M3/4A16 Assessed Coursework 2 **Darryl Holm**
Due in class Thursday November 27, 2008

Ask in class to clarify the exact meaning of a question if you're unsure.
No conferring or copying: it will be more obvious than you think.

#1 *Quadratic Casimirs*

A Hamiltonian flow in $\mathbb{R}^3$ is defined by the system of ordinary differential equations

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = x_1x_3\,, \quad \dot{x}_3 = -\,x_1x_2\,. \tag{1}$$

#1a *Express this system in three-dimensional vector notation as flow on the intersections of level sets for a family of circular cylinders with a family of parabolic cylinders. That is, express (1) in three-dimensional vector notation as*

$$\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2\,,$$

where H_1 and H_2 are two conserved functions corresponding to these level sets. Write two different Poisson matrices and Lie-Poisson brackets for system (1). (Be sure to check that these Poisson brackets both satisfy the Jacobi identity.)

The system (1) may be written in two equivalent forms. The first is

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \det\begin{pmatrix} \hat{\mathbf{1}} & \hat{\mathbf{2}} & \hat{\mathbf{3}} \\ 0 & x_2 & x_3 \\ x_1 & 0 & 1 \end{pmatrix} = \nabla\tfrac{1}{2}(x_2^2 + x_3^2) \times \nabla(\tfrac{1}{2}x_1^2 + x_3)$$

So, $H_1 = \frac{1}{2}(x_2^2 + x_3^2)$ and $H_2 = \frac{1}{2}x_1^2 + x_3$. The second form of (1) uses the hat map

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & 0 \\ -x_2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ 0 \\ 1 \end{pmatrix} = B_1\nabla H_2 \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & x_1 \\ 0 & -x_1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} = B_2\nabla H_1 \end{aligned}$$

Namely, $(B_1)_{ij} = -\epsilon_{ijk}\partial H_1/\partial x_k$ and $(B_2)_{ij} = +\epsilon_{ijk}\partial H_2/\partial x_k$. The Hamiltonian matrices B_1 and B_2 define the following Poisson brackets

$$\{F,\,G\}_1 = (\nabla F)^T B_1 \nabla G \quad \textit{and} \quad \{F,\,G\}_2 = (\nabla F)^T B_2 \nabla G$$

The entries of the Hamiltonian matrices each define the structure constants of a Lie algebra by identifying $B_{ij} = [x_i,\,x_j] = c_{ij}^k x_k$. Therefore, the 3×3 skew-symmetric matrices B_1 and B_2 each produce Lie-Poisson brackets that satisfy the Jacobi identity by being dual to a Lie algebra. Moreover, the two Hamiltonian matrices are ***compatible****. That is, for the same reason as before (linearity), the sum $B_1 + B_2$ also defines a valid Poisson bracket. This is no surprise because B_1 and B_2 also produce $\mathbb{R}^3$ brackets.*

For the remainder of the solution of this problem, see Section 6.2 of the text.

#1b *Reduce system (1) to canonical dynamics on a level set of H_1.*

#1c *Reduce system (1) to canonical dynamics on a level set of H_2.*

#1d $\boxed{\textbf{\textit{Fourth year students}}}$ *Find the conditions on the four real constants $\{\alpha, \beta, \mu, \nu\}$ for the system (1) to be expressed as flow on the intersections of the level sets of two families of quadric surfaces,*

$$\begin{aligned} \frac{2H}{\alpha} + \frac{\beta^2}{\alpha^2} &= \frac{\beta}{\alpha}x_1^2 + x_2^2 + \Big(x_3 + \frac{\beta}{\alpha}\Big)^2 \\ \frac{2C}{\mu} + \frac{\nu^2}{\mu^2} &= \frac{\nu}{\mu}x_1^2 + x_2^2 + \Big(x_3 + \frac{\nu}{\mu}\Big)^2 \end{aligned}$$

That is, find the conditions on constants $\{\alpha, \beta, \mu, \nu\}$ that allow (1) to be written in three-dimensional vector notation as

$$\dot{\mathbf{x}} = \nabla H \times \nabla C\,,$$

where H and C are the two conserved functions defined above.

The single condition for this is

$$\alpha\nu - \beta\mu = 1$$

#2 ***Hamilton's principle for geodesic flow on the symplectic group*** $Sp(2)$

Let the set of 2×2 matrices M_i with $i = 1, 2, 3$ satisfy the defining relation for the symplectic Lie group $Sp(2)$,

$$M_i J M_i^T = J \quad \textit{with} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that corresponding elements of its Lie algebra $m_i = \dot{M}_i M_i^{-1} \in sp(2)$ satisfy $(Jm_i)^T = Jm_i$ for each $i = 1, 2, 3$. Thus, $\mathsf{X}_i = Jm_i$ satisfying $\mathsf{X}_i^T = \mathsf{X}_i$ is a set of three symmetric 2×2 matrices. For definiteness, we may choose a basis given by

