M3/4A16 Assessed Coursework 1 **Darryl Holm**
Due in class Thursday November 6, 2008

#1 *Eikonal equation from Fermat's principle*

#1a *Prove that the 3D eikonal equation*

$$\frac{d}{ds}\left(n(\mathbf{r})\frac{d\mathbf{r}}{ds}\right) = |\dot{\mathbf{r}}|^2\frac{\partial n}{\partial \mathbf{r}} \tag{1}$$

preserves $|\dot{\mathbf{r}}| = 1$, *where* $\dot{\mathbf{r}} = d\mathbf{r}/ds$.

Expanding the 3D eikonal equation yields

$$\left(\frac{\partial n}{\partial \mathbf{r}}\cdot\dot{\mathbf{r}}\right)\dot{\mathbf{r}} + n(\mathbf{r})\ddot{\mathbf{r}} = |\dot{\mathbf{r}}|^2\frac{\partial n}{\partial \mathbf{r}}$$

Rearranging yields

$$\ddot{\mathbf{r}} = -\,\dot{\mathbf{r}}\times\left(\dot{\mathbf{r}}\times\frac{1}{n}\frac{\partial n}{\partial \mathbf{r}}\right).$$

Consequently, $\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} = 0$ *and the magnitude* $|\dot{\mathbf{r}}|$ *is preserved.*

Evolution under the eikonal equation tends to align $\dot{\mathbf{r}}$ *with* $\partial n/\partial\mathbf{r}$.
This is the continuum material version of Snell's Law.

#1b *Compute the Euler-Lagrange equations when Hamilton's principle is written in the form*

$$0 = \delta\int_A^B ds = \delta\int_A^B\left(\frac{dr^i}{ds}g_{ij}(\mathbf{r}(s))\frac{dr^j}{ds}\right)^{1/2}ds\,,$$

with $ds^2 = dr^i g_{ij} dr^j$ *for the Riemannian metric* $g_{ij}(\mathbf{r}(s))$ *in 3D with arclength parameter* s. *Show that these equations may be expressed as,*

$$\ddot{r}^{\,c} + \Gamma^c_{be}(\mathbf{r}(s))\dot{r}^b\dot{r}^e = 0 \quad \textit{with} \quad \dot{r}^b = \frac{dr^b}{ds} \quad b,\,c,\,e\in\{1,2,3\}, \tag{2}$$

in which the quantities $\Gamma^c_{be}(\mathbf{r})$ *are defined by*

$$\Gamma^c_{be}(\mathbf{r}) = \frac{1}{2}g^{ca}\left[\frac{\partial g_{ae}(\mathbf{r})}{\partial r^b} + \frac{\partial g_{ab}(\mathbf{r})}{\partial r^e} - \frac{\partial g_{be}(\mathbf{r})}{\partial r^a}\right],$$

and g^{ca} *is the inverse of the metric, so that* $g^{ca}g_{ab} = \delta^c_b$.
Does the eikonal equation emerge when $g_{ab} = n^2(\mathbf{r})\delta_{ab}$*? Prove it.*

$$
\begin{aligned}
0 &= \delta \int_A^B \left(\frac{dr^i}{ds} g_{ij}(\mathbf{r}(s)) \frac{dr^j}{ds} \right)^{1/2} ds =: \delta \int_A^B \left(\|\dot{\mathbf{r}}\|^2 \right)^{1/2} ds \\
&= \int_A^B \frac{1}{2\|\dot{\mathbf{r}}\|} \, \delta \left(\frac{dr^i}{ds} g_{ij}(\mathbf{r}(s)) \frac{dr^j}{ds} \right) ds \\
&= \int_A^B \frac{1}{\|\dot{\mathbf{r}}\|} \left(\frac{1}{2} \frac{\partial g_{ij}}{\partial r^k} \dot{r}^i \dot{r}^j - \frac{d}{ds} \left(g_{kj} \dot{r}^j \right) \right) \delta r^k \, ds \\
&= \int_A^B \frac{1}{\|\dot{\mathbf{r}}\|} \left(\frac{1}{2} \frac{\partial g_{ij}}{\partial r^k} \dot{r}^i \dot{r}^j - \left(g_{kl} \ddot{r}^l + \frac{\partial g_{kl}}{\partial r^m} \dot{r}^l \dot{r}^m \right) \right) \delta r^k \, ds \,,
\end{aligned}
$$

from which equation (2) emerges after rearranging.

