1 M3-4-5A16 Assessed Problems # 1

Exercise 1.1 Do the following exercises from the notes discussed in class.

(a) The matrix representation of the Galilean group is the linear transformation, for $g \in G(3)$,

$$\begin{bmatrix} O & \mathbf{v}_0 & \mathbf{r}_0 \\ \mathbf{0} & 1 & t_0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \tilde{t} \\ 1 \end{bmatrix} = \begin{bmatrix} O\mathbf{r} + \mathbf{v}_0\tilde{t} + \mathbf{r}_0 \\ \tilde{t} + t_0 \\ 1 \end{bmatrix}$$

Compute the matrix representation of the inverse Galilean group transformation, $g^{-1} \in G(3)$. Assume that the ten parameters of this matrix Lie group depend on time as $g(t) \in G(3)$, along a curve in $G(3)$. Take the time derivative $\dot{g}(t) \in T_gG(3)$ at $g(t)$ and compute $\dot{g}(t)g^{-1} \in T_eG(3)$.

$\boxed{\boxed{\text{Answer}}}$.

$$gg^{-1} = \begin{bmatrix} O & \mathbf{v}_0 & \mathbf{r}_0 \\ \mathbf{0} & 1 & t_0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \begin{bmatrix} O^{-1} & -O^{-1}\mathbf{v}_0 & -O^{-1}(\mathbf{r}_0 - \mathbf{v}_0 t_0) \\ \mathbf{0} & 1 & -t_0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \dot{g}g^{-1} &= \begin{bmatrix} \dot{O} & \dot{\mathbf{v}}_0 & \dot{\mathbf{r}}_0 \\ \mathbf{0} & 0 & \dot{t}_0 \\ \mathbf{0} & 0 & 0 \end{bmatrix} \begin{bmatrix} O^{-1} & -O^{-1}\mathbf{v}_0 & -O^{-1}(\mathbf{r}_0 - \mathbf{v}_0 t_0) \\ \mathbf{0} & 1 & -t_0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \dot{O}O^{-1} & \dot{\mathbf{v}}_0 - \dot{O}O^{-1}\mathbf{v}_0 & \dot{\mathbf{r}}_0 - \dot{\mathbf{v}}_0 t_0 - \dot{O}O^{-1}(\mathbf{r}_0 - \mathbf{v}_0 t_0) \\ \mathbf{0} & 0 & \dot{t}_0 \\ \mathbf{0} & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} g^{-1}\dot{g} &= \begin{bmatrix} O^{-1} & -O^{-1}\mathbf{v}_0 & -O^{-1}(\mathbf{r}_0 - \mathbf{v}_0 t_0) \\ \mathbf{0} & 1 & -t_0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{O} & \dot{\mathbf{v}}_0 & \dot{\mathbf{r}}_0 \\ \mathbf{0} & 0 & \dot{t}_0 \\ \mathbf{0} & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} O^{-1}\dot{O} & O^{-1}\dot{\mathbf{v}}_0 & O^{-1}(\dot{\mathbf{r}}_0 - \mathbf{v}_0 \dot{t}_0) \\ \mathbf{0} & 0 & \dot{t}_0 \\ \mathbf{0} & 0 & 0 \end{bmatrix} \end{aligned}$$

▲

(b) Prove Galilean equivariance of Newton's motion equation

$$m_j \ddot{\mathbf{r}}_j = \sum_{k \neq j} \mathbf{F}_{jk}$$

with inter-particle gravitational forces

$$\mathbf{F}_{jk} = \frac{\gamma\, m_j m_k}{|\mathbf{r}_{jk}|^3}\mathbf{r}_{jk}\,, \quad \text{where} \quad \mathbf{r}_{jk} = \mathbf{r}_j - \mathbf{r}_k\,.$$

$\boxed{\boxed{\text{Answer}}}$. The interparticle forces $\mathbf{F}_{jk}$ are plainly invariant under translations in space and time, and Galilean boosts. They are also equivariant under rotations in space, since

$$m_j O\ddot{\mathbf{r}}_j = \sum_{k \neq j} O\mathbf{F}_{jk} = \sum_{k \neq j} \frac{\gamma\, m_j m_k}{|O\mathbf{r}_{jk}|^3} O\mathbf{r}_{jk}\,,$$

for any orthogonal transformation O. ▲

(c) Define the sphere S^{n-1} in $\mathbb{R}^n$. Explicitly determine the dimension of its tangent space TS^{n-1}.

Answer.

$$TS^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \ |\mathbf{x}|^2 = 1\,,\ \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}\,, \quad \text{so} \quad \dim\left(TS^{n-1}\right) = 2n-2$$

▲

(d) Consider the following mixed tensor defined on a smooth manifold M with local coordinates q

$$T(q) = T^{abc}_{ijk}(q)\frac{\partial}{\partial q^a}\otimes\frac{\partial}{\partial q^b}\otimes\frac{\partial}{\partial q^c}\otimes dq^i\otimes dq^j\otimes dq^k,$$

in which $\otimes$ denotes direct (or, tensor) product. How do the components of the mixed tensor T transform under a change of coordinates $q \to y = \phi(q)$ where ϕ is a smooth function?

That is, write the components of $T(y) = T(\phi(q))$ in the new basis in terms of the Jacobian matrix for the change of coordinates and the components $T^{abc}_{ijk}(q)$ of $T(q)$.

Answer. Insert the Jacobian and its inverse as

$$\frac{\partial}{\partial q^a} = \left[\left(\frac{\partial y}{\partial q}\right)^{-1}\right]^\alpha_a \frac{\partial}{\partial y^\alpha}\,, \quad dq^i = \frac{\partial q^i}{\partial y^a}dy^a\,, \text{ etc.}$$

then regroup terms.

