Solutions to Assessed Homework 3 **Darryl Holm**

(#1) *Euler-Lagrange equations for geodesics on* $SO(3)$

#1a The skew symmetry of $\widehat{\Omega}(t) = O^{-1}\dot{O}(t)$ follows by taking the time derivative of the defining relation for orthogonal matrices.

The time derivative of $O^T(t)\mathbb{I}O(t) = \mathbb{I}$ yields $(O^{-1}(t)\mathbb{I}O(t))^{\boldsymbol{\cdot}} = 0$. This means

$$0 = \big[\dot{O}^T O^{-T}\big]\mathbb{I} + \mathbb{I}\big[O^{-1}\dot{O}\big]\,.$$

Consequently, if $\widehat{\Omega} = [\dot{O}O^{-1}]$ and $\mathbb{I}^T = \mathbb{I}$, the quantity $\mathbb{I}\widehat{\Omega}$ is skew. That is,

$$(\mathbb{I}\widehat{\Omega})^T = -\,\mathbb{I}\widehat{\Omega}\,.$$

When $\mathbb{I}$ is the identity, this is the expected condition for the angular velocity.

#1b The variational formula is

$$\delta\widehat{\Omega} = \widehat{\Xi}^{\boldsymbol{\cdot}} + \widehat{\Omega}\widehat{\Xi} - \widehat{\Xi}\widehat{\Omega}\,, \quad \text{in which} \quad \widehat{\Xi} = O^{-1}\delta O\,. \tag{1}$$

This formula follows by subtracting the time derivative $\widehat{\Xi}^{\boldsymbol{\cdot}} = (O^{-1}\delta O)^{\boldsymbol{\cdot}}$ from the variational derivative $\delta\widehat{\Omega} = \delta(O^{-1}\dot{O})$ in the relations

$$\begin{aligned}
\delta\widehat{\Omega} &= \delta(O^{-1}\dot{O}) = -\,(O^{-1}\delta O)(O^{-1}\dot{O}) + \delta\dot{O} = -\,\widehat{\Xi}\widehat{\Omega} + \delta\dot{O}\,,\\
\widehat{\Xi}^{\boldsymbol{\cdot}} &= (O^{-1}\delta O)^{\boldsymbol{\cdot}} = -\,(O^{-1}\dot{O})(O^{-1}\delta O) + (\delta O)^{\boldsymbol{\cdot}} = -\,\widehat{\Omega}\widehat{\Xi} + (\delta O)^{\boldsymbol{\cdot}}\,,
\end{aligned}$$

and using equality of cross derivatives $\delta\dot{O} = (\delta O)^{\boldsymbol{\cdot}}$.

#1c Hamilton's principle for this problem is

$$L(\widehat{\Omega}) = -\frac{1}{2}\mathrm{tr}(\widehat{\Omega}\mathbb{A}\widehat{\Omega})\,,$$

in which $\mathbb{A}$ is a symmetric, positive-definite 3×3 matrix. Taking matrix variations yields

$$\begin{aligned}
\delta S &=: -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\,\frac{\delta L}{\delta\widehat{\Omega}}\right)dt\\
&= -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\mathbb{A}\widehat{\Omega} + \delta\widehat{\Omega}\widehat{\Omega}\mathbb{A}\right)dt\\
&= -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\,(\mathbb{A}\widehat{\Omega} + \widehat{\Omega}\mathbb{A})\right)dt\\
&= -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\,\widehat{\Pi}\right)dt\,.
\end{aligned}$$

The first step defines the variational derivative of S in terms of the matrix pairing. The second step applies the variational derivative. After cyclically permuting the order of matrix multiplication under the trace in the third step, the fourth step substitutes

$$\widehat{\Pi} = \mathbb{A}\widehat{\Omega} + \widehat{\Omega}\mathbb{A} = \frac{\delta L}{\delta \widehat{\Omega}}\,.$$

