M3/4A16 Assessed Coursework 1 Solutions **Darryl Holm**
Due in class Tuesday November 10, 2009

Exercise.

1. *Show that the canonical bracket operation*

$$\{F,\, H\} := \frac{\partial F}{\partial \mathbf{q}}\cdot\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}}\cdot\frac{\partial H}{\partial \mathbf{q}}\,, \tag{1}$$

satisfies the conditions to be a Poisson bracket.

> ***Answer.*** By direct computation, the canonical bracket operation is skew symmetric, bilinear and satisfies Leibniz and Jacobi. ▲

2. *Show that the following transformations of coordinates are each canonical.*
   (a) $T^*\mathbb{R}^2\ \{0\} \to T^*\mathbb{R}_+ \times T^*S^1$, *given by* $x+iy = re^{i\theta}\,,\quad p_x + ip_y = (p_r + ip_\theta/r)e^{i\theta}$.
   (b) $T^*\mathbb{R}^2 \to \mathbb{C}^2$, *given by* $a_k = q_k + ip_k\,,\quad a_k^* = q_k - ip_k$, *with* $k = 1, 2$.
   (c) $T^*\mathbb{R}^2\ \{0\} \to \mathbb{R}^2_+ \times T^2$, *given by* $I_k = \frac{1}{2}(q_k^2 + p_k^2)\,,\quad \phi_k = \tan^{-1}(p_k/q_k)$, *with* $k = 1, 2$.

> ***Answer.***
> (a) Evaluating $\mathrm{Re}\big((p_x + ip_y)(dx - idy)\big) = \mathrm{Re}\big((p_r + ip_\theta/r)(dr - ird\theta)\big)$ yields
> $$p_x dx + p_y dy = p_r dr + p_\theta d\theta.$$
> Taking exterior derivative gives
> $$dp_x \wedge dx + dp_y \wedge dy = dp_r \wedge dr + dp_\theta \wedge d\theta.$$
> (b) $da \wedge da^* = (dq_k + idp_k)\wedge(dq_k - idp_k) = -2idq\wedge dp$
> (c) For each k define
> $$a_k = q_k + ip_k\,,\quad r_k = |a_k| = (q_k^2 + p_k^2)^{1/2}\,,\quad \phi_k = \tan(p_k/q_k)\quad \text{(no sum)}$$
> Then for each k (no sum) with $I_k = \frac{1}{2}r_k^2$ and
> $$\begin{aligned} dI_k \wedge d\phi_k = \tfrac{1}{2}dr_k^2 \wedge d\phi_k &= (q_k dq_k + p_k dp_k)\wedge \frac{q_k dp_k - p_k dq_k}{q_k^2 + p_k^2} \\ &= \frac{q_k^2 dq_k \wedge dp_k - p_k^2 dp_k \wedge dq_k}{q_k^2 + p_k^2} \\ &= dq_k \wedge dp_k \end{aligned}$$
> ***Exercise:*** Write the canonical transformation $(I,\phi) \to (a, a^*)$!
> ▲

3. *Write the Hamiltonian equations for PISHO using each of the canonical coordinate systems of the previous exercise.*

Answer.

(a) $H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + r^2\right)$,

$$\begin{aligned} \dot{r} &= \partial H/\partial p_r = p_r\,, \quad \dot{p}_r = -\partial H/\partial r = -r \\ \dot{\theta} &= \partial H/\partial p_\theta = p_\theta/r^2\,, \quad \dot{p}_\theta = -\partial H/\partial \theta = 0\,. \end{aligned}$$

(b) $H = \frac{1}{2}(|a_1|^2 + |a_2|^2)$ and $\{a,\, a^*\} = -2i$, so

$$\dot{a}_k = \{a_k,\, H\} = -ia_k \quad \text{and} \quad a_k(t) = a_k(0)e^{-it}\,.$$

In these variables, PISHO dynamics is just a linear phase shift at frequency -1.

(c) $H = I_1 + I_2$ and $\{I_k,\, \phi_l\} = \delta_{kl}$, so

$$\begin{aligned} \dot{I}_k &= \{I_k,\, H\} = \frac{\partial H}{\partial \phi_k} = 0\,, \\ \dot{\phi}_k &= \{\phi_k,\, H\} = -\frac{\partial H}{\partial I_k} = -1 \quad \text{and} \quad \phi(t) = \phi(0) - t\,. \end{aligned}$$

This is no surprise for us, since in the previous part $a_k = \sqrt{2}\, I_k e^{i\phi_k}$.