The quantity $\phi^*T := T\circ\phi$ is called the *pull back* of the tensor T by the smooth mapping ϕ.

▲

(e) Determine the Lie group symmetries of the action principle given by $\delta S = 0$ for the following action,

$$S = \int_{t_1}^{t_2} L(\dot{\mathbf{q}}(t))dt\,.$$

What conservation laws does Noether's theorem imply for these symmetries? Prove them.

Answer. The Lagrangian $L(\dot{\mathbf{q}}(t))$ is invariant under translations, $\mathbf{q} \to \mathbf{q} + \boldsymbol{\varepsilon}$, so that the Noether quantity is

$$\frac{\partial L}{\partial\dot{\mathbf{q}}}\cdot\delta\mathbf{q} = \frac{\partial L}{\partial\dot{\mathbf{q}}}\cdot\boldsymbol{\varepsilon}\,.$$

Therefore, each component of the linear momentum $\frac{\partial L}{\partial\dot{\mathbf{q}}}$ is conserved. This is borne out by the corresponding Euler-Lagrange equations, too. ▲

□

Exercise 1.2 Compute the Euler-Lagrange equations and explain any conservation laws that may exist for a 2D harmonic oscillator with Lagrangian

$$\begin{aligned} L(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2}\|\dot{\mathbf{q}}\|^2 - \frac{k}{2}\|\mathbf{q}\|^2 \\ &= \frac{1}{2}\dot{q}^b \mathrm{g}_{bc}(\mathbf{q})\dot{q}^c - \frac{k}{2}q^b \mathrm{g}_{bc}(\mathbf{q})q^c\,, \end{aligned} \tag{1}$$

where $\mathbf{q} = (q^1, q^2)^T \in M$ are coordinates on a smooth 2D Riemannian manifold M with metric g_{ab}, $a, b = 1, 2$, the quantities $(\mathbf{q}, \dot{\mathbf{q}}) \in TM$, and k is a constant real number.

$\boxed{\text{Answer}}$. The Lagrangian in this case has partial derivatives given by

$$\frac{\partial L}{\partial \dot{q}^a} = g_{ac}(q)\dot{q}^c \quad \text{and} \quad \frac{\partial L}{\partial q^a} = \frac{1}{2}\frac{\partial g_{bc}(q)}{\partial q^a}\dot{q}^b\dot{q}^c - \frac{k}{2}\frac{\partial}{\partial q^a}\Big(q^b \mathrm{g}_{bc}(q)q^c\Big)\,.$$

Consequently, its Euler–Lagrange equations are

$$\begin{aligned} \big[\,L\,\big]_{q^a} &:= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} \\ &= g_{ae}(q)\ddot{q}^{\,e} + \frac{\partial g_{ae}(q)}{\partial q^b}\dot{q}^b\dot{q}^e - \frac{1}{2}\frac{\partial g_{be}(q)}{\partial q^a}\dot{q}^b\dot{q}^e + \frac{k}{2}\frac{\partial}{\partial q^a}\Big(q^b \mathrm{g}_{be}(q)q^e\Big) = 0\,. \end{aligned} \tag{2}$$

Symmetrising the coefficient of the second term and contracting with co-metric g^{ca} satisfying $g^{ca}g_{ae} = \delta^c_e$ yields

$$\ddot{q}^{\,c} + \Gamma^c_{be}(q)\dot{q}^b\dot{q}^e = -\frac{k}{2}g^{ca}\frac{\partial}{\partial q^a}\Big(q^b \mathrm{g}_{be}(q)q^e\Big)\,, \tag{3}$$

with

$$\Gamma^c_{be}(q) = \frac{1}{2}g^{ca}\left[\frac{\partial g_{ae}(q)}{\partial q^b} + \frac{\partial g_{ab}(q)}{\partial q^e} - \frac{\partial g_{be}(q)}{\partial q^a}\right], \tag{4}$$

in which the Γ^c_{be} are the ***Christoffel symbols*** for the Riemannian metric g_{ab}.

▲

Specialise the Lagrangian in (1) to consider the following cases:

(a) Motion in a plane, with $k\mathrm{g}_{ab} = k_{ab} = \text{const}$, which corresponds to an anisotropic spring constant matrix,

$$\mathbf{k} = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$$

$\boxed{\text{Answer}}$. The Lagrangian becomes

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \tfrac{1}{2}\|\dot{\mathbf{q}}\|^2 - \tfrac{1}{2}K\mathbf{q}\cdot\mathbf{q}\,, \tag{5}$$

on the tangent bundle

$$T\mathbb{R}^2 = \{(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^2 \times \mathbb{R}^2\}\,,$$

where we can always rotate our coordinate system to diagonalise k to $K = \mathrm{diag}(k_1, k_2)$.

Calculating the **Euler–Lagrange equation** for this Lagrangian yields the following equation of motion,

$$\ddot{\mathbf{q}} = -K\mathbf{q}\,.$$

The conserved energy is

$$E(\mathbf{q}, \dot{\mathbf{q}}) = \tfrac{1}{2}\|\dot{\mathbf{q}}\|^2 + \tfrac{1}{2}K\mathbf{q}\cdot\mathbf{q}\,,$$

arising from time-translation invariance of the Lagrangian, or simply noticing that this is a closed conservative system.