Next, substituting formula (1) for $\delta\widehat{\Omega}$ from part **(#1b)** into the variation of the action (2) leads to

$$\delta S = -\frac{1}{2}\int_a^b \mathrm{tr}\,\big(\delta\widehat{\Omega}\widehat{\Pi}\big)\,dt = -\frac{1}{2}\int_a^b \mathrm{tr}\,\big((\widehat{\Xi}^{\,\cdot} + \widehat{\Omega}\widehat{\Xi} - \widehat{\Xi}\widehat{\Omega})\widehat{\Pi}\big)\,dt\,.$$

Permuting cyclically under the trace again yields $\mathrm{tr}(\widehat{\Omega}\widehat{\Xi}\widehat{\Pi}) = \mathrm{tr}(\widehat{\Xi}\widehat{\Pi}\widehat{\Omega})$. Integrating by parts (dropping endpoint terms) then yields the equation

$$\delta S = -\,\frac{1}{2}\int_a^b \mathrm{tr}\,\big(\widehat{\Xi}\,(\,-\,\widehat{\Pi}^{\,\cdot} + \widehat{\Pi}\widehat{\Omega} - \widehat{\Omega}\widehat{\Pi}\,)\big)\,dt\,.$$

Finally, invoking stationarity $\delta S = 0$ for an arbitrary variation $\widehat{\Xi} = O^{-1}\delta O$ yields geodesic dynamics on $SO(3)$ with respect to the metric $\mathbb{A}$ in the matrix commutator form

$$\frac{d\widehat{\Pi}}{dt} = -\,[\,\widehat{\Omega}\,,\,\widehat{\Pi}\,] \quad \text{with} \quad \widehat{\Pi} = \mathbb{A}\widehat{\Omega} + \widehat{\Omega}\mathbb{A} = \frac{\delta L}{\delta\widehat{\Omega}} = -\,\widehat{\Pi}^T\,.$$

#1d $\boxed{\textbf{\textit{Fourth year, MSc and MSci students}}}$

Identify vector components Ω_k, $k = 1,2,3$, with the components of the skew-symmetric matrix $\widehat{\Omega}_{ij}$, $i,j = 1,2,3$, as

$$\widehat{\Omega}_{ij} = -\,\epsilon_{ijk}\Omega_k\,.$$

This relation implies the Euler-Lagrange equations from **(#1c)** may be written in $\mathbb{R}^3$ vector form as

$$\dot{\mathbf{\Pi}} = -\,\mathbf{\Omega}\times\mathbf{\Pi}\,,$$

whose vector components are expressed as

$$\begin{aligned}(a_2 + a_3)\dot{\Omega}_1 &= -(a_2 - a_3)\Omega_2\Omega_3\,,\\ (a_3 + a_1)\dot{\Omega}_2 &= -(a_3 - a_1)\Omega_3\Omega_1\,,\\ (a_1 + a_2)\dot{\Omega}_3 &= -(a_1 - a_2)\Omega_1\Omega_2\,.\end{aligned}$$

(#2) *Modulation equations*

The real 3-wave modulation equations on $\mathbb{R}^3$ are

$$\dot{X}_1 = -X_2X_3\,, \quad \dot{X}_2 = -X_3X_1\,, \quad \dot{X}_3 = +X_1X_2\,.$$

#2a These equations may be written in the $\mathbb{R}^3$ bracket form as,

$$\dot{\mathbf{X}} = \nabla C \times \nabla H = \nabla\frac{1}{2}(X_1^2 + X_3^2) \times \nabla\frac{1}{2}(X_2^2 + X_3^2)\,,$$

where level sets of C and H are circular cylinders, oriented along the X_2 and X_1 axes, respectively.

Characterise the equilibrium points geometrically in terms of the gradients of C and H. How many are there? Which are stable?