$$\mathsf{X}_1 = Jm_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathsf{X}_2 = Jm_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \mathsf{X}_3 = Jm_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This basis corresponds to the vector of momentum maps given by quadratic phase-space functions $\mathbf{X} = (|\mathbf{q}|^2, |\mathbf{p}|^2, \mathbf{q}\cdot\mathbf{p})^T$ used in class for geometric optics. One sees this by using the symmetric matrices $\mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3$ above to compute the following three quadratic forms defined using $\mathbf{z} = (\mathbf{q}, \mathbf{p})^T$

$$\tfrac{1}{2}\mathbf{z}^T\mathsf{X}_1\mathbf{z} = |\mathbf{q}|^2 = X_1, \qquad \tfrac{1}{2}\mathbf{z}^T\mathsf{X}_2\mathbf{z} = |\mathbf{p}|^2 = X_2, \qquad \tfrac{1}{2}\mathbf{z}^T\mathsf{X}_3\mathbf{z} = \mathbf{q}\cdot\mathbf{p} = X_3\,.$$

#2a *For* $\mathsf{X} = Jm$, $\mathsf{Y} = Jn \in sym(2)$ *with* $m, n \in sp(2)$, *prove the middle equality in*

$$[\mathsf{X}, \mathsf{Y}]_J := \mathsf{X}J\mathsf{Y} - \mathsf{Y}J\mathsf{X} = -J(mn - nm) = -J[m, n]$$

Use this equality to show that the J-bracket $[\mathsf{X}, \mathsf{Y}]_J$ *satisfies the Jacobi identity.*

The first part is a straightforward calculation using $J^2 = -\mathrm{Id}_{2\times 2}$ *with the definitions of* X *and* Y. *The second part follows from the Jacobi identity for the symplectic Lie algebra and linearity in the definitions of* $\mathsf{X}, \mathsf{Y} \in sym(2)$ *in terms of* $m, n \in sp(2)$.

#2b *If* $\mathsf{X} = J\dot{M}M^{-1}$ *for derivative* $\dot{M} = \partial M(s,\sigma)/\partial s|_{\sigma=0}$ *and* $\mathsf{Y} = JM'M^{-1}$ *for variational derivative* $\delta M = M' = \partial M(s,\sigma)/\partial\sigma|_{\sigma=0}$, *show that equality of cross derivatives in* s *and* σ *implies the relation*

$$\delta\mathsf{X} = \mathsf{X}' = \dot{\mathsf{Y}} + [\mathsf{X}, \mathsf{Y}]_J$$

This is an important standard calculation in geometric mechanics. It begins by computing the time derivative of $MM^{-1} = Id$ *along the curve* $M(s)$ *yields* $(MM^{-1})^{\cdot} = 0$, *so that*

$$(M^{-1})^{\cdot} = -\,M^{-1}\dot{M}M^{-1}\,.$$

Next, one defines $m = \dot{M}M^{-1}$ *and* $n = M'M^{-1}$. *Then the previous relation yields*

$$\begin{aligned} m' &= \dot{M}'M^{-1} - \dot{M}M^{-1}M'M^{-1} \\ \dot{n} &= \dot{M}'M^{-1} - M'M^{-1}\dot{M}M^{-1} \end{aligned}$$

so that subtraction yields the relation

$$m' - \dot{n} = nm - mn =: -[m, n]$$

Hence, upon substituting the definitions of X *and* Y, *one finds*

$$\begin{aligned} \mathsf{X}' = Jm' &= J\dot{n} - J[m, n] \\ &= \dot{\mathsf{Y}} + [\mathsf{X}, \mathsf{Y}]_J = \dot{\mathsf{Y}} + 2\mathrm{sym}(\mathsf{X}J\mathsf{Y}) \end{aligned}$$

#2c *Use the previous relation to compute the Euler-Poincaré equation for evolution resulting from Hamilton's principle*

$$0 = \delta S = \delta \int \ell(\mathsf{X}(s))\, ds = \int \mathrm{tr}\left(\frac{\partial \ell}{\partial \mathsf{X}}\, \delta\mathsf{X}\right) ds$$

Integrating by parts and rearranging as follows,

$$\begin{aligned} 0=\delta S &= \int \mathrm{tr}\left(\frac{\partial \ell}{\partial \mathsf{X}}\mathsf{X}'\right) ds \\ &= \int \mathrm{tr}\left(\frac{\partial \ell}{\partial \mathsf{X}}\left(\dot{\mathsf{Y}}-\mathsf{Y}J\mathsf{X}+\mathsf{X}J\mathsf{Y}\right)\right) ds \\ &= \int \mathrm{tr}\left(\Big(-\frac{d}{ds}\frac{\partial \ell}{\partial \mathsf{X}}-J\mathsf{X}\frac{\partial \ell}{\partial \mathsf{X}}+\frac{\partial \ell}{\partial \mathsf{X}}\mathsf{X}J\Big)\mathsf{Y}\right) ds \\ &= \int \mathrm{tr}\left(\Big(-\frac{d}{ds}\frac{\partial \ell}{\partial \mathsf{X}}-2\mathrm{sym}\Big(J\mathsf{X}\frac{\partial \ell}{\partial \mathsf{X}}\Big)\Big)\mathsf{Y}\right) ds \end{aligned}$$

results in the ***Euler-Poincaré equation****,*

$$\frac{d}{ds}\frac{\partial \ell}{\partial \mathsf{X}} = -2\mathrm{sym}\Big(J\mathsf{X}\frac{\partial \ell}{\partial \mathsf{X}}\Big) = +2\mathrm{sym}\Big(\frac{\partial \ell}{\partial \mathsf{X}}\mathsf{X}J\Big) \tag{2}$$