We will treat the Hamiltonian formulation of this problem later. ▲

(b) Motion on a sphere, with the harmonic oscillator attached to the North pole

Answer. The Euler–Lagrange equations for motion restricted to a sphere may be obtained by modifying the Lagrangian in the previous case by using a Lagrange multiplier μ to enforce the restriction. Namely, for $\mathbf{x} \in \mathbb{R}^3$ and harmonic potential $\frac{1}{2}A\mathbf{x}\cdot\mathbf{x}$, with diagonal $A = \mathrm{diag}(a_1, a_2, a_3)$ in which we choose $0 < a_1 < a_2 < a_3$, the Lagrangian becomes

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) = \tfrac{1}{2}\|\dot{\mathbf{x}}\|^2 - \tfrac{1}{2}A\mathbf{x}\cdot\mathbf{x} - \mu(1 - \|\mathbf{x}\|^2)\,, \tag{6}$$

on the tangent bundle

$$TS^2 = \{(\mathbf{x}, \dot{\mathbf{x}}) \in \mathbb{R}^3 \times \mathbb{R}^3 |\ \|\mathbf{x}\|^2 = 1,\ \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}.$$

Calculating the Euler–Lagrange equations for this Lagrangian and then solving for the Lagrange multiplier μ by enforcing $\frac{d}{dt}(\mathbf{x}\cdot\dot{\mathbf{x}}) = 0$ yields the equation of motion,

$$\ddot{\mathbf{x}} = -A\mathbf{x} + (A\mathbf{x}\cdot\mathbf{x} - \|\dot{\mathbf{x}}\|^2)\mathbf{x} = -\|\dot{\mathbf{x}}\|^2\mathbf{x} - P_{\perp\mathbf{x}}(A\mathbf{x})\,, \tag{7}$$

where $P_{\perp\mathbf{x}}(\mathbf{x}) = 0$. In components, this is

$$\ddot{x}^c + x^c\delta_{ab}\dot{x}^a\dot{x}^b = -\left(\delta^{cb} - \frac{x^c x^b}{|\mathbf{x}|^2}\right)(Ax)_b$$

Constants of motion. We form the matrices

$$Q^{ij} = (x^i x^j) = Q^{ji} \quad\text{and}\quad L^{ij} = (x^i\dot{x}^j - x^j\dot{x}^i) = -L^{ji}\,,$$

and then notice that the Euler–Lagrange equations (7) for the Lagrangian in (6) are equivalent to the matrix commutator equations

$$\dot{Q} = [Q, L]\,,\quad \dot{L} = [Q, A]\,,\quad \|\mathbf{x}\|^2 = 1 \quad\text{and}\quad \mathbf{x}\cdot\dot{\mathbf{x}} = 0\,.$$

These matrix commutator equations may be combined by introducing a constant parameter λ, so that

$$\frac{d}{dt}\left(-Q + L\lambda + A\lambda^2\right) = \left[-Q + L\lambda + A\lambda^2, -L - A\lambda\right]\,.$$

This formula yields conservation of traces of powers of the matrix $M = -Q + L\lambda + A\lambda^2$, shown directly by computing

$$\frac{d}{dt}\mathrm{tr}(M^n) = n\,\mathrm{tr}(M^{n-1}[M, L - A\lambda]) = n\,\mathrm{tr}[M^n, L - A\lambda] = 0\,,$$

since the trace of a commutator always vanishes. For example, the λ^2-coefficient of the case $\frac{d}{dt}\mathrm{tr}(M^2) = 0$ yields conservation of the energy

$$E(Q, L) = -\,\mathrm{tr}(L^2) + 2\,\mathrm{tr}(AQ)$$

for this system. The coefficients of the other powers of λ in $\frac{d}{dt}\mathrm{tr}(M^2) = 0$ are conserved trivially.

`The problem of harmonic oscillator motion on a sphere may also be formulated as motion on` $SO(3)$`, as follows.`

The Lagrangian for this type of motion would be the difference of kinetic minus potential energy for a particle of unit mass,

$$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}|\dot{\mathbf{x}}|^2 - \frac{1}{2}A\mathbf{x}\cdot\mathbf{x} \quad \text{for} \quad (\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^3\,. \tag{8}$$

Now, motion on a sphere comprises *rotation*, which may be written as the action of the rotation group on a vector in $\mathbb{R}^3$, by setting

$$\mathbf{x}(t) = O(t)\mathbf{x}_0\,, \quad \dot{\mathbf{x}}(t) = \dot{O}(t)\mathbf{x}_0 \quad \text{for} \quad (O,\,\dot{O}) \in TSO(3)\,, \tag{9}$$

where $\mathbf{x}_0 = \mathbf{x}(0)$ is the initial position of the particle. Relations (9) replace motion on the sphere S^2 by motion on the group $SO(3)$, whose action $SO(3)\times\mathbb{R}^3 \to \mathbb{R}^3$ on vectors in $\mathbb{R}^3$ leaves their Euclidean lengths invariant and, thus, preserves the sphere,

$$|\mathbf{x}(t)|^2 = \text{tr}\left[\big(O(t)\mathbf{x}_0\big)^T\big(O(t)\mathbf{x}_0\big)\right] = \text{tr}\left(\mathbf{x}_0^T O^T O \mathbf{x}_0\right) = |\mathbf{x}_0|^2 \quad \text{since} \quad O^TO = O^{-1}O = \text{Id}\,.$$

That is, the $SO(3)$ rotations (9) of vectors in $\mathbb{R}^3$ are summoned for this problem, because they map the sphere into itself.

