M3-4-5 A34 Handout: What is Geometric Mechanics II?

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Text for the course M3-4-5 A34:

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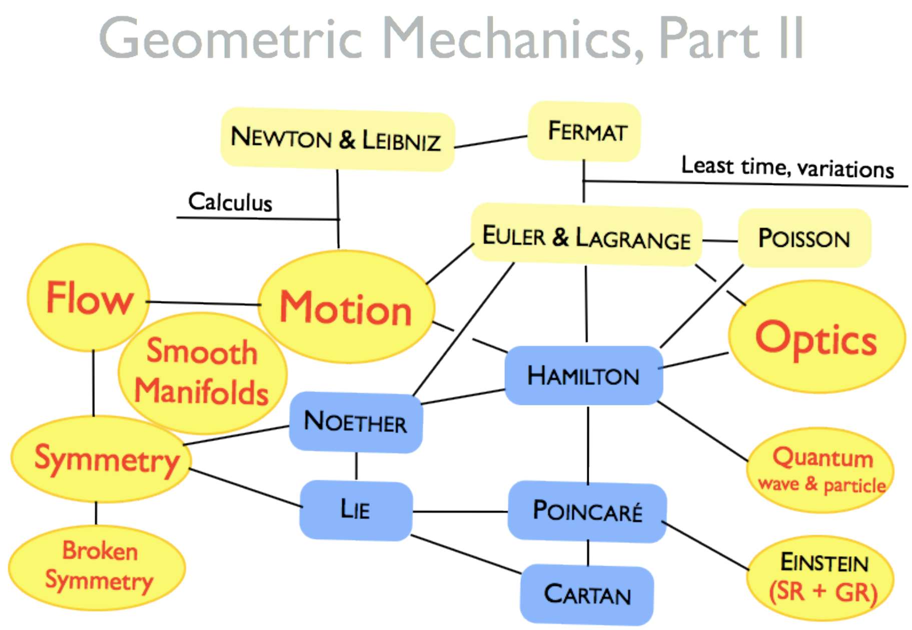

Figure 1: Geometric Mechanics has involved many great mathematicians!

What shall we study?

Hamilton: quaternions, AD, Ad, ad, Ad*, ad* actions, variational principles

Lie: Groups of transformations that depend smoothly on parameters

Poincaré: Mechanics on Lie groups, $SO(3)$, $SU(2)$, $Sp(2)$, $SE(3) \simeq SO(3)$Ⓢ$\mathbb{R}^3$

Noether: Implications of symmetry in variational principles

Cartan: Lie transformations of differential forms and fluid flows

Geometric Mechanics A34 deals with motion on smooth manifolds

The rest of this handout is meant to be a sort of un-alphabetized glossary, a list of words and concepts that will be introduced and studied later in the course, defined and used succinctly here in sentences.

Transformation theory

smooth manifold	equilibrium	tangent lift
tangent space	linearisation	commutator
motion equation	infinitesimal transformation	differential, d
vector field	pull-back	differential k-form
diffeomorphism	push-forward	wedge product, $\wedge$
flow	Jacobian matrix	Lie derivative, $\pounds_Q$
fixed point	directional derivative	product rule

- Let M be a *smooth manifold*, $\dim M = n$. That is, M is a smooth space that is locally $\mathbb{R}^n$.
- The *tangent space* TM contains velocity $v_q = \dot{q}(t) \in T_qM$, tangent to curve $q(t) \in M$ at point q. The coordinates are $(q, v_q) \in TM$.
  Note, $\dim TM = 2n$ and subscript q reminds us that v_q is an element of the tangent space at the point q and that on TM we must keep track of base points.

  The tangent space $TM := \cup_{q\in M} T_qM$ is also called the *tangent bundle* of the manifold M.