Specialise this evolution equation to the case that $\ell(\mathsf{X})=\frac{1}{2}\mathrm{tr}(\mathsf{X}^2)$*, where* tr *denotes trace of a matrix. (This is geodesic motion on the matrix Lie group* $Sp(2)$ *with respect to the trace norm of matrices.)*

When $\ell(\mathsf{X})=\frac{1}{2}\mathrm{tr}(\mathsf{X}^2)$ *we have* $\partial\ell/\partial\mathsf{X}=\mathsf{X}$*, so the Euler-Poincaré equation (2) becomes*

$$\dot{\mathsf{X}} = -2\mathrm{sym}\left(J\mathsf{X}^2\right) = \mathsf{X}^2J - J\mathsf{X}^2 = [\mathsf{X}^2,\, J] \tag{3}$$

This is called the ***Bloch-Iserles equation****.*

#2d $\boxed{\textbf{\textit{Fourth year students}}}$ *Write the Hamiltonian form of the Euler-Poincaré equation and identify the associated Lie-Poisson bracket. Write the Hamiltonian form of the corresponding geodesic equation when* $\ell(\mathsf{X})=\frac{1}{2}\mathrm{tr}(\mathsf{X}^2)$.

The Hamiltonian form of the Euler-Poincaré equation (2) is found from the Legendre transform via the dual relations

$$\mu = \frac{\partial \ell}{\partial \mathsf{X}} \quad \textit{and} \quad \mathsf{X} = \frac{\partial h}{\partial \mu} \quad \textit{with} \quad h(\mu) = \mathrm{tr}(\mu\mathsf{X}) - \ell(\mathsf{X})$$

Thus,

$$\dot{\mu} = -2\mathrm{sym}\Big(J\frac{\partial h}{\partial \mu}\mu\Big) = -J\frac{\partial h}{\partial \mu}\mu + \mu\frac{\partial h}{\partial \mu}J$$

The Lie-Poisson bracket is obtained from

$$\begin{aligned}\frac{d}{ds}f(\mu) &= \mathrm{tr}\left(\frac{\partial f}{\partial \mu}\frac{d\mu}{ds}\right)\\ &= -\,2\mathrm{tr}\left(\mu\,\mathrm{sym}\Big(\frac{\partial f}{\partial \mu}J\frac{\partial h}{\partial \mu}\Big)\right)\\ &= -\,\mathrm{tr}\left(\mu\left[\frac{\partial f}{\partial \mu}\,,\,\frac{\partial h}{\partial \mu}\right]_J\right)\\ &=: \Big\{f\,,h\Big\}_J\end{aligned}$$

It's Jacobi identity follows from that of the J-bracket discussed earlier.

For the geodesic equation $h = \frac{1}{2}\mu^2$ and equation (3) keeps its form with $\mathsf{X} \to \mu$.

#3 ***Structure equations from Fermat's principle***

$\boxed{\textit{Review}}$

Fermat's (or Hamilton's) principle for geometric optics is

$$0 = \delta\int_A^B\left(\frac{dr^i}{ds}g_{ij}(\mathbf{r}(s))\frac{dr^j}{ds}\right)^{1/2}ds\,.$$

For ray paths $\mathbf{r}(s)$ in an isotropic medium in 3D the Riemannian metric appearing in Fermat's principle is $g_{ij} = n^2(\mathbf{r}(s))\delta_{ij}$, where $n(\mathbf{r})$ is the refractive index of the medium and δ_{ij} is the Kronecker delta symbol. In terms of this metric, the length element $ds = ndl$ is defined by $ds^2 = n^2dl^2 = g_{ij}(\mathbf{r})dr^idr^j$, summing over $i, j = 1, 2, 3$. As we know, the eikonal equation may be expressed as a geodesic equation,

$$\ddot{r}^{\,c} + \Gamma^c_{be}(\mathbf{r}(s))\dot{r}^b\dot{r}^e = 0 \quad \textit{with} \quad \dot{r}^b = \frac{dr^b}{ds} \quad \textit{and} \quad b,\, c,\, e \in \{1,2,3\}\,.$$