*However, inspection shows that this is **not** the eikonal equation (1) when* $g_{ab} = n^2(\mathbf{r})\delta_{ab}$.

#1c *Prove that equation (2) preserves* $\|\dot{\mathbf{r}}\|^2 = \dot{r}^i g_{ij}(\mathbf{r}) \dot{r}^j$. *What does this tell us about the last part of question* **#1b***?*

Take the s-derivative of $\|\dot{\mathbf{r}}\|^2$ *and follow the path of the variational derivation of the equation.*

$$
\frac{1}{2} \frac{d}{ds} \|\dot{\mathbf{r}}\|^2 = \frac{1}{2} g_{ij,k} \dot{r}^k \dot{r}^i \dot{r}^j + \dot{r}^i g_{ij} \ddot{r}^j
$$

Now substitute from above

$$
\dot{r}^k g_{kl} \ddot{r}^l = \frac{1}{2} \dot{r}^k g_{lm,k} \dot{r}^l \dot{r}^m - \dot{r}^k g_{kl,m} \dot{r}^l \dot{r}^m
$$

and rearrange to prove the point that $\|\dot{\mathbf{r}}\|^2 = \dot{r}^i g_{ij}(\mathbf{r}) \dot{r}^j$ *is preserved. This tells us that substituting* $g_{ab} = n^2(\mathbf{r})\delta_{ab}$ *into the geodesic equation (2) will **not** recover the eikonal equation (1). Equations (1) and (2) are different. This is clear, for example, because their conservation laws differ. The eikonal equation (1) preserves the Euclidean condition* $|\dot{\mathbf{r}}| = 1$, *not* $\|\dot{\mathbf{r}}\| = 1$, *which is preserved by (2).*

#1d $\boxed{\textit{Fourth year students}}$

(i) *Compute the quantities* $\Gamma^c_{be}(\mathbf{r})$ *for* $g_{ij} = n^2(\mathbf{r})\delta_{ij}$ *when* $n = n(r)$ *with* $r^2 := r^a \delta_{ab} r^b$.

(ii) *Write the eikonal equation (1) when the index of refraction* $n(r)$ *de-*

pends only on r.
(iii) *Show that the eikonal equation conserves the vector $\mathbf{L} = \mathbf{r} \times n(r)\dot{\mathbf{r}}$ when index of refraction n depends only on the spherical radius $r = |\mathbf{r}|$.*

(i) *For $g_{ij}(r) = \delta_{ij} n^2(r)$ with $r^2 := r^a \delta_{ab} r^b$, the geodesic equation (2) becomes*

$$
\begin{aligned}
\ddot{r}^c &= -\Gamma^c_{be}(\mathbf{r}(s))\dot{r}^b \dot{r}^e \\
&= -\frac{1}{2} g^{ca} \left[\frac{\partial g_{ae}(r)}{\partial r^b} + \frac{\partial g_{ab}(r)}{\partial r^e} - \frac{\partial g_{be}(r)}{\partial r^a} \right] \dot{r}^b \dot{r}^e \\
&= -\frac{1}{nr}\frac{\partial n}{\partial r} \left[\delta^c_e \delta_{ab} r^a + \delta^c_b \delta_{ed} r^d - r_a \delta^{ac} \delta_{eb} \right] \dot{r}^e \dot{r}^b
\end{aligned}
$$

or, equivalently, in Euclidean vector form

$$
\ddot{\mathbf{r}} = -\frac{1}{nr}\frac{\partial n}{\partial r}\Big[\underbrace{\dot{\mathbf{r}} \times \big(\dot{\mathbf{r}} \times \mathbf{r}\big)}_{\textit{Eikonal}} + \underbrace{(\mathbf{r} \cdot \dot{\mathbf{r}})\dot{\mathbf{r}}}_{\textit{Extra}} \Big]
$$