Equilibria occur at points where the cross product of gradients $\nabla C \times \nabla H$ vanishes. In the orthogonal intersection of two circular cylinders as above, this may occur at points where the circular cylinders are tangent, and at points where the axis of one cylinder punctures normally through the surface of the other. The elliptic cylinders are tangent at one $\mathbb{Z}_2$-symmetric pair of points along the X_3 axis, and the elliptic cylinders have normal axial punctures at two other $\mathbb{Z}_2$-symmetric pairs of points along the X_1 and X_2 axes. There is a total of 6 equilibrium points. 4 are stable and 2 are unstable.

#2b Cylindrical polar coordinates are chosen along the axis of the circular cylinder level set of C by writing $(X_1, X_3) = (-r\cos\theta,\, r\sin\theta)$. Thus,

$$X_1^2 + X_3^2 = r^2(\cos^2\theta + \sin^2\theta) = 2C$$

Thus,

$$d^{\,3}X = -dX_2 \wedge dX_1 \wedge dX_3 = dX_2 \wedge d\frac{r^2}{2} \wedge d\theta = dC \wedge d\theta \wedge dX_2$$

Restricting the $\mathbb{R}^3$ Poisson bracket to a level set of C yields

$$\{F,\, H\}d^{\,3}X = dC \wedge \{F,\, H\}_C d\theta \wedge dX_2$$

where on a level set of C,

$$\{F,\, H\}_C = \frac{\partial F}{\partial\theta}\frac{\partial H}{\partial X_2} - \frac{\partial H}{\partial\theta}\frac{\partial F}{\partial X_2} \quad \text{so that} \quad \{\theta,\, X_2\}_C = 1\,.$$

#2c The Hamiltonian H on a level set of C is given by

$$H = \frac{1}{2}X_2^2 + 2C\sin^2\theta\,.$$

The equations of motion on a level set of C are given by

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial X_2} = X_2 \qquad \frac{dX_2}{dt} = -\,\frac{\partial H}{\partial \theta} = -\,2C\sin\theta\cos\theta$$

These reduce to the pendulum equation,

$$\frac{d^2\theta}{dt^2} = -\,C\sin 2\theta\,.$$

#2d $\boxed{\textbf{\textit{Fourth year, MSc and MSci students}}}$

The geometric phase for any closed orbit on the level set of C is the integral

$$\Delta\phi_{geom} = \frac{1}{C}\int_A d\theta\wedge dX_2 = -\frac{1}{C}\oint_{\partial A} X_2 d\theta\,,$$

by Stokes theorem. Here A is the area enclosed by the solution orbit ∂A on a level set of C. Then

$$\begin{aligned}\Delta\phi_{geom} &= -\frac{1}{C}\oint_{\partial A} X_2\,\dot\theta(t)\,dt = -\frac{1}{C}\oint_{\partial A} X_2\frac{\partial H}{\partial X_2}\,dt\\ &= = -\frac{1}{C}\oint_{\partial A} 2H - 4C\sin^2\theta\,dt = -\frac{2T}{C}\Big(H - 2C\Big\langle V\Big\rangle\Big)\,,\end{aligned}$$

where

$$\Big\langle V\Big\rangle = \frac{1}{T}\oint_{\partial A} 2C\sin^2\theta(t)\,dt$$

is the average of the potential energy over the orbit.

The dynamic phase is given by the formula,

$$\begin{aligned}\Delta\phi_{dyn} &= \frac{1}{C}\oint_{\partial A}\Big(X_2\dot\theta + C\dot\phi\Big)dt\\ &= \frac{1}{C}\oint_{\partial A}\Big(X_2\frac{\partial H}{\partial X_2} + C\frac{\partial H}{\partial C}\Big)dt\\ &= \frac{1}{C}\oint_{\partial A} X_2^2 + 2C\sin^2\theta\,dt\\ &= \frac{1}{C}\oint_{\partial A} 2H - 2C\sin^2\theta\,dt\\ &= \frac{2T}{C}\Big(H - C\Big\langle V\Big\rangle\Big)\end{aligned}$$

where ϕ is the angle conjugate to C and T is the period of the orbit around which the integration is performed. Thus, the total phase change around the orbit is

$$\Delta\phi_{tot} = \Delta\phi_{dyn} + \Delta\phi_{geom} = 2T\Big\langle V\Big\rangle\,.$$