  The curve $\dot{q}(t) \in TM$ is called the *tangent lift* of the curve $q(t) \in M$.
- A *motion* is defined as a smooth curve $q(t) \in M$ parameterised by $t \in \mathbb{R}$ that solves the *motion equation*, which is a system of differential equations

$$\dot{q}(t) = \frac{dq}{dt} = f(q) \in TM\,, \tag{1}$$

  or in components

$$\dot{q}^i(t) = \frac{dq^i}{dt} = f^i(q) \quad i = 1, 2, \dots, n\,, \tag{2}$$

- The map $f : q \in M \to f(q) \in T_qM$ is a *vector field*.

  According to standard theorems about differential equations that are not proven in this course, the solution, or integral curve, $q(t)$ exists, provided f is sufficiently smooth, which will always be assumed to hold.

  Vector fields can also be defined as *differential operators* that act on functions, as

$$\frac{d}{dt}G(q) = \dot{q}^i(t)\frac{\partial G}{\partial q^i} = f^i(q)\frac{\partial G}{\partial q^i} \quad i = 1, 2, \dots, n, \quad \text{(sum on repeated indices)} \tag{3}$$

  for any smooth function $G(q) : M \to \mathbb{R}$.
- To indicate the dependence of the solution of its initial condition $q(0) = q_0$, we write the motion as a smooth transformation

$$q(t) = \phi_t(q_0)\,.$$

Because the vector field f is independent of time t, for any fixed value of t we may regard ϕ_t as mapping from M into itself that satisfies the composition law

$$\phi_t \circ \phi_s = \phi_{t+s}$$

and

$$\phi_0 = \mathrm{Id}\,.$$

Setting $s = -t$ shows that ϕ_t has a smooth inverse. A smooth mapping that has a smooth inverse is called a *diffeomorphism*. Geometric mechanics deals with diffeomorphisms.

- The smooth mapping $\phi_t : \mathbb{R} \times M \to M$ that determines the solution $\phi_t \circ q_0 = q(t) \in M$ of the motion equation (1) with initial condition $q(0) = q_0$ is called the *flow* of the vector field Q.

  A point $q_e \in M$ at which $f(q_e) = 0$ is called a *fixed point* of the flow ϕ_t, or an *equilibrium*.

  Vice versa, the vector field f is called the *infinitesimal transformation* of the mapping ϕ_t, since

  $$\frac{d}{dt}\bigg|_{t=0} (\phi_t \circ q_0) = f(q)\,.$$

  That is, $f(q)$ is the *linearisation* of the flow map ϕ_t at the point $q \in M$.

  More generally, the *directional derivative* of the function h along the vector field f is given by the action of a differential operator, as

  $$\frac{d}{dt}\bigg|_{t=0} h \circ \phi_t = \left[\frac{\partial h}{\partial \phi_t} \frac{d}{dt} (\phi_t \circ q_0) \right]_{t=0} = \frac{\partial h}{\partial q^i} \dot{q}^i = \frac{\partial h}{\partial q^i} f^i(q) =: Qh\,.$$

- Under a smooth change of variables $q = c(r)$ the vector field Q in the expression Qh transforms as

  $$Q = f^i(q) \frac{\partial}{\partial q^i} \quad \mapsto \quad R = g^j(r) \frac{\partial}{\partial r^j} \quad \text{with} \quad g^j(r) \frac{\partial c^i}{\partial r^j} = f^i(q(r)) \quad \text{or} \quad g = c_r^{-1} f \circ c\,, \qquad (4)$$

  where c_r is the *Jacobian matrix* of the transformation. That is,

  $$(Qh) \circ c = R(h \circ c)\,.$$

  We express the transformation between the vector fields as $R = c^* Q$ and write this relation as

  $$(Qh) \circ c =: c^* Q(h \circ c)\,. \qquad (5)$$

  The expression $c^* Q$ is called the *pull-back* of the vector field Q by the map c. Two vector fields are equivalent under a map c, if one is the pull-back of the other, and fixed points are mapped into fixed points.

  The inverse of the pull-back is called the *push-forward*. It is the pull-back by the inverse map.