▲

4. Write the Hamiltonian forms of the PISHO equations in terms of S^1-invariant quantities, for the following cases:

(a) $(X_1, X_2, X_3) = (|\mathbf{q}|^2, |\mathbf{p}|^2, \mathbf{q}\cdot\mathbf{p})$ and $p_\theta^2 = |\mathbf{p}\times\mathbf{q}|^2$ with $(\mathbf{q},\mathbf{p}) \in \mathbb{R}^2\times\mathbb{R}^2$.

(b) $R = |a_1|^2+|a_2|^2$, $Y_1+iY_2 = 2a_1^*a_2$ and $Y_3 = |a_1|^2-|a_2|^2$, with $a_k := q_k+ip_k \in \mathbb{C}^1$ for $k = 1, 2$.

Answer.

(a) Set $p_\theta^2 = |\mathbf{p}\times\mathbf{q}|^2 = |\mathbf{q}|^2|\mathbf{p}|^2 - (\mathbf{p}\cdot\mathbf{q})^2 = X_1X_2 - X_3^2 =: S^2(\mathbf{X})$ and write the flow on $\mathbf{X}\in\mathbb{R}^3$ with $H(\mathbf{X}) = \frac{1}{2}(X_1+X_2)$ for PISHO as

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \dot{\mathbf{X}} = \nabla S^2(\mathbf{X}) \times \nabla H(\mathbf{X}) = \det \begin{bmatrix} \hat{\mathbf{1}} & \hat{\mathbf{2}} & \hat{\mathbf{3}} \\ X_2 & X_1 & -2X_3 \\ X_1 & X_2 & 0 \end{bmatrix} = \begin{bmatrix} 2X_3 \\ -2X_3 \\ 0 \end{bmatrix}$$

(b) In this case, $H = R$ for PISHO and $R^2 = |\mathbf{Y}|^2 = Y_1^2 + Y_2^2 + Y_3^2$

$$\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \\ \dot{Y}_3 \end{bmatrix} = \dot{\mathbf{Y}} = \nabla R^2(\mathbf{Y}) \times \nabla H(\mathbf{Y}) = \nabla R^2 \times \nabla R = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\,.$$

Substituting $a_k(t) = a_k(0)e^{-it}$ shows that indeed each of the S^1-invariant quantities in the vector $\mathbf{Y}$ are invariant under the dynamics of PISHO.

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Exercise.

1. *Use the canonical Poisson brackets in (1) to compute ($\{X_1, X_2\}$, etc.) among the three rotationally invariant quadratic phase space functions*

$$(X_1, X_2, X_3) = (|\mathbf{q}|^2, |\mathbf{p}|^2, \mathbf{q}\cdot\mathbf{p}) \,, \qquad (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^2 \times \mathbb{R}^2 \,. \tag{2}$$

> ***Answer.*** The canonical Poisson brackets among the axisymmetric variables X_1, X_2 and X_3 in (2) close among themselves,
>
> $$\{X_1, X_2\} = 4X_3 \,, \quad \{X_2, X_3\} = -\,2X_2 \,, \quad \{X_3, X_1\} = -\,2X_1 \,.$$
>
> ▲

2. *Show that these Poisson brackets may be expressed as a closed system*

$$\{X_i, X_j\} = c^k_{ij} X_k, \qquad i, j, k = 1, 2, 3, \tag{3}$$

in terms of these invariants, by computing the coefficients c^k_{ij}.

> ***Answer.*** The constants $c^k_{ij}$are immediately read off as
>
> $$c^3_{12} = 4 = -c^3_{21} \,, \quad c^1_{13} = c^2_{32} = 2 = -c^2_{23} = -c^1_{31} \,, \tag{4}$$
>
> and all the other c^k_{ij}'s vanish. ▲

3. *Write the Poisson brackets among the invariants $\{X_i, X_j\}$ as a* 3×3 *skew-symmetric table.*

> ***Answer.*** In tabular form, the Poisson brackets among the invariants $\{X_i, X_j\}$ are
>
> $$\{X_i,\, X_j\} = \begin{array}{|c|ccc|} \hline \{\,\cdot\,,\,\cdot\,\} & X_1 & X_2 & X_3 \\ \hline X_1 & 0 & 4X_3 & 2X_1 \\ X_2 & -4X_3 & 0 & -2X_2 \\ X_3 & -2X_1 & 2X_2 & 0 \\ \hline \end{array} \tag{5}$$
>
> ▲