The quantities $\Gamma^c_{be}(\mathbf{r})$ are symmetric in (b, e), as reflected in their definition

$$\Gamma^c_{be}(\mathbf{r}) = \frac{1}{2}g^{ca}\left[\frac{\partial g_{ae}(\mathbf{r})}{\partial r^b} + \frac{\partial g_{ab}(\mathbf{r})}{\partial r^e} - \frac{\partial g_{be}(\mathbf{r})}{\partial r^a}\right],$$

and g^{ca} is the inverse $g^{ca}g_{ab} = \delta^c_b$. [Details appear in Chapter 2 of the class text.] *The corresponding curvature tensor R^a_{bcd} may be expressed in terms of the Christoffel symbols Γ^a_{bc} by*

$$R^a_{bcd} = \frac{\partial\Gamma^a_{bc}}{\partial r^d} - \frac{\partial\Gamma^a_{bd}}{\partial r^c} + \Gamma^e_{bc}\Gamma^a_{de} - \Gamma^e_{bd}\Gamma^a_{ce}\,,$$

and $R_{abcd} = g_{ae}R^e_{bcd}$ possesses the expected symmetries

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}$$

Although these equations may look formidable, it turns out that they may be derived easily by using the language of differential forms.

$\boxed{\textbf{\textit{Differential forms}}}$ *Define a basis of 1-forms $e^a = e^a_i dr^i$ with $a = 1, 2, 3$, in terms of the Cartesian basis dr^i with $i = 1, 2, 3$. The corresponding dual basis is defined by $e_b = e^j_b \partial/\partial r^j$ with $e^j_b e^b_i = \delta^j_i$ and it satisfies the contraction relation*

$$e_b \lrcorner\, e^a := e^a_j e^j_b = \delta^a_b \quad \textit{for} \quad \partial/\partial r^j \lrcorner\, dr^i = \delta^i_j \,.$$

The induced metric defined by $\eta_{ab} = g_{ij} e^j_a e^i_b$ then satisfies $\eta_{ab} e^a_j e^b_i = g_{ij}$ which implies

$$ds^2 = dr^i g_{ij} dr^j = \eta_{ab} e^a_i dr^i e^b_j dr^j = \eta_{ab} e^a e^b \,.$$

For example, the path length in Fermat's principle for an isotropic medium may be expressed as,

$$ds^2 = n^2(dx^2 + dy^2 + dz^2) = (e^1)^2 + (e^2)^2 + (e^3)^2 = \delta_{ab} e^a e^b \,,$$

with $\eta_{ab} = \delta_{ab}$ and $e^a = n(\mathbf{r})dr^a$, that is,

$$e^1 = ndx, \quad e^2 = ndy, \quad e^3 = ndz \,.$$

The 1-forms e^a with $a = 1, 2, 3$ are called ***Cartan frames****. They satisfy* ***Cartan's structure equations****, namely*

$$\begin{aligned} de^a + \omega^a_b \wedge e^b &= 0 \,, \\ d\omega^a_b + \omega^a_c \wedge \omega^c_b &= -\tfrac{1}{2} R^a_b \,, \end{aligned}$$

where d denotes the exterior differential operator and $\omega_{ab} = \eta_{ac}\omega^c_b = -\omega_{ba}$. The first of these structure equations defines the Christoffel symbols. The second defines the curvature tensor. In the Cartan frame basis, the coefficients in Cartan's structure equations are given in terms of Γ^a_{bc} and R^a_{bcd} by

$$\omega^a_b = \Gamma^a_{bc} e^c \quad \textit{and} \quad R^a_b = R^a_{bcd} e^c \wedge e^d \,,$$

so that $R^a_{bdc} = -R^a_{bcd}$ (as expected from its definition) now follows from antisymmetry of the wedge product, $e^c \wedge e^d = -e^d \wedge e^c$. For example, in spherical coordinates, $e^r = dr$, $e^\theta = rd\theta$, $e^\phi = r\sin\theta d\phi$. A direct calculation then yields structure equations

$$de^r = 0 \,, \quad de^\theta - \frac{1}{r} e^r \wedge e^\theta = 0 \,, \quad de^\phi - \frac{1}{r} e^r \wedge e^\phi - \frac{\cot\theta}{r} e^\theta \wedge e^\phi = 0 \,,$$

whose coefficients are the well-known Christoffel symbols for spherical coordinates.

Problem statement (Solution only requires properties of d and $\wedge$!)

#3a *Compute ω^a_b and give an example of computing R^a_b when $e^a = n(\mathbf{r})dr^a$, $\eta_{ab} = \delta_{ab}$ and $ds^2 = \mathbf{e}(r) \cdot \mathbf{e}(r)$.*

By using the definition of ω^a_b *in* $de^a + \omega^a_b \wedge e^b = 0$ *and the relations* $e^a = n dr^a$, *we find by direct computation that*

$$\begin{aligned} \omega^1_2 = -\,\omega^2_1 &= n^{-1}(n_y dx - n_x dy)\,, \\ \omega^1_3 = -\,\omega^3_1 &= n^{-1}(n_z dx - n_x dz)\,, \\ \omega^2_3 = -\,\omega^3_2 &= n^{-1}(n_z dy - n_y dz)\,. \end{aligned}$$