(#3) *2D oscillators*

The 2D oscillator Hamiltonian $H : \mathbb{C}^2 \to \mathbb{R}$, with complex 2-vector $\mathbf{a} = (a_1, a_2) \in \mathbb{C}^2$ and constant frequencies ω_j,

$$H = \frac{1}{2}\sum_{j=1}^{2} \omega_j |a_j|^2 = \frac{1}{4}(\omega_1 + \omega_2)\big(|a_1|^2 + |a_2|^2\big) + \frac{1}{4}(\omega_1 - \omega_2)\big(|a_1|^2 - |a_2|^2\big)\,.$$

#3a The canonical Hamiltonian dynamics with $\{a_j, a_k^*\} = -2i\delta_{jk}$ is

$$H = \frac{1}{2}\sum_{j=1}^{2} \omega_j |a_j|^2 = \frac{1}{4}(\omega_1 + \omega_2)\underbrace{\big(|a_1|^2 + |a_2|^2\big)}_{1\,:\,1 \text{ resonance}} + \frac{1}{4}(\omega_1 - \omega_2)\underbrace{\big(|a_1|^2 - |a_2|^2\big)}_{1\,:\,-1 \text{ resonance}}\,.$$

This is the linear combination of Hamiltonians for a $1 : 1$ resonant oscillator and a $1 : -1$ oscillator.

#3b The infinitesimal transformations generated by X, Y, Z, R on a_1, a_2, are

$$\begin{aligned}
X_R a_j &= \{a_j, R\} = -\,2i\,\frac{\partial R}{\partial a_j^*} = -\,2i\,a_j\,, \\
X_Z a_1 &= \{a_1, Z\} = -\,2i\,a_1\,, \qquad X_Z a_2 = \{a_2, Z\} = 2i\,a_2\,, \\
X_X a_1 &= \{a_1, X\} = -\,2i\,a_2\,, \qquad X_X a_2 = \{a_2, X\} = -\,2i\,a_1\,, \\
X_Y a_1 &= \{a_1, Y\} = -2a_2\,, \qquad X_Y a_2 = \{a_2, Y\} = 2a_1\,.
\end{aligned}$$

These infinitesimal transformations may be expressed as matrix operations,

$$\begin{aligned}
X_Z \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= -2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{or} \quad X_Z\mathbf{a} = -\,2i\,\sigma_3\mathbf{a}\,, \\
X_X \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= -2i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{or} \quad X_X\mathbf{a} = -\,2i\,\sigma_1\mathbf{a}\,, \\
X_Y \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= -2i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \text{or} \quad X_Y\mathbf{a} = -\,2i\,\sigma_2\mathbf{a}\,.
\end{aligned}$$

From these expressions, one recognises that the finite transformations, or flows, of the Hamiltonian vector fields for (X, Y, Z) are rotations about the (X, Y, Z) axes, respectively.

#3c For the Hamiltonian,

$$\begin{aligned}
H &= \frac{\omega_1}{2}\,(R + Z) + \frac{\omega_2}{2}\,(R - Z) \\
&= \frac{1}{2}(\omega_1 + \omega_2)R + \frac{1}{2}(\omega_1 - \omega_2)Z\,,
\end{aligned} \tag{2}$$

the equations $\dot{X}, \dot{Y}, \dot{Z}, \dot{R}$ for the S^1 invariants X, Y, Z, R of the $1:1$ resonance may be written as

$$\dot{F} = \{F, H\}$$

which produces

$$\dot{R} = 0 = \dot{Z}\,, \quad \dot{X} = \frac{1}{2}\,(\omega_1 - \omega_2)Y \quad \text{and} \quad \dot{Y} = -\,\frac{1}{2}\,(\omega_1 - \omega_2)X\,.$$

