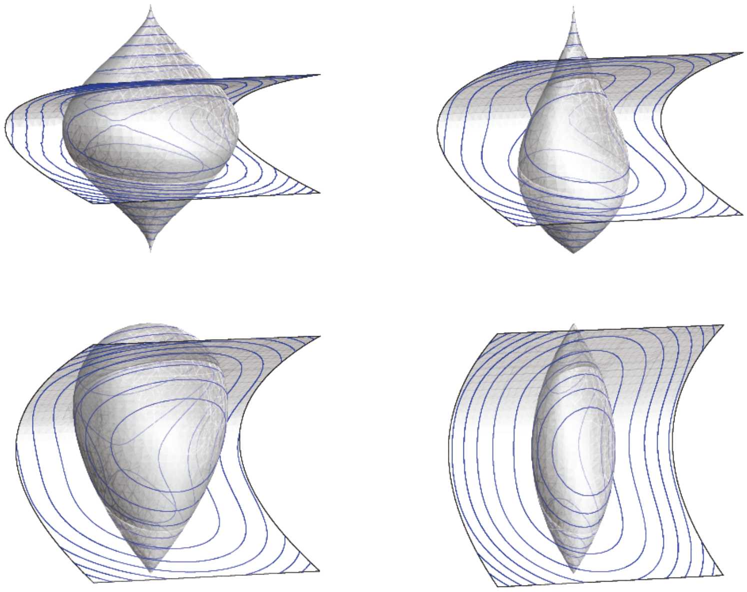

Figure 5: Motion along intersections of surfaces in $\mathbb{R}^3$.

(A) Show that for smooth functions $c, f, h : \mathbb{R}^3 \to \mathbb{R}$, the $\mathbb{R}^3$-bracket defined by

$$\{f, h\} = -\nabla c \cdot \nabla f \times \nabla h$$

satisfies the defining properties of a Poisson bracket. Is it also a derivation satisfying the Leibniz relation for a product of functions on $\mathbb{R}^3$? If so, why?

$\boxed{\textit{Answer}}$ The $\mathbb{R}^3$-bracket is plainly a skew-symmetric bilinear Leibniz operator. Its Hamiltonian vector fields are divergence free vector fields in $\mathbb{R}^3$. These vector fields in $\mathbb{R}^3$ satisfy the Jacobi identity under commutation. The identification of the $\mathbb{R}^3$-bracket with its Hamiltonian vector fields shows that it satisfies Jacobi. This will be made clearer below. ▲

(B) How is the $\mathbb{R}^3$-bracket related to the canonical Poisson bracket in the plane?

$\boxed{\textit{Answer}}$ The canonical Poisson bracket in the (x, y)-plane is given by the particular choice of the $\mathbb{R}^3$-bracket

$$\{f, h\} = -\nabla z \cdot \nabla f \times \nabla h$$

▲

(C) The Casimirs (or distinguished functions, as Lie called them) of a Poisson bracket satisfy

$$\{c, h\}(\mathbf{x}) = 0\,, \quad \textit{for all} \quad h(\mathbf{x})$$

Part (E) provides additional hints to proving that the $\mathbb{R}^3$-bracket satisfies the defining properties of a Poisson bracket. What are the Casimirs for the $\mathbb{R}^3$ bracket?

$\boxed{\textit{Answer}}$ Smooth functions of c are Casimirs for the $\mathbb{R}^3$-bracket given by

$$\{f, h\} = -\nabla c \cdot \nabla f \times \nabla h.$$

▲

(D) *Write the motion equation for the $\mathbb{R}^3$-bracket*

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\}$$

in vector form using gradients and cross products. Show that the corresponding Hamiltonian vector field $X_h = \{\,\cdot\,, h\}$ has zero divergence.

Answer

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\} = \nabla c \times \nabla h$$

The corresponding Hamiltonian vector field $X_h = \{\,\cdot\,, h\}$ has zero divergence because the vector $\nabla c \times \nabla h$ has zero divergence, since it's a curl. ▲

(E) *Show that under the $\mathbb{R}^3$-bracket, the Hamiltonian vector fields $X_f = \{\,\cdot\,, f\}$, $X_h = \{\,\cdot\,, h\}$ satisfy the following anti-homomorphism that relates the commutation of vector fields to the $\mathbb{R}^3$-bracket operation between smooth functions on $\mathbb{R}^3$,*

$$[X_f, X_h] = -X_{\{f,h\}}.$$

Hint: *commutation of divergenceless vector fields does satisfy the Jacobi identity and for the $\mathbb{R}^3$-bracket these vector fields are related to the Poisson bracket by*

$$\begin{aligned}
[X_G, X_H] &= X_G X_H - X_H X_G \\
&= \{G, \cdot\}\{H, \cdot\} - \{H, \cdot\}\{G, \cdot\} \\
&= \{G, \{H, \cdot\}\} - \{H, \{G, \cdot\}\}\,.
\end{aligned}$$

Answer **Lemma.** The $\mathbb{R}^3$-bracket defined on smooth functions (C, F, H) by

$$\{F, H\} = -\nabla C \cdot \nabla F \times \nabla H$$

may be identified with the divergenceless vector fields by

$$[X_G, X_H] = -X_{\{G,H\}}\,, \tag{13}$$

where $[X_G, X_H]$ is the Jacobi-Lie bracket of vector fields X_G and X_H.

Proof. Equation (13) may be verified by a direct calculation,

$$\begin{aligned}
[X_G, X_H] &= X_G X_H - X_H X_G \\
&= \{G, \cdot\}\{H, \cdot\} - \{H, \cdot\}\{G, \cdot\} \\
&= \{G, \{H, \cdot\}\} - \{H, \{G, \cdot\}\} \\
&= \{\{G, H\}, \cdot\} = -X_{\{G,H\}}\,.
\end{aligned}$$

■

Remark. The last step in the proof of the Lemma uses the Jacobi identity for the $\mathbb{R}^3$-bracket, which follows from the Jacobi identity for divergenceless vector fields, since

$$X_F X_G X_H = -\{F, \{G, \{H, \cdot\}\}\}$$

▲

(F) *Show that the motion equation for the $\mathbb{R}^3$-bracket is invariant under a certain linear combination of the functions c and h. Interpret this invariance geometrically.*

Answer

$$\nabla(\alpha c + \beta h) \times \nabla(\gamma c + \epsilon h) = \nabla c \times \nabla h \quad \text{for constants satisfying} \quad \alpha\epsilon - \beta\gamma = 1.$$

Under such a (volume-preserving) transformation, the level sets change, but their intersections remain invariant.

▲