This calculation proceeds as follows. Expanding $de^1 + \omega^1_2 \wedge e^2 + \omega^1_3 \wedge e^3 = 0$ *yields*

$$n_y dy \wedge dx + n_z dz \wedge dx + \omega^1_2 \wedge n dy + \omega^1_3 \wedge n dz = 0\,.$$

Likewise, expanding $de^2 + \omega^2_1 \wedge e^1 + \omega^2_3 \wedge e^3 = 0$ *yields*

$$n_x dx \wedge dy + n_z dz \wedge dy + \omega^2_1 \wedge n dx + \omega^2_3 \wedge n dz = 0\,.$$

Hence,

$$\begin{aligned} \omega^1_2 &= n^{-1}(n_y dx + S^1_2 dy)\,, \quad \omega^1_3 = n^{-1}(n_z dx + S^1_3 dz)\,, \\ \omega^2_1 &= n^{-1}(n_x dy + S^2_1 dx)\,, \quad \omega^2_3 = n^{-1}(n_z dy + S^2_3 dz)\,, \end{aligned}$$

where functions $S^1_2, S^1_3, S^2_1, S^2_3$ *are sought by expanding* de^2 *and* de^3. *For* $\eta_{ab} = \delta_{ab}$, *one has* $\omega_{21} = \omega^2_1 = -\,\omega^1_2 = -\omega_{12}$ *and this skew symmetry yields* $S^1_2 = -n_x$ *and* $S^2_1 = -n_y$. *The other* ω^a_b *may be obtained by cyclically permuting indices.*

Substituting the results for ω^a_b *into Cartan's second structure equation and identifying terms as in the previous computation leads to the following expressions for the curvature two-form*

$$\begin{aligned} -\frac{1}{2}R^1_2 = d\omega^1_2 + \omega^1_c \wedge \omega^c_2 &= -n^{-1}\Big(n_{xx} + n_{yy} - \frac{1}{n}(n_x^2 + n_y^2 - n_z^2)\Big) dx \wedge dy \\ &\quad + n^{-1}(n_{yz} dz \wedge dx + n_{yy} dy \wedge dz)\,, \\ -\frac{1}{2}R^1_3 = d\omega^1_3 + \omega^1_c \wedge \omega^c_3 &= \ldots \\ -\frac{1}{2}R^2_3 = d\omega^2_3 + \omega^2_c \wedge \omega^c_3 &= \ldots \end{aligned}$$

#3b *Solve for* Γ^1_{21}, Γ^1_{22} *and* Γ^1_{23} *using Cartan's first structure equation.*

By definition,

$$\begin{aligned} \omega^a_b &= \Gamma^a_{bc} e^c \\ \omega^1_2 &= \Gamma^1_{21} e^1 + \Gamma^1_{22} e^2 + \Gamma^1_{23} e^3 \\ &= n^{-2}\Big(n_y e^1 - n_x e^2\Big) \end{aligned}$$

Therefore, by identifying coefficients,

$$\Gamma^1_{21} = n^{-2} n_y, \quad \Gamma^1_{22} = -n^{-2} n_x, \quad \Gamma^1_{23} = 0, \quad etc.$$

#3c *Give an example of computing R^a_{bcd} in terms of $n(\mathbf{r})$ and its spatial derivatives using Cartan's second structure equation.*

Mimicking the calculation of the Γ's from the ωs yields

$$\begin{aligned} R^1_{212} &= n^{-3}(n_{xx} + n_{yy}) - n^{-4}(n_x^2 + n_y^2 - n_z^2)\,, \\ R^1_{313} &= n^{-3}(n_{xx} + n_{zz}) - n^{-4}(n_x^2 + n_z^2 - n_y^2)\,, \\ R^2_{323} &= n^{-3}(n_{yy} + n_{zz}) - n^{-4}(n_y^2 + n_z^2 - n_x^2)\,. \end{aligned}$$

#3d $\boxed{\textbf{\textit{Fourth year students}}}$ *Use the metric $g_{ae} = n^2 \delta_{ae}$ for Fermat's principle to lower the first index to $R_{abcd} = g_{ae} R^e_{bcd}$ and express the scalar curvature $R = R_{abab}$ in terms of $n(\mathbf{r})$ and its spatial derivatives. How does this formula simplify when the index of refraction is taken as $n(r) = 1/(1 \pm r^2)$? (The + case is called the Maxwell fish-eye lens. What geometrical observation can be made about the difference in scalar curvature for these two cases?)*

Lowering the first index by using $g_{ab} = n^2 \delta_{ab}$ yields

$$\begin{aligned} R_{1212} &= n^{-1}(n_{xx} + n_{yy}) - n^{-2}(n_x^2 + n_y^2 - n_z^2)\,, \\ R_{1313} &= n^{-1}(n_{xx} + n_{zz}) - n^{-2}(n_x^2 + n_z^2 - n_y^2)\,, \\ R_{2323} &= n^{-1}(n_{yy} + n_{zz}) - n^{-2}(n_y^2 + n_z^2 - n_x^2)\,. \end{aligned}$$

which possesses symmetries

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}$$

and vanishes when two indices match in either the first pair or the second pair.
The scalar curvature is calculated using these symmetries as

$$R = R_{abab} = 2(R_{1212} + R_{1313} + R_{2323}) = \frac{4}{n}\left[\Delta n - \frac{1}{2n}|\nabla n|^2\right]$$