(ii) *In contrast the eikonal equation (1) may be written as*

$$
n(\mathbf{r})\ddot{\mathbf{r}} = -\dot{\mathbf{r}} \times \left(\dot{\mathbf{r}} \times \frac{\partial n}{\partial \mathbf{r}} \right) = -\dot{\mathbf{r}} \left(\dot{\mathbf{r}} \cdot \frac{\partial n}{\partial \mathbf{r}} \right) + |\dot{\mathbf{r}}|^2 \frac{\partial n}{\partial \mathbf{r}}
$$

which implies for $n(r)$ that

$$
\Big(n(\mathbf{r})\dot{\mathbf{r}} \Big)^{\cdot} = |\dot{\mathbf{r}}|^2 \frac{\partial n}{\partial \mathbf{r}} \quad \textit{with} \quad \frac{\partial n}{\partial \mathbf{r}} = \frac{dn}{dr}\frac{\mathbf{r}}{r}
$$

(iii) *Consequently*

$$
\dot{\mathbf{L}} = \Big(\mathbf{r} \times n(\mathbf{r})\dot{\mathbf{r}} \Big)^{\cdot} = |\dot{\mathbf{r}}|^2 \, \mathbf{r} \times \frac{\partial n}{\partial \mathbf{r}}
$$

The RHS vanishes when $n = n(r)$ by the previous equation.

#2 *Hamiltonian formulations*

According to the text the eikonal equation also follows from Fermat's principle in the form

$$
0 = \delta S = \delta \int_A^B \frac{1}{2} n^2(\mathbf{r}(\tau)) \frac{d\mathbf{r}}{d\tau} \cdot \frac{d\mathbf{r}}{d\tau} \, d\tau = \delta \int_A^B L(\mathbf{r}, \dot{\mathbf{r}}) \, d\tau \,, \tag{3}
$$

with new arclength parameter $d\tau = nds$. (You may retain the dot notation for $d/d\tau$.) Use this version of Fermat's principle to write the Hamiltonian formulations of the solutions of question #1. (As usual, 3rd year students do parts a,b,c, while 4th year and MSc students do parts a,b,c,d.)

#2a *The fibre derivative of the Lagrangian in (3) above is*

$$\mathbf{p} = \frac{\partial L}{\partial (d\mathbf{r}/d\tau)} = n^2(\mathbf{r})\frac{d\mathbf{r}}{d\tau} \tag{4}$$

This defines the canonical momentum and yields the Legendre transformation for the Hamiltonian

$$H(\mathbf{r},\mathbf{p}) = \mathbf{p}\cdot\frac{d\mathbf{r}}{d\tau} - L(\mathbf{r}, d\mathbf{r}/d\tau) = \frac{|\mathbf{p}|^2}{2n^2(\mathbf{r})}$$

The canonical Hamilton equations are

$$\frac{d\mathbf{r}}{d\tau} = \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{n^2}\mathbf{p} \quad \textit{and} \quad \frac{d\mathbf{p}}{d\tau} = -\frac{\partial H}{\partial \mathbf{r}} = \frac{|\mathbf{p}|^2}{n^3}\frac{\partial n}{\partial \mathbf{r}}$$

Substituting the momentum-velocity relation (4) into the momentum equation yields

$$\frac{d}{d\tau}\left(n^2\frac{d\mathbf{r}}{d\tau}\right) = n^2\left|\frac{d\mathbf{r}}{d\tau}\right|^2\frac{\partial n}{\partial \mathbf{r}} \tag{5}$$