The kinetic and potential energies of the particle on the sphere may be written on the group $SO(3)$ by using the transformation (9). In these terms, the kinetic energy is given by

$$\begin{aligned}
|\dot{\mathbf{x}}(t)|^2 &= \text{tr}\left[\big(\dot{O}(t)\mathbf{x}_0\big)^T\big(\dot{O}(t)\mathbf{x}_0\big)\right] \\
&= \text{tr}\left(\mathbf{x}_0^T\dot{O}^T\dot{O}\mathbf{x}_0\right) = \text{tr}\left(\mathbf{x}_0^T\dot{O}^TOO^{-1}\dot{O}\mathbf{x}_0\right) = \text{tr}\left(\mathbf{x}_0^T(O^T\dot{O})^T(O^{-1}\dot{O})\mathbf{x}_0\right) \\
&= \text{tr}\left(\mathbf{x}_0^T(\widehat{\Omega}^T\widehat{\Omega})\mathbf{x}_0\right) \quad \text{on defining} \quad \widehat{\Omega} := O^{-1}\dot{O} \quad \text{and using} \quad O^T = O^{-1} \\
&= \text{tr}\left(I_0\widehat{\Omega}^T\widehat{\Omega}\right) \quad \text{where} \quad I_0 = \mathbf{x}_0\mathbf{x}_0^T = I_0^T\,.
\end{aligned}$$

Remark. By using the hat map $\widehat{\ }: \mathfrak{so}(3) \to \mathbb{R}^3$ given by $\widehat{\Omega} = \boldsymbol{\Omega}\times$, this expression for the kinetic energy may also be written as

$$|\dot{\mathbf{x}}(t)|^2 = \text{tr}\left((\widehat{\Omega}\mathbf{x}_0)^T\widehat{\Omega}\mathbf{x}_0\right) = |\widehat{\Omega}\mathbf{x}_0|^2 = |\boldsymbol{\Omega}\times\mathbf{x}_0|^2\,.$$

The potential energy is given by

$$\begin{aligned}
\frac{1}{2}A\mathbf{x}\cdot\mathbf{x} &= \frac{1}{2}\,\text{tr}\big(\mathbf{x}^TA\mathbf{x}\big) \\
&= \frac{1}{2}\,\text{tr}\big(\mathbf{x}_0^TO^T(t)AO(t)\mathbf{x}_0\big) \quad \text{with} \quad O^T(t) = O^{-1}(t) \\
&= \frac{1}{2}\,\text{tr}\big(\mathbf{x}_0^TI(t)\mathbf{x}_0\big) \quad \text{with} \quad I(t) := O^{-1}(t)AO(t)\,,
\end{aligned}$$

in which the evolution of the symmetric matrix $I(t)$ preserves its eigenvalues, which are the same as those of diagonal $A = \text{diag}(a_1, a_2, a_3)$, all assumed to be distinct and positive.

Remark. The symmetric tensor $I(t) = O^{-1}(t)AO(t)$ satisfies the evolution equation

$$\frac{dI}{dt} = [I(t), \widehat{\Omega}]\,.$$

Proof. This formula is obtained by directly computing the time derivative of $I(t) = O^{-1}(t)AO(t)$ and using the formula

$$\frac{d}{dt}O^{-1}(t) = -O^{-1}\frac{dO}{dt}O^{-1}(t)$$

■

These formulas transform the Lagrangian $L : T\mathbb{R}^3 \to \mathbb{R}$ in (8) into $\ell : \mathfrak{so}(3) \times sym(3) \to \mathbb{R}$, where

$$\ell(\widehat{\Omega}(t), I(t)) = \frac{1}{2}\mathrm{tr}\left(I_0\widehat{\Omega}^T\widehat{\Omega}\right) - \frac{1}{2}\mathrm{tr}\big(I_0 I(t)\big) \quad \text{for} \quad (\widehat{\Omega}, I) \in T_eSO(3) \times sym(3)\,. \tag{10}$$

Here we denote $\mathfrak{so}(3) := T_eSO(3)$ and $sym(3)$ is the vector space of 3×3 symmetric matrices, which transform under $SO(3)$ as $SO(3) \times sym(3) \to sym(3)$ by matrix conjugation.

We now substitute the transformed Lagrangian (10) into Hamilton's principle,

$$0 = \delta S = \delta \int_a^b \ell(\widehat{\Omega}(t), I(t))\, dt = \delta \int_a^b \frac{1}{2}\,\mathrm{tr}\left(I_0\widehat{\Omega}^T\widehat{\Omega}\right) - \frac{1}{2}\,\mathrm{tr}\big(I_0 I(t)\big)\, dt$$

Remark. A computation with $\widehat{\Omega} := O^{-1}\dot{O}\,,\quad \widehat{\Xi} := O^{-1}\delta O$ and $I(t) := O^{-1}(t)AO(t)$ yields the ***variational identities***

$$\delta\widehat{\Omega} - \frac{d\widehat{\Xi}}{dt} = \left[\widehat{\Omega},\, \widehat{\Xi}\right] \quad \text{and} \quad \delta I = \left[I(t), \widehat{\Xi}\right].$$

Proof. These formulas result from direct computations using the definitions $\widehat{\Omega} := O^{-1}\dot{O}\,,\quad \widehat{\Xi} := O^{-1}\delta O$ and $I(t) := O^{-1}(t)AO(t)$, and the relation

$$\frac{d}{dt}O^{-1}(t) = -O^{-1}\frac{dO}{dt}O^{-1}(t)$$

■

On substituting these variational identities in Hamilton's principle and using the trace pairing, for matrices $\langle A, B\rangle = \mathrm{tr}\,(A^TB)$, for example,

$$\mathrm{tr}\left(I_0\widehat{\Omega}^T\widehat{\Omega}\right) = \mathrm{tr}\left(\widehat{\Omega}^T\widehat{\Omega}I_0\right) = \mathrm{tr}\left((I_0\widehat{\Omega})^T\widehat{\Omega}\right) =: \left\langle I_0\widehat{\Omega}\,,\, \widehat{\Omega}\right\rangle$$

