M3/4A16 Assessed Coursework 3 **Darryl Holm**

Due in class Tuesday December 16, 2008

Ask in class about clarifying the exact meaning of a question if you're unsure.

No conferring or copying: it will be more obvious than you think.

#1 *Exercises in exterior calculus operations*

Vector notation for differential basis elements:
One denotes differential basis elements dx^i *and* $dS_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$, *for* $i, j, k = 1, 2, 3$, *in vector notation as*

$$\begin{aligned}
d\mathbf{x} &:= (dx^1, dx^2, dx^3)\,, \\
d\mathbf{S} &= (dS_1, dS_2, dS_3) \\
&:= (dx^2 \wedge dx^3,\ dx^3 \wedge dx^1,\ dx^1 \wedge dx^2)\,, \\
dS_i &:= \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k\,, \\
d^{\,3}x &= d\mathrm{Vol} := dx^1 \wedge dx^2 \wedge dx^3\,.
\end{aligned}$$

(1a) ***Vector algebra operations***

(i) *Show that contraction with the vector field* $X = X^j\partial_j =: \mathbf{X}\cdot\nabla$ *recovers the following familiar operations among vectors*

$$\begin{aligned}
X \lrcorner\, d\mathbf{x} &= \mathbf{X}\,, \\
X \lrcorner\, d\mathbf{S} &= \mathbf{X} \times d\mathbf{x}\,, \\
(\textit{or,}\quad X \lrcorner\, dS_i &= \epsilon_{ijk}X^j dx^k) \\
Y \lrcorner\, X \lrcorner\, d\mathbf{S} &= \mathbf{X} \times \mathbf{Y}\,, \\
X \lrcorner\, d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S} = X^k dS_k\,, \\
Y \lrcorner\, X \lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y} \cdot d\mathbf{x} = \epsilon_{ijk}X^iY^jdx^k\,, \\
Z \lrcorner\, Y \lrcorner\, X \lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y} \cdot \mathbf{Z}\,.
\end{aligned}$$

(ii) *Show that these are consistent with*

$$X \lrcorner\, (\alpha \wedge \beta) = (X \lrcorner\, \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \lrcorner\, \beta)\,,$$

for a k*-form* α.

(iii) *Use (ii) to compute* $Y \lrcorner\, X \lrcorner\, (\alpha \wedge \beta)$ *and* $Z \lrcorner\, Y \lrcorner\, X \lrcorner\, (\alpha \wedge \beta)$.

(1b) ***Exterior derivative examples in vector notation***
Show that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation

$$\begin{aligned}
df &= f_{,j}\, dx^j =: \nabla f \cdot d\mathbf{x} \\
0 = d^2 f &= f_{,jk}\, dx^k \wedge dx^j \\
df \wedge dg &= f_{,j}\, dx^j \wedge g_{,k}\, dx^k =: (\nabla f \times \nabla g)\cdot d\mathbf{S} \\
df \wedge dg \wedge dh &= f_{,j}\, dx^j \wedge g_{,k}\, dx^k \wedge h_{,l}\, dx^l =: (\nabla f \cdot \nabla g \times \nabla h)\, d^{\,3}x
\end{aligned}$$

Likewise, show that

$$\begin{aligned} d(\mathbf{v}\cdot d\mathbf{x}) &= (\mathrm{curl}\,\mathbf{v})\cdot d\mathbf{S} \\ d(\mathbf{A}\cdot d\mathbf{S}) &= (\mathrm{div}\,\mathbf{A})\,d^{\,3}x\,. \end{aligned}$$

Verify the compatibility condition $d^2=0$ *for these forms as*

$$\begin{aligned} 0 = d^2 f = d(\nabla f\cdot d\mathbf{x}) &= (\mathrm{curl}\,\mathrm{grad}\,f)\cdot d\mathbf{S}\,, \\ 0 = d^2(\mathbf{v}\cdot d\mathbf{x}) = d\big((\mathrm{curl}\,\mathbf{v})\cdot d\mathbf{S}\big) &= (\mathrm{div}\,\mathrm{curl}\,\mathbf{v})\,d^{\,3}x\,. \end{aligned}$$