- The *commutator*

  $$QR - RQ =: \big[Q,\, R\big]$$

  of two vector fields Q and R defines another vector field. Indeed, if

  $$Q = f^i(q) \frac{\partial}{\partial q^i} \quad \text{and} \quad R = g^j(q) \frac{\partial}{\partial q^j}$$

then

$$\big[Q,\,R\big] = \left(f^i(q)\frac{\partial g^j(q)}{\partial q^i} - g^i(q)\frac{\partial f^j(q)}{\partial q^i} \right)\frac{\partial}{\partial q^j}$$

because the second-order derivative terms cancel. By the pull-back relation (5)

$$c^*\big[Q,\,R\big] = \big[c^*Q,\,c^*R\big] \tag{6}$$

under a change of variables defined by a smooth map, c. This means the definition of the vector field commutator is independent of the choice of coordinates.[1]

- The *differential* of a smooth function $f : M \to M$ is defined as

$$df = \frac{\partial f}{\partial q^i} dq^i \,,$$

in which the set dq^i, $i = 1, 2, \dots, \dim M$, is called a *differential basis set* for the manifold M.

- Under a smooth change of variables $s = \phi \circ q = \phi(q)$ the differential of the composition of functions $d(f \circ \phi)$ transforms according to the chain rule as

$$df = \frac{\partial f}{\partial q^i} dq^i \,, \quad d(f\circ\phi) = \frac{\partial f}{\partial \phi^j(q)}\frac{\partial \phi^j}{\partial q^i} dq^i = \frac{\partial f}{\partial s^j} ds^j \quad \Longrightarrow \quad d(f\circ\phi) = (df)\circ\phi \tag{7}$$

That is, the differential d commutes with the pull-back ϕ^* of a smooth transformation ϕ,

$$d(\phi^* f) = \phi^* df \,. \tag{8}$$

In a moment, this pull-back formula will give us the rule for transforming differential forms of any order.

- Differential k-forms on an n-dimensional manifold are defined in terms of the differential d and the antisymmetric *wedge product* $(\wedge)$ satisfying

$$dq^i \wedge dq^j = -\, dq^j \wedge dq^i \,, \quad \text{for} \quad i, j = 1, 2, \dots, n \tag{9}$$

By using wedge product, any k-form $\alpha \in \Lambda^k$ on M may be written locally at a point $q \in M$ in the differential basis dq^j as

$$\alpha_m = \alpha_{i_1 \dots i_k}(m)\, dq^{i_1} \wedge \cdots \wedge dq^{i_k} \in \Lambda^k \,, \quad i_1 < i_2 < \cdots < i_k \,, \tag{10}$$

where the sum over repeated indices is ordered, so that it must be taken over all i_j satisfying $i_1 < i_2 < \cdots < i_k$. Roughly speaking differential forms Λ^k are objects that can be integrated. As we shall see, vector fields also act on differential forms in interesting ways.

- Pull-backs of other differential forms may be built up from their basis elements, the dq^{i_k}.

By equation (8),

Theorem 1 (Pull-back of a wedge product)**.** *The pull-back of a wedge product of two differential forms is the wedge product of their pull-backs:*

$$\phi_t^*(\alpha \wedge \beta) = \phi_t^*\alpha \wedge \phi_t^*\beta \,. \tag{11}$$

[1] Letting the map c depend smoothly on a parameter t as c_t and taking the tangent to the relation $c_t^*\big[Q,\,R\big] = \big[c_t^*Q,\,c_t^*R\big]$ at the identity $t = 0$ results in the *Jacobi condition* for the vector fields to form an algebra. The Jacobi condition is discussed further below.

Definition 1 (Lie derivative of a differential k-form)**.** *The* Lie derivative *of a differential k-form Λ^k by a vector field Q is defined by linearising its flow ϕ_t around the identity $t = 0$,*

$$\pounds_Q \Lambda^k = \frac{d}{dt}\Big|_{t=0} \phi_t^* \Lambda^k \quad \textit{maps} \quad \pounds_Q \Lambda^k \mapsto \Lambda^k \,.$$

Hence, by equation (11), the Lie derivative satisfies the product rule for the wedge product.