we find that

$$\begin{aligned}
0 = \delta S &= \int_a^b \left\langle \frac{\partial \ell}{\partial\widehat{\Omega}}\,,\, \delta\widehat{\Omega}\right\rangle + \left\langle \frac{\partial \ell}{\partial I}\,,\, \delta I\right\rangle\, dt \\
&= \int_a^b \left\langle \widehat{\Pi}, \delta\widehat{\Omega}\right\rangle - \frac{1}{2}\left\langle I_0, \delta I\right\rangle dt \quad \text{with} \quad \widehat{\Pi} := \frac{\partial \ell}{\partial\widehat{\Omega}} = I_0\widehat{\Omega} \\
&= \int_a^b \left\langle \widehat{\Pi}[\widehat{\Omega},\, \widehat{\Pi}], \widehat{\Xi}\right\rangle - \frac{1}{2}\left\langle I_0, [I(t), \widehat{\Xi}]\right\rangle dt \\
&= -\int_a^b \left\langle \frac{d\widehat{\Pi}}{dt} + \left[\widehat{\Omega}, \widehat{\Pi}\right] + \frac{1}{2}[I, I_0],\, \widehat{\Xi}\right\rangle dt + \mathrm{tr}\left(\widehat{\Pi}^T\widehat{\Xi}\right)
\end{aligned}$$

This yields the Euler-Poincaré system of equations for the *body angular momentum* $\widehat{\Pi}(t)$ and the symmetric tensor $I(t) := O^{-1}(t)AO(t)$,

$$\frac{d\widehat{\Pi}}{dt} + \left[\widehat{\Omega}, \widehat{\Pi}\right] + \frac{1}{2}[I, I_0] = 0 \quad \text{and} \quad \frac{dI}{dt} + [\widehat{\Omega}\,,\, I(t)] = 0\,, \tag{11}$$

The system (11) conserves the sum of the kinetic and potential energies,

$$E(\widehat{\Omega}(t), I(t)) := \frac{1}{2}\,\mathrm{tr}\left(I_0\widehat{\Omega}^T\widehat{\Omega}\right) + \frac{1}{2}\,\mathrm{tr}\big(I_0 I(t)\big)\,.$$

Remark about the Hamiltonian formulation of the C Neumann problem.
The solution of the problem of harmonic motion on a sphere is due to C Neumann [1859]. Later, we will Legendre transform the Lagrangian for the C Neumann problem and also find the Lie-Poisson Hamiltonian structure for the system of equations (11). For now, we define

$$H(\widehat{\Pi}(t), I(t)) := \frac{1}{2}\left\langle \widehat{\Pi}\,,\, I_0^{-1}\widehat{\Pi}\right\rangle + \frac{1}{2}\left\langle I(t), I_0\right\rangle\,.$$

with variations

$$\delta H(\widehat{\Pi}(t), I(t)) := \left\langle \delta\widehat{\Pi}\,,\, I_0^{-1}\widehat{\Pi}\right\rangle + \frac{1}{2}\left\langle \delta I(t), I_0\right\rangle\,.$$

Consequently, equations (11) may be rewritten equivalently as

$$\begin{aligned} \frac{d\widehat{\Pi}}{dt} &= -\left[\frac{\delta H}{\delta\widehat{\Pi}}, \widehat{\Pi}\right] - \left[I, \frac{\delta H}{\delta I}\right] = \mathrm{ad}^*_{\frac{\delta H}{\delta\widehat{\Pi}}}\widehat{\Pi} - I\diamond\frac{\delta H}{\delta I}\,, \\ \frac{dI}{dt} &= -\left[\frac{\delta H}{\delta\widehat{\Pi}}\,,\, I(t)\right] = -\,\pounds_{\frac{\delta H}{\delta\widehat{\Pi}}} I\,, \end{aligned} \tag{12}$$

where the $\diamond$ operation is defined by

$$\left\langle I\diamond\frac{\delta H}{\delta I}\,,\,\widehat{\Xi}\right\rangle = \left\langle\frac{\delta H}{\delta I}\,,\,\pounds_{\widehat{\Xi}} I\right\rangle = \left\langle\frac{\delta H}{\delta I}\,,\, -\left[I, \widehat{\Xi}\right]\right\rangle = \left\langle\left[I, \frac{\delta H}{\delta I}\right]\,,\,\widehat{\Xi}\right\rangle,$$

for an arbitrary $\widehat{\Xi}\in\mathfrak{so}(3)$ and the pairing is the trace pairing of skew-symmetric matrices.

Remark. Later, equations (12) will be discovered to comprise a Lie-Poisson Hamiltonian system on the dual of the semidirect-product Lie algebra $\mathfrak{so}(3)Ⓢsym(3)$, given by

$$\left[(\widehat{\Xi}_1, I_1)\,,(\widehat{\Xi}_2, I_2)\right] = \left(\left[\widehat{\Xi}_1, \widehat{\Xi}_2\right]\,,\,\widehat{\Xi}_1 I_2 - \widehat{\Xi}_1 I_2\right) = \left(\left[\widehat{\Xi}_1, \widehat{\Xi}_2\right]\,,\,\left[\widehat{\Xi}_1, I_2\right] - \left[\widehat{\Xi}_2, I_1\right]\right).$$

▲

(c) Extra credit Compute the Euler-Lagrange equations and explain any conservation laws for the Lagrangian above in (1) with the following isotropic metric:

$$\mathrm{g}_{ab}(\mathbf{q}) = Q^2(\mathbf{q})\delta_{ab}$$

where $Q : M \to \mathbb{R}$ and δ_{ab} is the 2×2 identity matrix.