When $n(r) = 1/(1 \pm r^2)$, this becomes

$$R = \frac{8(\mp 2r^2 - 1)}{(1 \pm r^2)^2}\,.$$

Thus, for the fish-eye $n(r) = 1/(1 + r^2)$, the scalar curvature is always negative*. This means the rays diverge away from the center.*

$\boxed{\textbf{Axisymmetric translation invariant media in 3D}}$ *These computations in general terms may look a bit daunting. Let's look at a simpler case, but still in 3D. The index of refraction in axisymmetric media is $n(r)$ and one may formulate the problem for geodesic ray paths in* full 3D cylindrical coordinates (r, ϕ, z). *The metric for optics in cylindrical coordinates is given by $ds^2 = n^2(r)(dr^2 + r^2 d\phi^2 + dz^2)$. This metric is diagonal with $g_{rr} = n^2(r)$, $g_{\phi\phi} = r^2 n^2(r)$, $g_{zz} = n^2(r)$, or, in matrix form,*

$$g_{ij} = n^2(r) \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In this case, $\eta_{ab} = \delta_{ab}$ and $ds^2 = \mathbf{e}(r) \cdot \mathbf{e}(r)$, with

$$e^r = n(r) dr, \quad e^\phi = n(r) r d\phi, \quad e^z = n(r) dz\,.$$

One calculates the ω^a_b as before from Cartan's first structure equation. Hence, $\omega^a_b = \Gamma^a_{bc} e^c$ yields the nonzero Christoffel symbols. In the $(dr, d\phi, dz)$ basis, these take a slightly simpler form,

$$\begin{aligned}
\Gamma^r_{rr} = \frac{1}{n}\frac{dn}{dr}\,, \quad \Gamma^r_{\phi\phi} &= -\Big(\frac{1}{r} + \frac{1}{n}\frac{dn}{dr}\Big) r^2\,, \quad \Gamma^r_{zz} = -\,\frac{1}{n}\frac{dn}{dr}\,, \\
\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} &= \frac{1}{r} + \frac{1}{n}\frac{dn}{dr}\,, \\
\Gamma^z_{rz} = \Gamma^z_{zr} &= \frac{1}{n}\frac{dn}{dr}\,.
\end{aligned}$$

These nonzero Christoffel symbols also arise from their Levi-Civita definition in terms of derivatives of the metric. The Γ's produce the geodesic equations

$$\begin{aligned}
\ddot{r} + \frac{1}{n}\frac{dn}{dr}\Big(\dot{r}^2 - \dot{z}^2\Big) &- \Big(\frac{1}{r} + \frac{1}{n}\frac{dn}{dr}\Big) r^2 \dot{\phi}^2 = 0\,, \\
\ddot{\phi} &+ 2\Big(\frac{1}{r} + \frac{1}{n}\frac{dn}{dr}\Big) \dot{r}\dot{\phi} = 0\,, \\
\ddot{z} &+ \frac{2}{n}\frac{dn}{dr}\, \dot{r}\dot{z} = 0\,.
\end{aligned}$$

The same geodesic equations of motion arise from the stationary principle,

$$\begin{aligned}
0 = \delta S &= \delta \int L(r, \dot{r}, \dot{\phi}, \dot{z}) ds = \delta \int \frac{1}{2} n^2(r) \Big(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\Big) ds \\
&= \int \Bigg[\bigg(-\frac{d}{ds}\Big(n^2(r)\dot{r}\Big) + r\dot{\phi}^2 n^2(r) + n\frac{dn}{dr}\Big(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\Big)\bigg) \delta r \\
&\quad - \frac{d}{ds}\Big(n^2(r) r^2 \dot{\phi}\Big)\delta\phi - \frac{d}{ds}\Big(n^2(r)\dot{z}\Big)\delta z \Bigg] ds
\end{aligned}$$

Invariance of $L(r,\dot r,\dot\phi,\dot z)$ in Hamilton's principle under translations in the coordinates ϕ and z yields conservation of the two corresponding canonical momenta,

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = n^2(r) r^2 \dot\phi \quad \textit{and} \quad p_z = \frac{\partial L}{\partial \dot z} = n^2(r)\dot z\,.$$