Corollary 1 (Product rule for the Lie derivative of a wedge product)**.**

$$\pounds_Q(\alpha \wedge \beta) = \pounds_Q \alpha \wedge \beta + \alpha \wedge \pounds_Q \beta \,. \tag{12}$$

Proof. Linearise (11) around the identity, $t = 0$, using the product rule for the derivative. □

Variational principles

kinetic energy	Hamilton's principle	momentum
Riemannian metric	variational derivative	fibre derivative
Lagrangian	Legendre transformation	pairing

- Define *kinetic energy*, $KE : TM \to \mathbb{R}$, via a *Riemannian metric* $g_q(\,\cdot\,,\,\cdot\,) : TM \times TM \to \mathbb{R}$.
- Choose *Lagrangian* $L : TM \to \mathbb{R}$. (For example, one could choose L to be KE.)
- *Hamilton's principle* is $\delta S = 0$ with $S = \int_a^b L(q, \dot q) dt$, where for a family of curves parameterised smoothly by (t, ϵ) the linearisation

$$\delta S = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_a^b L(q(t,\epsilon), \dot q(t,\epsilon)) dt$$

  defines the *variational derivative* δS of S near the identity $\epsilon = 0$. The variations in q are assumed to vanish at endpoints in time, so that $q(a, \epsilon) = q(a)$ and $q(b, \epsilon) = q(b)$.

- *Legendre transformation* $LT : (q, \dot q) \in TM \to (q, p) \in T^*M$ defines *momentum* p as the *fibre derivative* of L, namely

$$p := \frac{\partial L(q, \dot q)}{\partial \dot q} \in T^*M \,.$$

  The LT is invertible for $\dot q = f(q, p)$, provided *Hessian* $\partial^2 L(q, \dot q)/\partial \dot q \partial \dot q$ has nonzero determinant. Note, $\dim T^*M = 2n$.

  In terms of LT, the *Hamiltonian* $H : T^*M \to \mathbb{R}$ is defined by

$$H(q, p) = \langle p, \dot q \rangle - L(q, \dot q)$$

  in which the expression $\langle p, \dot q \rangle$ in this calculation identifies a *pairing* $\langle\,\cdot\,,\,\cdot\,\rangle : T^*M \times TM \to \mathbb{R}$.

  Taking the differential of this definition yields

$$dH = \langle H_p, dp \rangle + \langle H_q, dq \rangle = \langle dp, \dot q \rangle + \langle p - L_{\dot q}, d\dot q \rangle - \langle L_q, dq \rangle$$

  from which Hamilton's principle $\delta S = 0$ for $S = \int_{t_0}^{t_1} \langle p, \dot q \rangle - H(q, p)\, dt$ produces Hamilton's canonical equations,

$$H_p = \dot q \quad \text{and} \quad H_q = -\,L_q = -\,\dot p \,.$$

- **Exercise:** Show that Hamilton's principle $\delta S = 0$ with $S = \int_a^b L(q, \dot{q})dt$ implies Euler-Lagrange (EL) equations:

$$\dot{p}(q, \dot{q}) = \frac{d}{dt}\frac{\partial L(q, \dot{q})}{\partial \dot{q}} = \frac{\partial L(q, \dot{q})}{\partial q}\,.$$

  What are the results for $\delta S = 0$ with $S = \int_a^b \langle p, \dot{q}\rangle - H(q,p)\,dt$?

- When $L = KE = \frac{1}{2}g_q(\dot{q}, \dot{q}) =: \frac{1}{2}\|\dot{q}\|^2$, the solution $q(t)$ of the EL equations that passes from point $q(a)$ to $q(b)$ is a *geodesic* with respect to the metric g_q.