Answer. This is a mechanical analogue of Fermat's principle,

$$0 = \delta S = \delta\int_a^b L(\mathbf{q},\,\dot{\mathbf{q}})\,dt = \delta\int_a^b \sqrt{Q^2(\mathbf{q})\,\dot{q}^b\delta_{bc}\dot{q}^c}\,dt = \delta\int_a^b Q(\mathbf{q}(s))ds$$

with $ds^2 = dq^b\delta_{bc}dq^c$. That is, the Lagrangian takes the same form as for a (reparametrised) Fermat's principle; namely

$$L(\mathbf{q},\,\dot{\mathbf{q}}) = \frac{1}{2}Q^2(\mathbf{q})\,\dot{q}^b\delta_{bc}\dot{q}^c\,, \tag{13}$$

in *Euclidean* coordinates $\mathbf{q}\in\mathbb{R}^3$ with a prescribed index of refraction $Q(\mathbf{q})$. Conversely, the geometry of ray optics may be regarded as geodesic motion of particles "coasting" through a manifold with an isotropic, but spatially varying metic (index of refraction). ▲

□

Exercise 1.3 Compute the Euler-Lagrange equations in the following four cases.

(a) *A charged particle in a constant magnetic field.* Define the magnetic field $\mathbf{B} = \nabla \times \mathbf{A} = \text{const}$, where $\mathbf{B}$, $\mathbf{A}$ are vector fields on $M = \mathbb{R}^3$. $\mathbf{A}$ is called the magnetic vector potential; and $\mathbf{A}$ is not a constant! The Lagrangian is given by:

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}\|\dot{\mathbf{q}}\|^2 + e\dot{\mathbf{q}} \cdot \mathbf{A}(\mathbf{q})$$

where e is the electron charge and $\| \cdot \|$ is the usual Euclidean norm on $\mathbb{R}^3$.

Answer.

(i) **Fibre derivative**
The fibre derivative gives a linear relation

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{q}} + \frac{e}{c}\mathbf{A}(\mathbf{q})$$

so this Lagrangian is hyperregular.

(ii) **Euler-Lagrange equations**
In vector form, this is

$$\ddot{\mathbf{q}} = \frac{e}{mc}\dot{\mathbf{q}} \times \mathbf{B}(\mathbf{q}) \quad \text{with} \quad \mathbf{B}(\mathbf{q}) := \frac{\partial}{\partial \mathbf{q}} \times \mathbf{A}(\mathbf{q})$$

and the terms on the right comprise the Lorentz force.

▲

(b) *The Kepler problem.* Consider a planet of mass m in a gravitational potential generated by a star of larger mass M. Fix the star's position at the origin of coordinates and use Newton's gravitational potential:

$$V(\mathbf{q}) = -G\frac{mM}{\|\mathbf{q}\|}$$

where $\mathbf{q} \in \mathbb{R}$ denotes here the position of the planet w.r.t the star.

Answer. The Lagrangian in this case is

$$L = \frac{1}{2}m\|\dot{\mathbf{q}}\|^2 + G\frac{mM}{\|\mathbf{q}\|}$$

(i) **Fibre derivative**
The fibre derivative gives a linear relation

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{q}}$$

so this Lagrangian is hyperregular (velocity may be obtained from momentum and position).

(ii) **Euler-Lagrange equations**
In 3D vector form, the Euler-Lagrange equation is

$$\ddot{\mathbf{q}} = -\frac{GM}{\|\mathbf{q}\|^3}\mathbf{q}$$

and the term on the right is Newton's gravitational force.

Planar Kepler problem: For planar motion in polar coordinates

$$(r, \dot{r}, \theta, \dot{\theta}) \in T\mathbb{R}_+ \times TS^1\,,$$

the Lagrangian is

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}$$

and the Euler-Lagrange equation is

$$\ddot{r} = -\frac{GM}{r^2} + \frac{J^2}{r^3} \quad \text{with} \quad J = r^2\dot{\theta} = const$$

The conserved energy is

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GMm}{r}$$

See Appendix B of the text [GM1] for more information (and revision) about the Kepler problem.

▲

(c) Free motion on a hyperboloid of revolution around the z-axis. We recall the equation for such a hyperboloid

$$\|\mathbf{x}\|_H^2 = \mathbf{x}\cdot A\mathbf{x} = \frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 1\,,$$

where $\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$, $(a, b) \in \mathbb{R}^2$ and $A = \mathrm{diag}(a^{-2}, a^{-2}, -b^{-2})$.

Answer. This can be done with a Lagrange multiplier, too. The Lagrangian becomes

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \tfrac{1}{2}\|\dot{\mathbf{x}}\|^2 - \mu(1 - \mathbf{x}\cdot A\mathbf{x})\,, \tag{14}$$

on the tangent bundle

$$TH^2 = \{(\mathbf{x}, \dot{\mathbf{x}}) \in \mathbb{R}^3 \times \mathbb{R}^3 |\ \mathbf{x}\cdot A\mathbf{x} = 1,\ \dot{\mathbf{x}}\cdot A\mathbf{x} = 0\}.$$

(i) **Fibre derivative**
The fibre derivative gives the linear relation, $\frac{\partial L}{\partial \dot{\mathbf{x}}} = \dot{\mathbf{x}}$.

(ii) **Euler-Lagrange equations**
Calculating the Euler–Lagrange equations for this Lagrangian and then solving for the Lagrange multiplier μ by enforcing $\frac{d}{dt}(\dot{\mathbf{x}}\cdot A\mathbf{x}) = 0$ yields the equation of motion,

$$\ddot{\mathbf{x}} = -\left(\frac{\dot{\mathbf{x}}\cdot A\dot{\mathbf{x}}}{|A\mathbf{x}|^2}\right)\mathbf{x} \quad \text{or, in components,} \quad \ddot{x}^c = -\left(\frac{x^c A_{ab}}{|A\mathbf{x}|^2}\right)\dot{x}^a\dot{x}^b =: -\Gamma^c_{ab}\dot{x}^a\dot{x}^b\,. \tag{15}$$

▲

(d) *Springs and masses.* Consider a one-dimensional system composed of three particles with masses m_1, m_2, m_3 interacting through two springs with spring constants k_{12} and k_{23}.