Verify the exterior derivatives of these contraction formulas for $X=\mathbf{X}\cdot\nabla$

(i) $d(X \,\lrcorner\, \mathbf{v}\cdot d\mathbf{x}) = d(\mathbf{X}\cdot\mathbf{v}) = \nabla(\mathbf{X}\cdot\mathbf{v})\cdot d\mathbf{x}$

(ii) $d(X \,\lrcorner\, \boldsymbol{\omega}\cdot d\mathbf{S}) = d(\boldsymbol{\omega}\times\mathbf{X}\cdot d\mathbf{x}) = \mathrm{curl}\,(\boldsymbol{\omega}\times\mathbf{X})\cdot d\mathbf{S}$

(iii) $d(X \,\lrcorner\, f\,d^{\,3}x) = d(f\mathbf{X}\cdot d\mathbf{S}) = \mathrm{div}\,(f\mathbf{X})\,d^{\,3}x$

(1c) *Use Cartan's formula,*

$$\pounds_X\alpha = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha)$$

for a $k-$*form* α, $k=0,1,2,3$ *in* $\mathbb{R}^3$ *to verify the Lie derivative formulas:*

(i) $\pounds_X f = X \,\lrcorner\, df = \mathbf{X}\cdot\nabla f$

(ii) $\pounds_X\,(\mathbf{v}\cdot d\mathbf{x}) = \big(-\,\mathbf{X}\times\mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{X}\cdot\mathbf{v})\big)\cdot d\mathbf{x}$

(iii) $\pounds_X(\boldsymbol{\omega}\cdot d\mathbf{S}) = \big(\mathrm{curl}\,(\boldsymbol{\omega}\times\mathbf{X}) + \mathbf{X}\,\mathrm{div}\,\boldsymbol{\omega}\big)\cdot d\mathbf{S}$
$\qquad = \big(-\,\boldsymbol{\omega}\cdot\nabla\mathbf{X} + \mathbf{X}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\,\mathrm{div}\,\mathbf{X}\big)\cdot d\mathbf{S}$

(iv) $\pounds_X(f\,d^{\,3}x) = (\mathrm{div}\,f\mathbf{X})\,d^{\,3}x$

(v) *Derive these formulas from the dynamical definition of Lie derivative.*

(1d) $\boxed{\textbf{\textit{Fourth year students}}}$ *Verify the following Lie derivative identities both by using Cartan's formula and by using the dynamical definition of Lie derivative:*

(i) $\pounds_{fX}\alpha = f\pounds_X\alpha + df\wedge(X \,\lrcorner\, \alpha)$

(ii) $\pounds_X d\alpha = d\big(\pounds_X\alpha\big)$

(iii) $\pounds_X(X \,\lrcorner\, \alpha) = X \,\lrcorner\, \pounds_X\alpha$

(iv) $\pounds_X(Y \,\lrcorner\, \alpha) = (\pounds_X Y) \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X\alpha)$

(v) $\pounds_X(\alpha\wedge\beta) = (\pounds_X\alpha)\wedge\beta + \alpha\wedge\pounds_X\beta$

#2 *Operations among vector fields*

The Lie derivative of one vector field by another is called the ***Jacobi-Lie bracket****, defined as*

$$\pounds_X Y := [X\,,\,Y] := \nabla Y\cdot X - \nabla X\cdot Y = -\,\pounds_Y X$$

In components, the Jacobi-Lie bracket is

$$[X\,,\,Y] = \left[X^k\frac{\partial}{\partial x^k}\,,\,Y^l\frac{\partial}{\partial x^l}\right] = \left(X^k\frac{\partial Y^l}{\partial x^k} - Y^k\frac{\partial X^l}{\partial x^k}\right)\frac{\partial}{\partial x^l}$$

The Jacobi-Lie bracket among vector fields satisfies the Jacobi identity,

$$[\,X\,,\,[Y\,,\,Z]\,] + [\,Y\,,\,[Z\,,\,X]\,] + [\,Z\,,\,[X\,,\,Y]\,] = 0$$

Verify the following formulas

(2a) $X \lrcorner (Y \lrcorner \alpha) = -Y \lrcorner (X \lrcorner \alpha)$

(2b) $[X\,,\,Y] \lrcorner \alpha = \pounds_X(Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha)$, *for zero-forms (functions) and one-forms.*

(2c) $\pounds_{[X\,,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$, *as a result of* **(b)**. *Use* **2(c)** *to verify the Jacobi identity.*

(2d) $\boxed{\textbf{\textit{Fourth year students}}}$
Verify formula 2(b) for arbitrary k−forms.