  In mechanics the point $q(b)$ is determined at time $t = b$ from the solution $q(t)$ to the initial value problem for EL equations with q and $\dot{q}$ specified at the initial time, e.g., at $t = a$.

  It is also possible to phrase this as a boundary value problem in time, by specifying endpoint positions $q(a)$ and $q(b)$ instead of the initial values of q and $\dot{q}$.

Geometric Mechanics is exemplified by mechanics on Lie groups

This is a topic invented by H. Poincaré in 1901 [Po1901].

group	conjugation map	structure constants
Lie group, G	Lie algebra bracket,	reduced Lagrangian
identity element, e	$[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$	dual Lie algebra, $\mathfrak{g}^*$
Lie algebra, $\mathfrak{g}$	Jacobi identity	dual basis, $e^k \in \mathfrak{g}^*$
tangent vectors	basis vectors, $e_k \in \mathfrak{g}$	pairing, $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$

- A *group* is a set of elements with an associative binary product that has a unique inverse and identity element.

- A *Lie group* G is a group that depends smoothly on a set of parameters in $\mathbb{R}^{\dim(G)}$.

  A Lie group is also a manifold, so it is an interesting arena for geometric mechanics.

- Choose the manifold M for mechanics as discussed above to be the Lie group G and denote the *identity element* as the point e. The identity element e satisfies $eg = g = ge$ for all $g \in G$, where the group product denoted by concatenation.

- The *Lie algebra* $\mathfrak{g}$ of the Lie group G is defined as the space of *tangent vectors* $\mathfrak{g} \cong T_eG$ at the identity e of the group.

  The Lie algebra has a *bracket* operation $[\,\cdot\,,\,\cdot\,] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, which it inherits from linearisation at the identity e of the *conjugation map* $h \cdot g = hgh^{-1}$ for $g, h \in G$. For this, one begins with the conjugation map $h(t) \cdot g(s) = h(t)g(s)h(t)^{-1}$ for curves $g(s), h(t) \in G$, with $g(0) = e = h(0)$. One linearises at the identity, first in s to get the operation $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ and then in t to get the operation $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which yields the Lie bracket. The bracket operation is antisymmetric $[a, b] = -[b, a]$ and satisfies the *Jacobi condition* for $a, b, c \in \mathfrak{g}$,

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0\,. \tag{13}$$

  The bracket operation among the *basis vectors* $e_k \in \mathfrak{g}$ with $k = 1, 2, \ldots, \dim(\mathfrak{g})$ defines the Lie algebra by its *structure constants* ${c_{ij}}^k$ in (summing over repeated indices)

$$[e_i\,,\, e_j] = {c_{ij}}^k e_k\,. \tag{14}$$

The requirement of skew-symmetry and the Jacobi condition put constraints on the structure constants. These constraints are

– skew-symmetry

$$c_{ji}^k = -\, c_{ij}^k \,, \tag{15}$$

– Jacobi identity

$$c_{ij}^k c_{lk}^m + c_{li}^k c_{jk}^m + c_{jl}^k c_{ik}^m = 0 \,. \tag{16}$$

Conversely, any set of constants c_{ij}^k that satisfy relations (15)–(16) defines a Lie algebra $\mathfrak{g}$.

Exercise: Prove that the Jacobi condition requires the relation (16).

Hint: the Jacobi condition involves summing three terms of the form

$$[\,\mathbf{e}_l \,,\, [\,\mathbf{e}_i \,,\, \mathbf{e}_j\,]\,] = c_{ij}^k [\,\mathbf{e}_l \,,\, \mathbf{e}_k] = c_{ij}^k c_{lk}^m \mathbf{e}_m \,.$$

Exercise: Prove that the Jacobi condition (13) arises from the linearisation of (6).

H. Poincaré's contribution [Po1901].

To understand [Po1901], let's begin by endowing the Lie algebra $\mathfrak{g}$ with a constant Riemannian metric $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ and introducing two more definitions.