Draw a diagram showing that the three particles are aligned on a single axis and attached with the two springs such that the spring k_{12} attaches the mass m_1 to m_2 and the spring k_{23} attaches the mass m_2 to m_3. In this case $M = \mathbb{R}^3$ and the spring potential energy in terms of the separation $x_{12} = x_1 - x_2$ for example is given by

$$V(x_{12}) = -k_{12}(x_{12})^2.$$

Answer. The Lagrangian for this system is $L : T\mathbb{R}^n$ given by

$$L(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) = \frac{1}{2}\sum_{i=1}^n m_i\dot{x}_i^2 - \frac{1}{2}\sum_{i,j=1}^n k_{ij}x_{ij}^2\,,$$

where n is the number of particles with masses m_j, $j = 1, 2, \ldots, n$, and $x_{ij} = x_i - x_j$ are the separations between the particles.

However, since the particles are only connected to their nearest neighbours we may set

$$L(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{x}_i^2 - \frac{1}{2}\sum_{i=1}^{n} k_{i,i+1}(x_i - x_{i+1})^2 \,.$$

The boundary conditions at the endpoints may either be fixed ($x_1 = 0, x_{n+1} = 1$), free ($x_{n+1} = 0$), or periodic ($x_{n+1} = x_1$). For $n = 3$, we have

$$\begin{aligned} L(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3) &= \frac{1}{2}\sum_{i=1}^{3} m_i \dot{x}_i^2 - \frac{1}{2}\sum_{i=1}^{3} k_{i,i+1}(x_i - x_{i+1})^2 \\ &= \frac{1}{2}\left(m_1\dot{x}_1^2 + m_1\dot{x}_2^2 + m_1\dot{x}_3^2\right) \\ &\quad - \frac{1}{2}\Big(k_{12}(x_1 - x_2)^2 + k_{23}(x_2 - x_3)^2 + k_{31}(x_3 - x_1)^2\Big)\,. \end{aligned}$$

(i) **Fibre derivative**
The fibre derivative gives the linear relation $p_i = \partial L/\partial \dot{x}_i = m_i \dot{x}_i$, $i = 1, 2, 3$.

(ii) **Euler-Lagrange equations** for the periodic case are

$$\begin{aligned} m_1 \ddot{x}_1 &= \frac{\partial L}{\partial x_1} = -k_{12}(x_1 - x_2) + k_{31}(x_3 - x_1)\,, \\ m_2 \ddot{x}_2 &= \frac{\partial L}{\partial x_2} = k_{12}(x_1 - x_2) - k_{23}(x_2 - x_3)\,, \\ m_3 \ddot{x}_3 &= \frac{\partial L}{\partial x_3} = k_{23}(x_2 - x_3) - k_{31}(x_3 - x_1)\,. \end{aligned}$$

Note that the total momentum is conserved,

$$\frac{dP_{tot}}{dt} = 0\,, \quad \text{with} \quad P_{tot} := \sum_{i=1}^{3} p_i = m_1\dot{x}_1 + m_2\dot{x}_2 + m_3\dot{x}_3\,.$$

The total energy,

$$E_{tot} = \frac{1}{2}\sum_{i=1}^{3} m_i \dot{x}_i^2 + \frac{1}{2}\sum_{i=1}^{3} k_{i,i+1}(x_i - x_{i+1})^2\,,$$

is also conserved, since these coupled harmonic oscillators comprise a closed conservative system.

▲

(e) $\boxed{\textit{Extra credit}}$ Recover the above equations of motion in parts (a)–(d) using Newton's 2nd law.

$\boxed{\textbf{Answer}}$. The forces in Newton's 2nd law for this problem are the right-hand sides of the Euler-Lagrange equations above (for the periodic case). ▲

□

Exercise 1.4 (Oscillator variables)
The Hamiltonian for the 2D isotropic harmonic oscillator in canonical variables $(\mathbf{q},\mathbf{p}) \in T^*\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2$ is given by

$$H = \frac{1}{2}|\mathbf{p}|^2 + \frac{1}{2}|\mathbf{q}|^2$$

For simplicity, we have chosen units in which the mass m and spring constant k satisfy $m = 1 = k$.

(a) Write the Hamiltonian H in *oscillator variables* given by

$$\mathbf{q} + i\mathbf{p} = \begin{bmatrix} q_1 + ip_1 \\ q_2 + ip_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} =: \mathbf{a} \in \mathbb{C}^2 \quad \text{with} \quad |\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a}^* = |a_1|^2 + |a_2|^2$$

Answer. In oscillator variables

$$\mathbf{q} + i\mathbf{p} = \begin{bmatrix} q_1 + ip_1 \\ q_2 + ip_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{a} \in \mathbb{C}^2$$

we may express the Hamiltonian H in terms of variables $\mathbf{a}$ by using

$$|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a}^* = |a_1|^2 + |a_2|^2 = q_1^2 + p_1^2 + q_2^2 + p_2^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2 = 2H$$

The transformation to oscillator variables is canonical: its symplectic two-form is

$$da \wedge da^* = (dq + idp) \wedge (dq - idp) = -2i\, dq \wedge dp$$

where we ignore subscripts for brevity.
Likewise, the Poisson bracket transforms as

$$\{a, a^*\} = \{q + ip\,,\, q - ip\} = -2i\,\{q\,,\,p\} = -2i\,\mathrm{Id}$$

Thus, in oscillator variables Hamilton's canonical equations become

$$\dot{a} = \{a, H\} = -2i\,\frac{\partial H}{\partial a^*} \quad \text{and} \quad \dot{a}^* = \{a^*, H\} = 2i\,\frac{\partial H}{\partial a}$$

The corresponding Hamiltonian vector field is

$$X_H = \{\,\cdot\,, H\} = -2i\,\frac{\partial H}{\partial a^*}\frac{\partial}{\partial a} + 2i\,\frac{\partial H}{\partial a}\frac{\partial}{\partial a^*}$$

satisfying

$$X_H \,\lrcorner\, (da \wedge da^*) = -2i\, dH$$

where $\lrcorner$ is the contraction sign from differential geometry.