> *Problems #1 and #2 are solved in the text. Most of the various parts of these problems were also discussed in class.*

#3 *Poisson brackets on* $\mathbb{C}^3/S^1$

(3a) *By using the canonical Poisson brackets for* $a_j = q_j + ip_j$ *and* $a_k^* = q_k - ip_k$

$$\{a_j\,,\,a_k^*\} = -2i\,\delta_{jk} \quad \textit{derived from} \quad \{q_j\,,\,p_k\} = \delta_{jk}\,,$$

compute the Poisson brackets among the nine quadratic quantities

$$Q_{jk} = a_j a_k^* \in \mathbb{C}^3/S^1 \quad \textit{for} \quad j,k = 1,2,3\,.$$

Hint: These are related to the canonical $\{q_j\,,\,p_k\}$ *coordinates by,*

$$\begin{aligned} Q_{jk} &= a_j a_k^* = (q_j + ip_j)(q_k - ip_k) \\ &= \underbrace{(q_jq_k + p_jp_k)}_{\textit{symmetric}} + i\,\underbrace{(p_jq_k - q_jp_k)}_{\textit{skew-symmetric}} \\ &= S_{jk} + i\,A_{jk}\,. \end{aligned}$$

where $S = \Re Q$ *and* $A = \Im Q$.

> *The Poisson bracket relations may also be read off from the Poisson commutators of the real and imaginary components of* $Q_{jk} \in \mathbb{C}^3/S^1$ *among themselves as*
>
> $$\begin{aligned} \{S_{jk}\,,\,S_{lm}\} &= \delta_{jl}A_{mk} + \delta_{kl}A_{mj} - \delta_{jm}A_{kl} - \delta_{km}A_{jl} \\ \{S_{jk}\,,\,A_{lm}\} &= \delta_{jl}S_{mk} + \delta_{kl}S_{mj} - \delta_{jm}S_{kl} - \delta_{km}S_{jl} \\ \{A_{jk}\,,\,A_{lm}\} &= \delta_{jl}A_{mk} - \delta_{kl}A_{mj} + \delta_{jm}A_{kl} - \delta_{km}A_{jl} \end{aligned}$$

(3b) *Define* $L_a = \epsilon_{ajk}A_{jk} = (\mathbf{p} \times \mathbf{q})_a$ *from the imaginary (skew-symmetric) part of* Q_{jk} *and compute the Poisson brackets:*

$$\{L_a\,,\,L_b\} \quad \textit{and} \quad \{L_a\,,\,Q_{jk}\}\,,$$

for $a,b,j,k = 1,2,3$.
Do these Poisson brackets for S^1*-invariant functions on* $\mathbb{C}^3$ *close among themselves? Why is that?*

One defines $L_a = -\frac{1}{2}\,\epsilon_{ajk}A_{jk} = (\mathbf{p}\times\mathbf{q})_a$ *and finds the Poisson bracket relations*

$$\begin{aligned} \{L_a\,,\,L_b\} &= [A_{ab} - A_{ba}] = \epsilon_{abc}L_c\,, \\ \{L_a\,,\,Q_{jk}\} &= \frac{1}{2}\left[\epsilon_{ajc}Q_{ck} - \epsilon_{akc}Q_{jc}\right]. \end{aligned}$$

These close among themselves because they are quadratic invariants of the symmetry $\mathbb{C}^3/S^1$. *The quadratic quantities close among themselves under Poisson bracket and the left and right sides of the calculation must both be* S^1*-invariant.*

(3c) *Write* $Q = S + iA$ *in the particular form to define* L_a, M_b *and* N_c *for indices* $a,b,c = 1,2,3$, *as*

$$Q = \begin{bmatrix} M_1 & N_3 - iL_3 & N_2 + iL_2 \\ N_3 + iL_3 & M_2 & N_1 - iL_1 \\ N_2 - iL_2 & N_1 + iL_1 & M_3 \end{bmatrix} = \begin{bmatrix} M_1 & N_3 & N_2 \\ N_3 & M_2 & N_1 \\ N_2 & N_1 & M_3 \end{bmatrix} + i \begin{bmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{bmatrix}$$

Compute the Poisson brackets $\{M_a\,,\,M_b\}$ *and* $\{N_a\,,\,N_b\}$.