Hence, using $d\tau = n(\mathbf{r})ds$ so that $d/ds = n(\mathbf{r})d/d\tau$ one finds

$$\begin{aligned}
\frac{d}{ds}\left(\frac{d\mathbf{r}}{ds}\right) &= n^2\frac{d^2\mathbf{r}}{d\tau^2} + \left(\frac{\partial n}{\partial \mathbf{r}}\cdot\frac{d\mathbf{r}}{d\tau}\right)\frac{d\mathbf{r}}{d\tau} \\
\textit{By equation (5)} &= -n\frac{d\mathbf{r}}{d\tau}\times\left(\frac{d\mathbf{r}}{d\tau}\times\frac{\partial n}{\partial \mathbf{r}}\right) \\
&= -\frac{1}{n}\frac{d\mathbf{r}}{ds}\times\left(\frac{d\mathbf{r}}{ds}\times\frac{\partial n}{\partial \mathbf{r}}\right)
\end{aligned}$$

This shows that the geodesic Euler-Lagrange equation for the Lagrangian (3) does recover the eikonal equation in canonical Hamiltonian form.

#2b *The eikonal equation **does** emerge when $g_{ab} = n^2(\mathbf{r})\delta_{ab}$ for this Lagrangian.*

#2c *Preservation of the Hamiltonian ensures preservation of $n^2|d\mathbf{r}/d\tau|^2 = |d\mathbf{r}/ds|^2$, just as for the eikonal equation, since $d\tau = n(\mathbf{r})ds$.*

This conservation law bodes well for these equations to recover the eikonal equation.

#2d $\boxed{\textbf{\textit{Fourth year students}}}$
Once we have recovered the eikonal equation, parts (i) *and* (ii) *follow as before, except now we must change the independent variable by using* $d\tau = nds$.

For (iii), *it remains to compute the Hamiltonian vector field for* $\mathbf{L} = \mathbf{r}\times\mathbf{p}$

$$\Big\{ \,\cdot\,,\, \mathbf{L} \Big\} = \mathbf{r}\times\frac{\partial}{\partial\mathbf{r}} + \mathbf{p}\times\frac{\partial}{\partial\mathbf{p}}$$

Consequently

$$\dot{\mathbf{L}} = \Big\{ \mathbf{L}\,,\, H \Big\} = \left(\frac{|\mathbf{p}|^2}{n^3(\mathbf{r})}\right)\mathbf{r}\times\frac{\partial n}{\partial\mathbf{r}}$$

and the RHS vanishes when $n = n(r)$.

#3 $\mathbb{R}^3-$***reduction for axisymmetric, translation invariant optical media***

#3a *Compute by chain rule that*

$$\frac{dF}{dt} = \{F,H\} \quad = \quad \nabla F\cdot\nabla S^2\times\nabla H = \frac{\partial F}{\partial X_k}\,\epsilon_{klm}\,\frac{\partial S^2}{\partial X_l}\,\frac{\partial H}{\partial X_m}\,,$$

for

$$X_1 = |\mathbf{q}|^2 \ge 0\,,\quad X_2 = |\mathbf{p}|^2 \ge 0\,,\quad X_3 = \mathbf{p}\cdot\mathbf{q}\,.$$

Done in notes. That $S^2 = \mathbf{p}\times\mathbf{q} \ge 0$ *is evident.*

#3b *Show that the Poisson bracket* $\{F,H\} = \nabla F\cdot\nabla S^2\times\nabla H$, *with definition* $S^2 = X_1X_2 - X_3^2$ *satisfies the Jacobi identity.*

This Poisson bracket may be written equivalently as

$$\{F,H\} = X_k\, c^k_{ij}\,\frac{\partial F}{\partial X_i}\frac{\partial H}{\partial X_j}$$