4. *Write the Poisson brackets for functions of these three invariants (X_1, X_2, X_3) as a vector cross product of gradients of functions of $\mathbf{X} \in \mathbb{R}^3$.*

> ***Answer.*** The Poisson bracket among the variables $\mathbf{X} = (X_1, X_2, X_3) \in \mathbb{R}^3$ may be expressed by using the chain rule as,
>
> $$\frac{dF}{dt} = \{F, H\} = \frac{\partial F}{\partial X_i}\,\{X_i,\, X_j\}\,\frac{\partial H}{\partial X_j} \,. \tag{6}$$

This may be expressed for $\frac{1}{2}S^2 = X_1X_2 - X_3^2$ as

$$\begin{aligned} \frac{dF}{dt} = \{F, H\} &= -\nabla S^2 \cdot \nabla F \times \nabla H \\ &= -\frac{\partial S^2}{\partial X_l}\epsilon_{ljk}\frac{\partial F}{\partial X_j}\frac{\partial H}{\partial X_k}\,. \end{aligned} \tag{7}$$

This is a direct verification using formula (6). For example,

$$\epsilon_{123}\frac{\partial S^2}{\partial X_3} = -4X_3, \quad \epsilon_{132}\frac{\partial S^2}{\partial X_2} = 2X_1, \quad \epsilon_{231}\frac{\partial S^2}{\partial X_1} = -2X_2\,.$$

The quantity ∇S^2 with $\frac{1}{2}S^2 = X_1X_2 - X_3^2$ defines the symmetric matrix

$$\nabla S^2 = \begin{bmatrix} X_2 \\ X_1 \\ -2X_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} =: \mathsf{K}\mathbf{X}\,.$$

The $\mathbb{R}^3$ bracket in vector form (7) may be written equivalently in several forms by using the matrix K as a Riemannian metric,

$$\begin{aligned} \{F,\, H\}_{\mathsf{K}} &= -\nabla S^2 \cdot \nabla F \times \nabla H \\ &= -X_l\mathsf{K}^{li}\epsilon_{ijk}\frac{\partial F}{\partial X_j}\frac{\partial H}{\partial X_k} \\ &= -\mathbf{X}\cdot\mathsf{K}\left(\frac{\partial F}{\partial \mathbf{X}} \times \frac{\partial H}{\partial \mathbf{X}}\right) \\ &=: -\left\langle \mathbf{X}\,,\,\left[\frac{\partial F}{\partial \mathbf{X}}\,,\,\frac{\partial H}{\partial \mathbf{X}}\right]_{\mathsf{K}}\right\rangle\,. \end{aligned} \tag{8}$$

This is the Lie-Poisson bracket for the Lie algebra structure represented on $\mathbb{R}^3$ by the vector product,

$$[\mathbf{u}, \mathbf{v}]_{\mathsf{K}} = \mathsf{K}(\mathbf{u}\times\mathbf{v}) \quad \text{for} \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^3\,. \tag{9}$$

One may verify that the bracket

$$\{X_j,\, X_k\}_{\mathsf{K}} := -X_l\mathsf{K}^{li}\epsilon_{ijk} = c^l_{jk}X_l\,, \tag{10}$$

matches the c^l_{jk} constants in (4).

Exercise. What is the hat map for this bracket? ★

▲

5. *Take the Poisson brackets of the three invariants* (X_1, X_2, X_3) *with the function,*

$$\tfrac{1}{2}S^2 = X_1X_2 - X_3^2\,. \tag{11}$$

Explain your answers geometrically in terms of vectors in $\mathbb{R}^3$*.*

Answer.

$$\{S^2, X_i\} = -\nabla S^2\cdot\nabla S^2\times\nabla X_i = 0\,, \quad i = 1,2,3, \quad \text{since} \quad \nabla S^2\times\nabla S^2 = 0\,.$$

▲

6. *Write the results of applying the Poisson brackets in the form*

$$\{\,\mathbf{z}\,,\, X_k\} = m_k\mathbf{z}\,,$$

for $\mathbf{z} = (\mathbf{q},\mathbf{p})^T \in \mathbb{R}^2\times\mathbb{R}^2$ and 2×2 matrices m_k, with $k = 1,2,3$. Identify the type of matrix that results (symmetric, skew symmetric, etc.) Write a 3×3 skew-symmetric table of their matrix commutation relations $[m_i, m_j]$, etc. Compare it with the table in Part 3.