1. Define a *reduced Lagrangian* $l : \mathfrak{g} \to \mathbb{R}$ and an associated variational principle $\delta S = 0$ with $S = \int_a^b l(\xi) dt$ where $\xi = \xi^k e_k \in \mathfrak{g}$ has components ξ^k in the set of basis vectors e_k.

2. Define the *dual Lie algebra* $\mathfrak{g}^*$ by using the fibre derivative of the Lagrangian $l : \mathfrak{g} \to \mathbb{R}$ as

$$\mu := \frac{\partial l(\xi)}{\partial \xi} \in \mathfrak{g}^* \,.$$

The relation $dl = \langle \mu, d\xi \rangle$ defines a pairing $\langle \,\cdot\,,\,\cdot\, \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$. A natural *dual basis* for $\mathfrak{g}^*$ would satisfy $\langle e^j, e_k \rangle = \delta^j_k$ in this pairing and an element $\mu \in \mathfrak{g}^*$ would have components in this dual basis given by $\mu = \mu_k e^k$, again with with $k = 1, 2, \ldots, \dim(\mathfrak{g})$.

- ***Exercise:***

  (a) Show that Hamilton's principle $\delta S = 0$ with $S = \int_a^b l(\xi) dt$ implies the Euler-Poincaré (EP) equations:

  $$\frac{d}{dt}\mu_i(\xi) = \frac{d}{dt}\frac{\partial l(\xi)}{\partial \xi^i} = -\, {c_{ij}}^k \xi^j \mu_k(\xi) \,,$$

  for variations given by $\delta\xi = \dot{\eta} + [\xi, \eta]$ with $\xi, \eta \in \mathfrak{g}$.

  (b) Show that this variational formulation recovers Poincaré's equations introduced in [Po1901].

- ***Exercise:*** The Lie algebra $\mathfrak{so}(3)$ of the Lie group $SO(3)$ of rotations in three dimensions has structure constants ${c_{ij}}^k = {\epsilon_{ij}}^k$, where ${\epsilon_{ij}}^k$ with $i, j, k \in \{1, 2, 3\}$ is totally antisymmetric under pairwise permutations of its indices, with ${\epsilon_{12}}^3 = 1$, ${\epsilon_{21}}^3 = -1$, etc.

  (a) Identify the Lie bracket $[a, b]$ of two elements $a = a^i e_i, b = b^j e_j \in \mathfrak{so}(3)$ with the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ according to

  (b) Show that this formula implies the Jacobi identity for the cross product of vectors in $\mathbb{R}^3$.

This is no surprise because, that familiar cross product relation for vectors may be proven by using the antisymmetric tensor $\epsilon_{ij}{}^k$.

$$[a,b] = [a^i e_i, b^j e_j] = a^i b^j \epsilon_{ij}{}^k e_k = (\mathbf{a}\times\mathbf{b})^k e_k \,.$$

(c) Show that for vectors in $\mathbb{R}^3$ the EP equation

$$\dot\mu_i = -\epsilon_{ij}{}^k \xi^j \mu_k$$

is equivalent to the vector equation for $\boldsymbol{\xi}, \boldsymbol{\mu} \in \mathbb{R}^3$

$$\dot{\boldsymbol{\mu}} = -\,\boldsymbol{\xi}\times\boldsymbol{\mu}\,.$$

(d) Show that when the Lagrangian is given by the quadratic expression

$$l(\boldsymbol{\xi}) = \frac{1}{2}\|\boldsymbol{\xi}\|_K^2 = \frac{1}{2}\boldsymbol{\xi}\cdot K\boldsymbol{\xi} = \frac{1}{2}\xi^i K_{ij}\xi^j$$

for a symmetric constant Riemannian metric $K^T = K$, then Euler's equations for a rotating rigid body are recovered.

(d) Identify the functional dependence of $\boldsymbol{\mu}$ on $\boldsymbol{\xi}$ and give the physical meanings of the symbols $\boldsymbol{\xi}, \boldsymbol{\mu}$ and K in Euler's rigid body equations.

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