From the table

$$\{X_i,\,X_j\} = \begin{array}{|c|ccc|}\hline \{\,\cdot\,,\,\cdot\,\} & X_1 & X_2 & X_3 \\ \hline X_1 & 0 & 4X_3 & 2X_1 \\ X_2 & -4X_3 & 0 & -2X_2 \\ X_3 & -2X_1 & 2X_2 & 0 \\ \hline \end{array}$$

one identifies c_{ij}^k

$$\{X_i,\,X_j\} = c_{ij}^k X_k\,.$$

We also have

$$\{X_l,\,\{X_i,\,X_j\}\} = c_{ij}^k\{X_l,\,X_k\} = c_{ij}^k c_{lk}^m X_m\,.$$

Hence, the Jacobi identity is satisfied as a consequence of

$$\begin{aligned}&\{X_l,\,\{X_i,\,X_j\}\} + \{X_i,\,\{X_j,\,X_l\}\} + \{X_j,\,\{X_l,\,X_i\}\}\\ &= c_{ij}^k\{X_l,\,X_k\} + c_{jl}^k\{X_i,\,X_k\} + c_{li}^k\{X_j,\,X_k\}\\ &= \Big(c_{ij}^k c_{lk}^m + c_{jl}^k c_{ik}^m + c_{li}^k c_{jk}^m\Big)X_m = 0\,,\end{aligned}$$

This is the condition required for the Jacobi identity to hold in terms of the structure constants.

Remark. *This calculation provides an independent proof of the Jacobi identity for the* $\mathbb{R}^3$ *bracket in the case of quadratic distinguished functions.* ■

#3c *Consider the Hamiltonian*

$$H = aY_1 + bY_2 + cY_3 \tag{6}$$

with the linear combinations

$$Y_1 = \frac{1}{2}(X_1 + X_2)\,, \quad Y_2 = \frac{1}{2}(X_2 - X_1)\,, \quad Y_3 = X_3\,,$$

and constant values of (a, b, c). *Compute the canonical dynamics generated by Hamiltonian (6) on level sets of* $S^2 > 0$ *and* $S^2 = 0$.

This amounts to computing the intersections of the planes

$$H = aY_1 + bY_2 + cY_3 = constant$$

with the hyperboloids of revolution about the Y_1*-axis,*

$$S^2 = Y_1^2 - Y_2^2 - Y_3^2 = constant\,.$$

One may solve this problem graphically, algebraically by choosing special cases for the orientations of the family of planes, or most generally by restricting the equations to a level set of either $S^2 =$ constant, or $H =$ constant.

Restricting to** $S^2 =$ constant **hyperboloids of revolution.

Each of the family of hyperboloids of revolution $S^2 =$ constant comprises a layer in the "hyperbolic onion" preserved by axisymmetric ray optics. We use hyperbolic polar coordinates on these layers in analogy to spherical coordinates,

$$Y_1 = S\cosh u\,, \quad Y_2 = S\sinh u\cos\psi\,, \quad Y_3 = S\sinh u\sin\psi\,.$$

The $\mathbb{R}^3-$bracket thereby transforms into hyperbolic coordinates as

$$\{F,\,H\}\,dY_1\wedge dY_2\wedge dY_3 = -\,\{F,\,H\}_{hyperb}\,S^2\,dS\wedge d\psi\wedge d\cosh u\,.$$

Note that the oriented quantity

$$S^2 d\cosh u\wedge d\psi = -S^2 d\psi\wedge d\cosh u\,,$$

*is the **area element on the hyperboloid** corresponding to the constant S^2.*

On a constant level surface of S^2 the function $\{F,H\}_{hyperb}$ only depends on $(\cosh u,\psi)$ so the Poisson bracket for optical motion on any particular *hyperboloid is then*

$$\begin{aligned}\{F,\,H\}\,d^{\,3}Y &= -\,S^2 dS\wedge dF\wedge dH = -\,S^2 dS\wedge\{F,\,H\}_{hyperb}\,d\psi\wedge d\cosh u\\ &= -\,S^2 dS\wedge\left(\frac{\partial F}{\partial\psi}\frac{\partial H}{\partial\cosh u} - \frac{\partial H}{\partial\cosh u}\frac{\partial F}{\partial\psi}\right)d\psi\wedge d\cosh u\,.\end{aligned}$$