In vector form, with $\mathbf{X} = (X, Y, Z)^T$, this is

$$\dot{\mathbf{X}} = \frac{1}{2}\,(\omega_1 - \omega_2)\mathbf{X} \times \widehat{\mathbf{Z}}\,,$$

where $\widehat{\mathbf{Z}}$ is the unit vector in the Z-direction ($\cos\theta = 0$). This motion is uniform rotation in the positive direction along a latitude of the Poincaré sphere $R = const$.

This azimuthal rotation on a latitude at fixed polar angle on the sphere occurs along the intersections of level sets of the Poincaré sphere $R = const$ and the planes $Z = const$, which are level sets of the Hamiltonian for a fixed value of R.

#3d $\boxed{\textbf{\textit{Fourth year, MSc and MSci students}}}$

The following are quadratic $1:-1$ invariants: $|a_1|^2, |a_2|^2, a_1a_2, a_1^*a_2^*$.

Then the following linear combinations of these are also invariant

$$\begin{aligned} S &= |a_1|^2 - |a_2|^2, \\ Y_1 &= |a_1|^2 + |a_2|^2, \\ Y_2 + iY_3 &= 2a_1a_2\,. \end{aligned} \tag{3}$$

Thus,

$$Y_2^2 + Y_3^2 = 4|a_1|^2|a_2|^2 = Y_1^2 - S^2\,,$$

and the level sets of the orbital manifold are the hyperboloids of revolution around the Y_1-axis parameterised by S. That is,

$$S^2 = Y_1^2 - Y_2^2 - Y_3^2\,. \tag{4}$$

We remark that:
$S = |a_1|^2 - |a_2|^2 = const$ is an ***hyperboloid*** in both $\mathbb{C}^2$ and $\mathbb{R}^3$.
$Y_1 = |a_1|^2 + |a_2|^2 = const$ is a ***sphere*** $S^3 \in \mathbb{C}^2$, and it is a ***plane*** in $\mathbb{R}^3$.

One may write the starting Hamiltonian as,

$$H \;=\; \frac{1}{2}(\omega_1 + \omega_2)Y_1 + \frac{1}{2}(\omega_1 - \omega_2)S\,,$$

in terms of the $1:-1$ invariants and thereby write the equations of motion in vector form, with $\mathbf{Y} = (Y_1, Y_2, Y_3)^T$.

For the $1:-1$ resonance Hamiltonian $H = Y_1$, the evolution of S, Y_2, Y_3 is described by

$$\dot{\mathbf{Y}} = \nabla S^2 \times \widehat{\mathbf{Y}}_\mathbf{1} = 2\widehat{\mathbf{Y}}_\mathbf{1} \times \mathbf{Y} \tag{5}$$

$$\begin{aligned} \dot{S} &= \{S, H\} = 0, \\ \dot{Y}_1 &= \{Y_1, H\} = \{Y_1, Y_1\} = 0, \\ \dot{Y}_2 &= \{Y_2, H\} = \{Y_2, Y_1\} = -2Y_3, \\ \dot{Y}_3 &= \{Y_3, H\} = \{Y_3, Y_1\} = 2Y_2\,. \end{aligned} \tag{6}$$

Thus, Y_2 and Y_3 rotate clockwise around the Y_1-axis in a plane at $Y_1 = const.$

This is the same motion as for the paraxial harmonic guide. Looking more closely, one sees that the Lie-Poisson bracket for the paraxial rays is identical to that for the $1:-1$ resonance. This is a coincidence that occurs because the Lie algebras $sp(2,\mathbb{R})$ and $su(1,1)$ happen to be identical.