$\{M_a\,,\,M_b\} = 0$ *and* $\{N_a\,,\,N_b\} = -\epsilon_{abc}L_c = -\{L_a\,,\,L_b\}$

(3d) ***Fourth year students***

Complete the Poisson bracket tables and use them to compute the Poisson bracket relations for,

$$\{N_a - iL_a\,,\,N_b - iL_b\}\,, \quad \{N^2 + L^2\,,\,N_b - iL_b\} \quad \text{and} \quad \{M_a\,,\,N_b - iL_b\}\,,$$

for indices $a,b = 1,2,3$.

These complete tables of Poisson brackets are,

$\{\cdot,\cdot\}$	L_1	L_2	L_3
L_1	0	L_3	$-L_2$
L_2	$-L_3$	0	L_1
L_3	L_2	$-L_1$	0

$\{\cdot,\cdot\}$	L_1	L_2	L_3
M_1	0	$2N_2$	$-2N_3$
M_2	$-2N_1$	0	$2N_3$
M_3	$2N_1$	$-2N_2$	0

$\{\cdot,\cdot\}$	N_1	N_2	N_3
N_1	0	$-L_3$	L_2
N_2	L_3	0	$-L_1$
N_3	$-L_2$	L_1	0

$\{\cdot,\cdot\}$	N_1	N_2	N_3
M_1	0	$-2L_2$	$2L_3$
M_2	$2L_1$	0	$-2L_3$
M_3	$-2L_1$	$2L_2$	0

$\{\cdot,\cdot\}$	L_1	L_2	L_3
N_1	$M_2 - M_3$	$-N_3$	N_2
N_2	N_3	$M_3 - M_1$	$-N_1$
N_3	$-N_2$	N_1	$M_1 - M_2$

As expected, the system is closed and it has the angular momentum Poisson bracket table as a closed subset. This is because the Lie algebra $su(2)$ is a subalgebra of $su(3)$.

These Poisson brackets may be consolidated into

$$
\begin{aligned}
\{M_a\,,\,M_b\} &= 0\,, \quad \{N_a\,,\,N_b\} = \epsilon_{abc}L_c = -\{L_a\,,\,L_b\}\,, \\
\{N_a\,,\,L_b\} &= -\,\epsilon_{abc}N_c + \delta_{ab}\mathrm{diag}(\Delta M)_b\,,
\end{aligned}
$$

where the traceless diagonal matrix $\mathrm{diag}(\Delta M)$ *has entries*

$$
\mathrm{diag}(\Delta M) = \begin{pmatrix} M_2 - M_3 & 0 & 0 \\ 0 & M_3 - M_1 & 0 \\ 0 & 0 & M_1 - M_2 \end{pmatrix}
$$

The required (interesting!) set of Poisson bracket relations among the M's, N's and L's is then,

$$
\begin{aligned}
\{N_a - iL_a\,,\,N_b - iL_b\} &= 2i\,\epsilon_{abc}(N_c + iL_c)\,, \\
\{N^2 + L^2\,,\,N_b - iL_b\} &= -2i(N_b - iL_b)\mathrm{diag}(\Delta M)_b\,, \\
\{M_a\,,\,N_b - iL_b\} &= 2i\,\mathrm{sgn}(b-a)(-1)^{a+b}(N_b - iL_b)\,.
\end{aligned}
$$

Note placements of $\pm i$ in the first equation. In deriving the second equation we used

$$
\begin{aligned}
\{L^2\,,\,L_b\} &= 0 \\
\{N^2\,,\,N_b\} &= -2(\mathbf{L}\times\mathbf{N})_b \\
\{N^2\,,\,L_b\} &= 2N_b\mathrm{diag}(\Delta M)_b\,, \\
\{L^2\,,\,N_b\} &= 2(\mathbf{L}\times\mathbf{N})_b + 2L_b\mathrm{diag}(\Delta M)_b\,.
\end{aligned}
$$

In the equation for $\{M_a\,,\,N_b - iL_b\}$, the quantity sgn$(b-a)$ is the sign of the difference $(b-a)$, which vanishes when $b=a$.