Legendre transforming yields the Hamiltonian

$$\begin{aligned} H &= p_r \dot r + p_\phi \dot\phi + p_z \dot z - L(r,\dot r,\dot\phi,\dot z) \\ &= \frac{1}{2n^2(r)}\Big(p_r^2 + \frac{p_\phi^2}{r^2} + p_z^2\Big) \end{aligned}$$

whose canonical equations are

$$\begin{aligned} \dot r &= \frac{\partial H}{\partial p_r} = \frac{p_r}{n^2(r)}\,, \qquad \dot p_r = -\frac{\partial H}{\partial r} = \frac{2H}{n}\frac{dn}{dr} + \frac{1}{n^2(r)}\frac{p_\phi^2}{r^3}\,, \\ \dot\phi &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{r^2 n^2(r)}\,, \qquad \dot p_\phi = -\frac{\partial H}{\partial \phi} = 0\,, \\ \dot z &= \frac{\partial H}{\partial p_z} = \frac{p_z}{n^2(r)}\,, \qquad \dot p_z = -\frac{\partial H}{\partial z} = 0\,. \end{aligned}$$

Notice that the (z,p_z) equations completely decouple *from the rest of the system. This justifies the approach taken in class of transforming the independent variable from ray-path coordinates s to axial coordinates z for paraxial optics. In class, we neglected the possibility that the ray would turn back into the opposite axial direction. Now we see that this neglect was justified, because the (z,p_z) equations imply that axial ray reversal is impossible for cylindrically symmetric materials. That is, the z-velocity keeps its sign.*

The level-set contours of H in the (r,p_r) reduced phase plane at constant (p_ϕ,p_z) for this system may be obtained by solving for $p_r(r)$ from the definition of the Hamiltonian. Namely, the level-sets of H in the (r,p_r) phase plane obey

$$p_r^2(r) = 2Hn^2(r) - \frac{p_\phi^2}{r^2} - p_z^2$$

for constant values of (H,p_ϕ,p_z). These level sets in the phase plane at constant p_z have turning points whenever

$$\frac{1}{2}\frac{dp_r^2}{dr}\bigg|_{H,p_\phi} = -\frac{p_r}{2}\frac{\partial H/\partial r}{\partial H/\partial p_r} = H\frac{dn^2}{dr} + \frac{p_\phi^2}{r^3} = 0\,.$$

If a level set of H crosses though $p_r = 0$, the radial direction of the ray reverses; so periodic motion in the radial coordinate is possible. However, as mentioned above, periodic motion in the other coordinates ϕ and z is not possible. Those coordinates must increase or decrease monotonically along the ray path, as dictated by the initial signs of the conserved canonical momenta p_ϕ and p_z.

Directionally dependent index of refraction *Consider light rays in a crystal whose index of refraction depends on direction as*

$$ds^2 = g_{ij}dr^i dr^j = (n_1 dr^1)^2 + (n_2 dr^2)^2 + (n_3 dr^3)^2 \,,$$

where the metric

$$g(\mathbf{r}(s)) = \mathrm{diag}(n_1^2, n_2^2, n_3^3) =: D^{-1}$$

is a 3×3 *diagonal matrix whose positive-definite entries depend on position along the ray path parameterized as* $\mathbf{r}(\tau)$*.*

Write Hamilton's canonical equations for rays in this medium, assuming the dynamical equations follow from Fermat's principle in the form

$$0 = \delta S = \delta \int L(\mathbf{r}, \dot{\mathbf{r}}) d\tau \quad \textit{with} \quad L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}\dot{\mathbf{r}}^T D^{-1}(\mathbf{r})\dot{\mathbf{r}}$$

where $\dot{\mathbf{r}} = d\mathbf{r}/d\tau$ *and* $D^T = D$ *is a symmetric matrix. The fiber derivative of this Lagrangian is the canonical momentum*

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = D^{-1}(\mathbf{r})\dot{\mathbf{r}}$$

consistent with the wave-front velocity relation

$$\dot{\mathbf{r}} = D(\mathbf{r})\nabla S$$

and its Euler-Lagrange equations are

$$\frac{d}{dt}\Big(D^{-1}_{lm}(\mathbf{r})\dot{r}^m \Big) = \dot{r}^i \frac{\partial D^{-1}_{ij}}{\partial r^l} \dot{r}^j$$

These are just the usual geodesic equations for the case of a diagonal metric D^{-1}*.*

The corresponding Hamiltonian is calculated from the Legendre transformation as

$$H(\mathbf{r}, \mathbf{p}) = \mathbf{p} \cdot \dot{\mathbf{r}} - L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}\mathbf{p} \cdot D(\mathbf{r})\mathbf{p}$$

Hamilton's canonical equations for this problem are then

$$\begin{aligned} \dot{\mathbf{r}} &= \frac{\partial H}{\partial \mathbf{p}} = D(\mathbf{r})\mathbf{p} \\ \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{r}} = -\frac{1}{2} p_i \left(\frac{\partial D^{ij}}{\partial \mathbf{r}} \right) p_j \end{aligned}$$

which of course recovers the Euler-Lagrange equations.