*Being a constant of the motion, the value of S^2 may be absorbed by a choice of units for any given initial condition and the Poisson bracket for the optical motion thereby becomes **canonical on each hyperboloid**,*

$$\frac{d\psi}{dz} = \{\psi,\,H\}_{hyperb} = \frac{\partial H}{\partial\cosh u}\,, \qquad \frac{d\cosh u}{dz} = \{\cosh u,\,H\}_{hyperb} = -\,\frac{\partial H}{\partial\psi}\,.$$

In the Cartesian variables $(Y_1,Y_2,Y_3)\in\mathbb{R}^3$, one has $\cosh u = Y_1/S$ and $\psi = \tan^{-1}(Y_3/Y_2)$. The Hamiltonian $H = aY_1 + bY_2 + cY_3$ becomes

$$H = aS\cosh u + bS\sinh u\cos\psi + cS\sinh u\sin\psi\,,$$

with

$$\sinh u = \sqrt{\cosh^2 u - 1} \quad \textit{and} \quad \sin\psi = \sqrt{1-\cos^2\psi}$$

so that

$$\frac{\partial \sinh u}{\partial \cosh u} = \coth u\,,$$

and

$$\begin{aligned}
\frac{1}{S}\frac{d\psi}{dz} = \frac{1}{S}\frac{\partial H}{\partial \cosh u} &= a + b\coth u\cos\psi + c\coth u\sin\psi\,,\\
\frac{1}{S}\frac{d\cosh u}{dz} = \frac{1}{S}\frac{\partial H}{\partial \psi} &= -b\sinh u\sin\psi + c\sinh u\cos\psi\,.
\end{aligned}$$

Restricting to the conical surface $S^2 = 0$

To restrict to the conical surface $S^2 = Y_1^2 - Y_2^2 - Y_3^2 = 0$ *one chooses coordinates*

$$Y_1 = Z\,, \quad Y_2 = Z\cos\psi\,, \quad Y_3 = Z\sin\psi$$

The Poisson bracket for the optical motion thereby becomes ***canonical on the cone****,*

$$\frac{d\psi}{dz} = \{\psi,\,H\}_{cone} = \frac{\partial H}{\partial Z}\,, \qquad \frac{dZ}{dz} = \{Z,\,H\}_{cone} = -\frac{\partial H}{\partial \psi}\,.$$

and

$$\begin{aligned}
\frac{1}{S}\frac{d\psi}{dz} = \frac{1}{S}\frac{\partial H}{\partial Z} &= a + b\cos\psi + c\sin\psi\,,\\
\frac{1}{S}\frac{dZ}{dz} = \frac{1}{S}\frac{\partial H}{\partial \psi} &= -bZ\sin\psi + cZ\cos\psi\,.
\end{aligned}$$

The equations on the hyperboloids and the cone are a bit complicated. It turns out that a very simple solution is possible on the level sets of the Hamiltonian planes.

Restricting to $H = constant$ ***planes.***

One may also restrict the equations to a planar level set of $H =$ *constant. This tactic is very insightful and particularly simple, because only linear equations arise.*

The latter approach summons the linear transformation with constant coefficients,

$$\begin{pmatrix} dY_1 \\ dY_2 \\ dY_3 \end{pmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \begin{pmatrix} dH \\ dx \\ dy \end{pmatrix}$$

One finds the constant Jacobian from $d^3Y = (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\, dH\wedge dx\wedge dy$. *Then one transforms to level sets of* H *by writing*

$$\begin{aligned} \frac{dF}{dz} = \{F,\, H\}d^3Y &= -dC\wedge dF\wedge dH = dH\wedge dF\wedge dC \\ &= (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})dH\wedge\{F,\, C\}_{xy}\, dx\wedge dy \end{aligned}$$