#4 $GL(n,\mathbb{R})-$invariant motions and infinitesimal generators

Begin with the Lagrangian

$$
L(S,\dot S,\dot{\mathbf{q}}) = \frac{1}{2}\,\mathrm{tr}\Big(\dot S S^{-1}\dot S S^{-1}\Big) + \frac{1}{2}\,\dot{\mathbf{q}}^T S^{-1}\dot{\mathbf{q}}
$$

where S is an $n\times n$ symmetric matrix and $\mathbf{q}\in\mathbb{R}^n$ is an $n-$component column vector. Note that the Lagrangian $L(S,\dot S,\dot{\mathbf{q}})$ is conveniently independent of the coordinate $\mathbf{q}$.

(4a) *Legendre transform to find the Hamiltonian for this system and write its* canonical *equations.*

(4b) *Show that the Lagrangian and Hamiltonian for this system are both invariant under the group action*

$$
\mathbf{q}\to G\mathbf{q} \quad \textit{and} \quad S\to GSG^T
$$

for any constant invertible $n\times n$ matrix, G.

(4c) **(i)** *Linearise this group action around the identity in terms of* $A = G'G^{-1}$ *and construct the infinitesimal transformations* $X_A\mathbf{q}$ *and* X_AS *for the linearised action of* G *on the configuration space* $(\mathbf{q}, S)$*.*

(ii) *Find the phase space function (*infinitesimal generator*) whose canonical Poisson brackets produce these infinitesimal transformations by pairing* $X_A\mathbf{q}$ *and* X_AS *with the corresponding canonical momenta and summing.*

(iii) *Compute the Poisson bracket of the canonical momenta with the infinitesimal generator. (This is the* cotangent lift *to the full phase space of the infinitesimal action of* G *on the configuration space.)*

(4d) $\boxed{\textbf{\textit{Fourth year students}}}$

(i) *Verify directly that the infinitesimal generator of the* G*-action is a conserved* $n \times n$ *matrix quantity by using the equations of motion.*

(ii) *Determine whether this Hamiltonian system has sufficiently many conservation laws in involution to be completely integrable, for any dimension* n*.*

(4a) *The Legendre transform is*

$$P = \frac{\partial L}{\partial \dot{S}} = S^{-1}\dot{S}S^{-1} \quad \textit{and} \quad \mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = S^{-1}\dot{\mathbf{q}}$$

Thus, the Hamiltonian $H(\mathbf{q}, \mathbf{p}, S, P)$ *is*

$$H(\mathbf{q}, \mathbf{p}, S, P) = \frac{1}{2}\mathrm{tr}\left(PS \cdot PS\right) + \frac{1}{2}\mathbf{p} \cdot S\mathbf{p}\,,$$

and its canonical equations are:

$$\dot{S} = \frac{\partial H}{\partial P} = SPS\,, \quad \dot{P} = -\frac{\partial H}{\partial S} = -\Big(PSP + \frac{1}{2}\mathbf{p} \otimes \mathbf{p}\Big),$$

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} = S\mathbf{p}\,, \quad \dot{\mathbf{p}} = \frac{\partial H}{\partial \mathbf{q}} = 0\,.$$