▲

(b) Consider the three quadratic quantities Y_1, Y_2, Y_3, given by

$$Y_1 + iY_2 = 2a_1a_2^* \quad \text{and} \quad Y_3 = |a_1|^2 - |a_2|^2\,.$$

(i) Show that these quantities are invariant under the S^1-transformation $\mathbf{a} \to \mathbf{a}e^{i\phi}$ for any ϕ.
(ii) Are these quadratic quantities conserved by the 2D isotropic harmonic oscillator? Prove it.
(iii) Compute the Poisson brackets among Y_1, Y_2, Y_3.

Answer.

(i) By inspection, these quantities are invariant under the S^1−transformation

(ii) The Hamiltonian for the 2D isotropic harmonic oscillator is

$$H = \frac{1}{2}(\,|a_1|^2 + |a_2|^2) = \frac{1}{2}|\mathbf{a}|^2$$

The corresponding canonical equations are

$$\dot{\mathbf{a}} = \{\mathbf{a}, H\} = -2i\,\mathbf{a} \quad \text{and} \quad \dot{\mathbf{a}}^* = \{\mathbf{a}^*, H\} = 2i\,\mathbf{a}^*$$

whose solutions are immediately found to be S^1 phase shifts, linear in time:

$$\mathbf{a}(t) = e^{-2i\,t}\mathbf{a}(0) \quad \text{and} \quad \mathbf{a}^*(t) = e^{2i\,t}\mathbf{a}^*(0)$$

Consequently, being invariant under such S^1 phase shifts, the three quadratic quantities

$$Y_1 + iY_2 = 2a_1a_2^* \quad \text{and} \quad Y_3 = |a_1|^2 - |a_2|^2$$

are conserved by the 2D isotropic harmonic oscillator.

(iii) The three quadratic S^1−invariants form a vector $\mathbf{Y}$ with components $(Y_1, Y_2, Y_3) \in \mathbb{R}^3$ whose magnitude satisfies

$$|\mathbf{Y}|^2 := Y_1^2 + Y_2^2 + Y_3^2 = (|a_1|^2 + |a_2|^2)^2 = (2H)^2$$

The Poisson brackets of the components $(Y_1, Y_2, Y_3) \in \mathbb{R}^3$ are computed by the chain rule to close among themselves as

$$\{Y_k\,,\,Y_l\} = -\,\epsilon_{klm}Y_m$$

Thus, functions of these S^1−invariants satisfy

$$\{F\,,\,H\}(\mathbf{Y}) = -\,\mathbf{Y}\cdot\frac{\partial F}{\partial \mathbf{Y}}\times\frac{\partial H}{\partial \mathbf{Y}}$$

The Hamiltonian for the 2D isotropic harmonic oscillator as expressed as $H = |\mathbf{Y}|^2/2$. This Hamiltonian has derivative $\partial H/\partial\mathbf{Y} = \mathbf{Y}$; so it is a ***Casimir*** for this Poisson bracket. That is, $H = |\mathbf{Y}|^2/2$ Poisson-commutes with any function of $\mathbf{Y}$. In particular, it Poisson-commutes with each of the components (Y_1, Y_2, Y_3). Hence, as expected, each component of $\mathbf{Y}$ is conserved under the dynamics generated by this Hamiltonian.

Remark. Perhaps surprisingly, the Poisson brackets among the three S^1 invariants $\mathbf{Y} \in \mathbb{R}^3$ are the same as the brackets among the vector components of angular momentum $\mathbf{J} := \mathbf{q}\times\mathbf{p}$. Why is this?

(iv) For the 2D *anisotropic* case, the harmonic oscillator Hamiltonian is

$$H = \frac{1}{2}(\,\omega_1|a_1|^2 + \omega_2|a_2|^2) = \frac{1}{4}\,((\omega_1+\omega_2)|\mathbf{Y}| + (\omega_1-\omega_2)Y_3)$$

and functions of the S^1−invariants (Y_1, Y_2, Y_3) satisfy

$$\frac{dF}{dt} = \{F\,,\,H\}(\mathbf{Y}) = \frac{\partial F}{\partial \mathbf{Y}}\cdot\mathbf{Y}\times\frac{\partial H}{\partial \mathbf{Y}} = \frac{1}{4}(\omega_1-\omega_2)\frac{\partial F}{\partial \mathbf{Y}}\cdot\mathbf{Y}\times\widehat{\mathbf{3}}$$

so the motion equation is

$$\frac{d\mathbf{Y}}{dt} = \{\mathbf{Y}\,,\,H\}(\mathbf{Y}) = \frac{1}{4}(\omega_1-\omega_2)\mathbf{Y}\times\widehat{\mathbf{3}}$$

This describes precession of $\mathbf{Y}$ about the $\widehat{\mathbf{3}}$-axis at constant frequency $\frac{1}{4}(\omega_2-\omega_1)$.

▲