Snell's Law

The wave-front velocity relation $\dot{\mathbf{r}} = D(\mathbf{r})\nabla S$ consistent with Hamilton's equation implies

$$\oint_C \mathbf{p} \cdot d\mathbf{r} = \oint_C \frac{\partial S}{\partial \mathbf{r}} \cdot d\mathbf{r} = \oint_C dS = 0$$

for any closed contour C. If the contour C encloses an interface between two media, shrinking the contour implies continuity of the tangential components of the momentum. Hence, Snell's Law becomes

$$(\mathbf{p} - \mathbf{p}') \times \hat{\mathbf{n}} = 0$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the surface and the prime distinguishes media on its two sides. Thus, the ray directions $\dot{\mathbf{r}}$ and $\dot{\mathbf{r}}'$ at a point on an interface satisfy,

$$\Big((D^{-1}\dot{\mathbf{r}}) - \big(D'^{\,-1}\dot{\mathbf{r}}' \big) \Big) \times \hat{\mathbf{n}} = 0$$

So that in its usual form Snell's Law at that point becomes

$$|D^{-1}\dot{\mathbf{r}}| \sin\theta = |D'^{\,-1}\dot{\mathbf{r}}'| \sin\theta'$$

Consequently, the direction of the ray at the interface will bend into the faster medium, but the sense of "faster" depends on the anisotropy parameters. Roughly speaking though, the wave speeds up when it enters a region of higher conductance D, since $\dot{\mathbf{r}} = D\nabla S$. For example, if we specify D^{-1} in diagonal form for a planar anisotropic medium as

$$D^{-1} = \begin{pmatrix} n_1^2(\mathbf{r}) & 0 \\ 0 & n_2^2(\mathbf{r}) \end{pmatrix},$$

then in Snell's Law one has

$$|D^{-1}\dot{\mathbf{r}}|^2 = (n_1^2 \dot{r}_1)^2 + (n_2^2 \dot{r}_2)^2$$

In this example, the quadratic conserved Hamiltonian $H : T^*\mathbb{R}^2 \to \mathbb{R}$ above is given by the following function defined on 4D phase space,

$$2H = \frac{p_1^2}{n_1^2(\mathbf{r})} + \frac{p_2^2}{n_2^2(\mathbf{r})}$$

For the choice $2H = 1$ we may define an angle θ from the center of an ellipse by

$$p_1 = n_1(\mathbf{r})\cos\theta \quad \textit{and} \quad p_2 = n_2(\mathbf{r})\sin\theta$$

The corresponding eikonal equation is

$$\|\nabla S\|^2 = |\nabla S \cdot D(\mathbf{r}) \cdot \nabla S|^2 = \frac{(S_{,1})^2}{n_1^2(\mathbf{r})} + \frac{(S_{,2})^2}{n_2^2(\mathbf{r})} = 1$$

The forward problem (IVP) is to find the find the wave-front evolution from a specified initial condition. This evolution provides a model of the isochronal surfaces. The inverse problem (data assimilation for tomography of the heart) would use these solutions to infer the spatial distribution of conductance.

Study sheet for the eikonal equation *November 12, 2008*

a *Write Fermat's principle for ray paths in 3D and derive the eikonal equation as its Euler-Lagrange equation.*

b *Write the 3D eikonal equation in Snell's Law form, as a double cross product of vectors.*

c *Show that the 3D eikonal equation in Snell's Law form may be written as a geodesic equation and identify its dependent and independent variables.*

d *Write the 3D eikonal equation as Cartan's first structure equation for a certain Riemannian manifold. Identify the Riemannian metric and its Christoffel symbols.*

e *Legendre transform the Lagrangian formulation of the 3D eikonal equation into canonical Hamiltonian form.*

f *Write the canonical Hamiltonian form of the eikonal equation for a material that is invariant under translations and rotations about an optical axis, by using the axis coordinate as an independent variable.*

g *Show that the flows of the Hamiltonian vector fields arising from the three rotationally invariant quadratic phase space functions may be written as symplectic matrix transformations of the phase space coordinates.*

h *Compute the matrices tangent to these three symplectic transformations at the identity and write a* 3×3 *skew-symmetric table of their commutation relations. Explain why it is skew-symmetric.*

i *Compute the canonical Poisson brackets (*$\{X_1, X_2\}$*, etc.) among the three rotationally invariant quadratic phase space functions*

$$(X_1, X_2, X_3) = (|\mathbf{q}|^2, |\mathbf{p}|^2, \mathbf{q} \cdot \mathbf{p})$$

Show that these Poisson brackets may be expressed as a closed system $\{X_i, X_j\} = c_{ij}^k X_k$*,* $i, j, k = 1, 2, 3,$ *in terms of these invariants.*

j *Write the Poisson brackets among these invariants as a* 3×3 *skew-symmetric table and compare it with the table in* **h**.

k *Write the Poisson brackets for functions of these three invariants* (X_1, X_2, X_3) *as a vector cross product of gradients of functions of* $\mathbf{X} \in \mathbb{R}^3$.

l *Relate the result of* **k** *to the Poisson bracket for the geodesic flow in problem* **2**. *Are these Poisson brackets equivalent, or not? Prove it.*