(4b) *Under the group action* $\mathbf{q} \to G\mathbf{q}$ *and* $S \to GSG^T$ *for any constant invertible* $n \times n$ *matrix,* G*, one finds* $\dot{S}S^{-1} \to G\dot{S}S^{-1}G^{-1}$ *and* $\dot{\mathbf{q}} \cdot S^{-1}\dot{\mathbf{q}} \to \dot{\mathbf{q}} \cdot S^{-1}\dot{\mathbf{q}}$*. Hence,* $L \to L$*. Likewise,* $P \to G^{-T}PG^{-1}$ *so* $PS \to G^{-T}PSG^T$ *and* $\mathbf{p} \to G^{-T}\mathbf{p}$ *so that* $S\mathbf{p} \to GS\mathbf{p}$*. Hence,* $H \to H$*, as well; so both* L *and* H *for the system are invariant.*

(4c) *The infinitesimal actions for* $G(\epsilon) = Id + \epsilon A + O(\epsilon^2)$*, where* $A \in gl(n)$ *are*

$$X_A\mathbf{q} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} G(\epsilon)\mathbf{q} = A\mathbf{q} \quad \textit{and} \quad X_AS = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\Big(G(\epsilon)SG(\epsilon)^T\Big) = AS + SA^T$$

Pairing $X_A\mathbf{q}$ *and* X_AS *with their corresponding canonical momenta and summing yields*

$$\langle J, A\rangle := \mathrm{tr}\left(PX_AS\right) + \mathbf{p} \cdot X_A\mathbf{q} = \mathrm{tr}\left(P(AS + SA^T)\right) + \mathbf{p} \cdot A\mathbf{q}$$

Hence,

$$\langle J, A\rangle := \operatorname{tr}\left(JA^T\right) = \operatorname{tr}\Big((2SP + \mathbf{q}\otimes\mathbf{p})A\Big)\,, \quad \textit{so} \quad J = (2PS + \mathbf{p}\otimes\mathbf{q})$$

For any chice of the matrix A, the Poisson bracket with $\langle J, A\rangle$ generates the Hamiltonian vector field

$$\begin{aligned}
\Big\{\,\cdot\,, \langle J, A\rangle\Big\} &= \operatorname{tr}\left(\frac{\partial\langle J, A\rangle}{\partial P}\frac{\partial}{\partial S}\right) + \frac{\partial\langle J, A\rangle}{\partial \mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{q}} \\
&\quad - \operatorname{tr}\left(\frac{\partial\langle J, A\rangle}{\partial S}\frac{\partial}{\partial P}\right) - \frac{\partial\langle J, A\rangle}{\partial \mathbf{q}}\cdot\frac{\partial}{\partial \mathbf{p}} \\
&= \operatorname{tr}\left((AS + SA^T)\frac{\partial}{\partial S}\right) + A\mathbf{q}\cdot\frac{\partial}{\partial \mathbf{q}} \\
&\quad - \operatorname{tr}\left((PA + A^TP)\frac{\partial}{\partial P}\right) - A^T\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{p}}
\end{aligned}$$

which recovers the infinitesimal action on $(S, \mathbf{q})$ and provides the cotangent-lifted action on the canonical momenta $(P, \mathbf{p})$.

(4d) *Conservation of $\langle J, A\rangle$ is verified directly in*

$$\frac{d}{dt}\langle J, A\rangle = \langle \dot{J}, A\rangle$$

by computing

$$\begin{aligned}
\dot{J} &= \Big(2\dot{P}S + 2P\dot{S} + \dot{\mathbf{p}}\otimes\mathbf{q} + \mathbf{p}\otimes\dot{\mathbf{q}}\Big) \\
&= -\Big(2PSP + (\mathbf{p}\otimes\mathbf{p})\Big)S + 2P\Big(SPS\Big) + \mathbf{0}\otimes\mathbf{q} + \mathbf{p}\otimes S\mathbf{p} \\
&= 0\,.
\end{aligned}$$

The system has $n(n+1)/2 + n = n(n+3)/2$ degrees of freedom. It conserves the n components of linear momentum $\mathbf{p}$ and the $n(n+1)/2$ components of J. Thus, there is one constant of motion for each degree of freedom. However, these two sets of independent conservation laws do not Poisson commute, since

$$\Big\{\mathbf{p}\,, \langle J, A\rangle\Big\} = -A^T\mathbf{p}\,.$$

This means that the naive count of degrees of freedom will not produce complete integrability, because the constants of motion are not in involution. In general, something more would be needed for complete integrability of this system to hold. This is a potential research question.