Answer. The traceless constant matrices

$$m_1 = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix},\; m_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},\; m_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (12)$$

satisfy a commutator table that corresponds to the Poisson bracket table (5). In particular,

$$[m_1, m_2] = 4m_3, \quad [m_2, m_3] = -2m_2, \quad [m_3, m_1] = -2m_1\,.$$

▲

7. *Show that the flows*

$$\phi_k : \mathbf{z}(t) = e^{m_k t}\mathbf{z}(0) = \sum_{n=0}^{\infty}\frac{1}{n!}(m_k t)^n\mathbf{z}(0)$$

of the Hamiltonian vector fields $\{\,\cdot\,,\, X_k\}$ arising from the three rotationally invariant quadratic phase space functions in (2) acting on the phase space vector $\mathbf{z} = (\mathbf{q},\mathbf{p})^T$ may be written as symplectic matrix transformations $\mathbf{z}(t) = S(t)\mathbf{z}(0)$, with $S^TJS = J$.

Answer. Taking the derivative of the definition of symplectic matrix transformations $\mathbf{z}(t) = S(t)\mathbf{z}(0)$, with $S^T(t)JS(t) = J$, yields

$$Jm_i + m_i^TJ = 0\,, \quad \text{where} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \qquad (13)$$

That is, if $(Jm_i) = (Jm_i)^T$ is a symmetric matrix, then its flow ϕ_k : $\mathbf{z}(t) = e^{m_k t}\mathbf{z}(0)$ is a symplectic matrix transformation. This is easily verified, as

$$Jm_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},\; Jm_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},\; Jm_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

are all symmetric. ▲

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Exercise.

1. *Use the canonical Poisson brackets in (1) to compute ($\{Y_1, Y_2\}$, etc.) among the three S^1-invariant quadratic phase space functions*

$$Y_1 + iY_2 = 2a_1^* a_2 \quad \textit{and} \quad Y_3 = |a_1|^2 - |a_2|^2 \,, \tag{14}$$

with $a_k := q_k + ip_k \in \mathbb{C}^1$ for $k = 1, 2$.

> ***Answer.*** Denote $z_A = (a_A, a_A^*)$, with $A = 1, 2, 3$, and use the chain rule for Poisson brackets twice; first as $\{a_k, a_l^*\} = -2i\delta_{kl}$ and second as
>
> $$\{Y_i,\, Y_j\} = \frac{\partial Y_i}{\partial z_A}\{z_A,\, z_B\}\frac{\partial Y_j}{\partial z_B} = 4\epsilon_{ijk}Y_k \,, \tag{15}$$
>
> for the invariant bilinear functions Y_i in (15). ▲

2. *Show that these Poisson brackets may be expressed as a closed system*

$$\{Y_i, Y_j\} = c_{ij}^k Y_k, \qquad i, j, k = 1, 2, 3, \tag{16}$$

in terms of these invariants, by computing the coefficients c_{ij}^k.

> ***Answer.***
>
> $$c_{ij}^k = 4\epsilon_{ijk}$$
>
> ▲

3. *Write the Poisson brackets $\{Y_i, Y_j\}$ among these invariants as a 3×3 skew-symmetric table.*

> ***Answer.*** This is also expressed in tabular form as
>
> $$\{Y_i,\, Y_j\} = \begin{array}{c|ccc} \{\,\cdot\,,\,\cdot\,\} & Y_1 & Y_2 & Y_3 \\ \hline Y_1 & 0 & 4Y_3 & -4Y_2 \\ Y_2 & -4Y_3 & 0 & 4Y_1 \\ Y_3 & 4Y_2 & -4Y_1 & 0 \end{array}$$
>
> ▲

4. *Write the Poisson brackets for functions of these three invariants (Y_1, Y_2, Y_3) as a vector cross product of gradients of functions of $\mathbf{Y} \in \mathbb{R}^3$.*

> ***Answer.***
>
> $$\{F\,,\, H\}(\mathbf{Y}) = 4\mathbf{Y}\cdot\frac{\partial F}{\partial \mathbf{Y}} \times \frac{\partial H}{\partial \mathbf{Y}}\,,$$
>
> ▲

5. *Take the Poisson brackets of the three invariants* (Y_1, Y_2, Y_3) *with the function,*

$$R = |a_1|^2 + |a_2|^2 \,. \tag{17}$$

Explain your answers geometrically in terms of vectors in $\mathbb{R}^3$.