Thus, on one of the planes in the family of level sets of H, *one finds*

$$\begin{aligned} \frac{dx}{dz} &= \frac{\partial C}{\partial y} \\ \frac{dy}{dz} &= -\frac{\partial C}{\partial x} \end{aligned}$$

Because C *is quadratic and the transformation from* (Y_1, Y_2, Y_3) *to* (H, x, y) *is linear, these canonical equations on any of the planar level sets of* H *are linear. That is,*

$$\begin{pmatrix} dx/dz \\ dy/dz \end{pmatrix} = \begin{bmatrix} \alpha & \beta \\ \bar\alpha & \bar\beta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \gamma \\ \bar\gamma \end{pmatrix}$$

with constants $(\alpha, \beta, \bar\alpha, \bar\beta, \gamma, \bar\gamma)$. *These are linear equations.*

Direct solution in canonical variables. *For the Hamiltonian*

$$H = aY_1 + bY_2 + cY_3 = \frac{a-b}{2}|\mathbf{q}|^2 + \frac{a+b}{2}|\mathbf{p}|^2 + c\,\mathbf{p}\cdot\mathbf{q}$$

one has

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{\partial H}{\partial \mathbf{p}} = (a+b)\mathbf{p} + c\mathbf{q} \\ \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{q}} = (b-a)\mathbf{q} - c\mathbf{p} \end{aligned}$$

or

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} c & a+b \\ b-a & -c \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}$$

As expected, this is a linear symplectic transformation and the matrix is given by

$$\begin{pmatrix} c & a+b \\ b-a & -c \end{pmatrix} = \frac{a-b}{2} m_1 + \frac{a+b}{2} m_2 + c\, m_3 \,.$$

#3d $\boxed{\textit{\textbf{Fourth year students}}}$

Derive the formula for reconstructing the angle canonically conjugate to S for the canonical dynamics generated by the planar Hamiltonian (6).

The volume elements corresponding to the Poisson brackets are

$$d^{\,3}Y =: dY_1 \wedge dY_2 \wedge dY_3 \quad = \quad d\frac{S^3}{3} \wedge d\cosh u \wedge d\psi$$

On a level set of $S = p_\phi$ this implies canonical variables $(\cosh u,\ \psi)$ *with symplectic form,*

$$d\cosh u \wedge d\psi \,,$$

and since $(S = p_\phi, \phi)$ are also canonically conjugate, one has

$$dp_j \wedge dq_j = dS \wedge d\phi + \, d\cosh u \wedge d\psi \,.$$

One recalls Stokes Theorem on phase space

$$\iint_A dp_j \wedge dq_j = \oint_{\partial A} p_j dq_j \,,$$

where the boundary of the phase space area ∂A is taken around a loop on a closed orbit. On an invariant hyperboloid S this loop integral becomes

$$\oint \mathbf{p}\cdot d\mathbf{q} := \oint p_j dq_j = \oint \Big(S d\phi + \cosh u \, d\psi \Big) \,.$$

Thus we may compute the total phase change around a closed periodic orbit on the level set of hyperboloid S from

$$\oint S d\phi \quad = \quad S\Delta\phi = \underbrace{-\oint \cosh u \, d\psi}_{\textit{Geometric } \Delta\phi} + \underbrace{\oint \mathbf{p}\cdot d\mathbf{q}}_{\textit{Dynamic } \Delta\phi} \tag{7}$$

Evidently, one may denote the total change in phase as the sum

$$\Delta\phi = \Delta\phi_{geom} + \Delta\phi_{dyn} \,,$$

by identifying the corresponding terms in the previous formula. By Stokes theorem, one sees that the geometric phase associated with a periodic motion on a particular hyperboloid is given by the hyperbolic solid angle enclosed by the orbit. Thus, the name: ***geometric phase****.*