Answer. The $\mathbb{R}^3$-bracket for functions of the vector $\mathbf{Y}$ on the sphere $R^2 = |\mathbf{Y}|^2 = const$ is

$$\{F,\, H\}(\mathbf{Y}) = \frac{\partial C}{\partial \mathbf{Y}} \cdot \frac{\partial F}{\partial \mathbf{Y}} \times \frac{\partial H}{\partial \mathbf{Y}} \quad \text{with} \quad C(\mathbf{Y}) = 2|\mathbf{Y}|^2 \,,$$

and

$$\{C,\, \mathbf{Y}\} = \frac{\partial C}{\partial \mathbf{Y}} \times \frac{\partial C}{\partial \mathbf{Y}} = 0, \quad \text{with} \quad C(\mathbf{Y}) = 2|\mathbf{Y}|^2 \,.$$

▲

6. *Write the results of applying the Poisson brackets in the form*

$$\{\, \mathbf{a}\,,\, Y_k\} = c_k \mathbf{a} \quad k = 1, 2, 3\,,$$

for $\mathbf{a} = (a_1, a_2)^T \in \mathbb{C}^2$ *and* 2×2 *matrices* c_k*, with* $k = 1, 2, 3$*. Identify the type of matrix that results (symmetric, skew symmetric, etc.) Write a* 3×3 *skew-symmetric table of their matrix commutation relations* $[c_i,\, c_j]$*, etc. Compare it with the table in Part 3.*

Answer. One writes

$$\mathbf{Y} = a_k \boldsymbol{\sigma}_{kl} a_l^* \,.$$

where $\boldsymbol{\sigma} = (\sigma_1,\, \sigma_2,\, \sigma_3)$ is the 'vector' of Pauli Spin Matrices, given by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{18}$$

Then $\{a_k, a_l^*\} = -2i\delta_{kl}$ implies

$$\{\, \mathbf{a}\,,\, Y_k\} = -2i\sigma_k \mathbf{a} = c_k \mathbf{a} \quad k = 1, 2, 3. \tag{19}$$

▲

7. *Show that the flows*

$$\phi_k : \mathbf{z}(t) = e^{c_k t}\mathbf{z}(0) = \sum_{n=0}^{\infty} \frac{1}{n!}(c_k t)^n \mathbf{z}(0)$$

of the Hamiltonian vector fields $\{\,\cdot\,,\, Y_k\}$ *arising from the three* S^1 *phase invariant quadratic phase space functions in (15) acting on the phase space vector* $\mathbf{a} = (a_1, a_2)^T \in \mathbb{C}^2$ *may be written as* $SU(2)$ *matrix transformations* $\mathbf{a}(t) = U(t)\mathbf{a}(0)$*, with* $U^\dagger U = \mathrm{Id}$.

Answer. Taking the derivative of the definition of unitary matrix transformations $\mathbf{a}(t) = U(t)\mathbf{a}(0)$, with $U(t)^\dagger U(t) = \mathrm{Id}$, yields

$$c_k + c_k^\dagger = 0$$

where superscript dagger (†) denotes Hermitian conjugate. In fact, the matrices $c_k = -2i\sigma_k$ in (20) satisfy $c_k^\dagger = -c_k$. The flows of such matrices, $\phi_k : \mathbf{a}(t) = e^{c_k t}\mathbf{a}(0)$ are unitary matrix transformations. ▲

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Exercise.

1. *Compute the canonical Poisson brackets ($\{L_1, L_2\}$, etc.) among the three components of the vector* $\mathbf{L} = \mathbf{q} \times \mathbf{p}$ *for* $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^3 \times \mathbb{R}^3$ *so that* $\mathbf{L} \in \mathbb{R}^3$ *also.*

Answer.

$$\{L_1, L_2\} = L_3, \quad \{L_2, L_3\} = L_1, \quad \{L_3, L_1\} = L_2\,.$$

▲

2. *Show that these Poisson brackets may be expressed as a closed system*

$$\{L_i, L_j\} = c_{ij}^k L_k, \qquad i, j, k = 1, 2, 3, \tag{20}$$

in terms of these invariants, by computing the coefficients c_{ij}^k.

Answer.

$$\{L_i, L_j\} = \epsilon_{ijk} L_k, \qquad i, j, k = 1, 2, 3.$$

▲

3. *Write the Poisson brackets* $\{L_i, L_j\}$ *among the* (L_1, L_2, L_3) *as a* 3×3 *skew-symmetric table.*

Answer.

$$\{L_i,\, L_j\} = \begin{array}{|c|ccc|}\hline \{\,\cdot\,,\,\cdot\,\} & L_1 & L_2 & L_3 \\ \hline L_1 & 0 & L_3 & -L_2 \\ L_2 & -L_3 & 0 & L_1 \\ L_3 & L_2 & -L_1 & 0 \\ \hline \end{array}$$

▲

4. *Write the Poisson brackets for functions of the three components* (L_1, L_2, L_3) *as a vector cross product of gradients of functions of* $\mathbf{L} \in \mathbb{R}^3$.

Answer.

$$\{F\,,\,H\}(\mathbf{L})=\mathbf{L}\cdot\frac{\partial F}{\partial \mathbf{L}}\times\frac{\partial H}{\partial \mathbf{L}}=\frac{1}{2}\frac{\partial L^2}{\partial \mathbf{L}}\cdot\frac{\partial F}{\partial \mathbf{L}}\times\frac{\partial H}{\partial \mathbf{L}}\,.$$

▲

5. *Take the Poisson brackets of the three components (L_1, L_2, L_3) with the function,*

$$L^2=L_1^2+L_2^2+L_3^2\,. \tag{21}$$

Explain your answers geometrically in terms of vectors in $\mathbb{R}^3$.

Answer.

$$\{\mathbf{L}\,,\,L^2\}=-\mathbf{L}\times\mathbf{L}=0\,.$$

▲

6. *Write the results of applying the Poisson brackets in the form*

$$\{\,\mathbf{z}\,,\,L_k\}=r_k\mathbf{z}\,,$$

for $\mathbf{z}=(\mathbf{q},\mathbf{p})^T\in\mathbb{R}^3\times\mathbb{R}^3$ and 3×3 matrices r_k, with $k=1,2,3$. Identify the type of matrix that results (symmetric, skew symmetric, etc.) Write a 3×3 skew-symmetric table of their matrix commutation relations $[r_i,\,r_j]$, etc. Compare it with the table in Part 3.

Answer.

$$\{\,\mathbf{z}\,,\,\boldsymbol{\xi}\cdot\mathbf{L}\}=\boldsymbol{\xi}\times\mathbf{z}=\widehat{\boldsymbol{\xi}}\,\mathbf{z}\,,$$

with

$$\widehat{\boldsymbol{\xi}}=\begin{bmatrix}0&\xi_3&-\xi_2\\-\xi_3&0&\xi_1\\\xi_2&-\xi_1&0\end{bmatrix}=\xi_k r_k$$

where

$$r_1=\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix},\quad r_2=\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix},\quad r_3=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}, \tag{22}$$

whose matrix commutators satisfy

$$[r_i,r_j]=\epsilon_{ijk}r_k,\qquad i,j,k=1,2,3.$$

▲

7. *Show that the flows*

$$\phi_k:\mathbf{z}(t)=e^{r_kt}\mathbf{z}(0)=\sum_{n=0}^{\infty}\frac{1}{n!}(r_kt)^n\mathbf{z}(0)$$

of the Hamiltonian vector fields $\{\,\cdot\,,\,L_k\}$ *arising from the three components of the vector* $\mathbf{L}=\mathbf{q}\times\mathbf{p}$ *acting on the phase space vector* $\mathbf{z}=(\mathbf{q},\mathbf{p})^T$ *may be written as* $SO(3)$ *orthogonal matrix transformations* $\mathbf{z}(t)=O(t)\mathbf{z}(0)$, *with* $O^TO=\mathrm{Id}$.

Answer. The matrices r_k in (23) are skew symmetric; that is, they satisfy $r_k^T=-r_k$. The flows of such matrices, $\phi_k:\mathbf{z}(t)=e^{r_kt}\mathbf{z}(0)$ are orthogonal matrix transformations, as one may see by taking the time derivative of the relation $O^T(t)O(t)=\mathrm{Id}$ with $O(t)=e^{r_kt}$ to find

$$0=\dot{O}O^T+O\dot{O}^T=\dot{O}O^T+(\dot{O}O^T)^T=r_k+r_k^T\,.$$

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