Solutions to Assessed Homework1

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1. Eikonal equation & Snell's Law from Fermat's principle

#1a Show explicitly that the eikonal equation for axial ray optics follows from the Euler-Lagrange equation.

> Direct verification, as in the notes.

#1b Compute the 3D Eikonal equation from Fermat's principle

> As in the calculation leading to the Euler-Lagrange equation in the notes, one finds
>
> $$0 = \delta \int_A^B n(\mathbf{r}(s))\,\sqrt{\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}}\,ds = \int_A^B \left[|\dot{\mathbf{r}}|\frac{\partial n}{\partial \mathbf{r}} - \frac{d}{ds}\left(n(\mathbf{r}(s))\frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}\right)\right]\cdot \delta\mathbf{r}\,ds$$
>
> where one denotes $\dot{\mathbf{r}} := d\mathbf{r}/ds$. The 3D eikonal equation emerges, upon choosing the arclength variable $ds^2 = d\mathbf{r}\cdot d\mathbf{r}$, so that $|\dot{\mathbf{r}}| = 1$. (Note that $d|\dot{\mathbf{r}}|/ds = 0$.)

#1c Verify that the *same* three-dimensional Eikonal equation *also* follows from Fermat's principle in the form

$$0 = \delta S = \delta \int_A^B \frac{1}{2} n^2(\mathbf{r}(\tau))\,\frac{d\mathbf{r}}{d\tau}\cdot\frac{d\mathbf{r}}{d\tau}\,d\tau\,,$$

with $d\tau = nds$ for the arclength parameter s. What features are different about the canonical momenta in the two versions of Fermat's principle?
How do their Hamiltonian formulations differ?

Denoting $\mathbf{r}'(\tau) = d\mathbf{r}/d\tau$ one computes,

$$0 = \delta S = \int_A^B \frac{ds}{d\tau}\left[\frac{nds}{d\tau}\frac{\partial n}{\partial \mathbf{r}} - \frac{d}{ds}\left(\frac{nds}{d\tau}n\frac{d\mathbf{r}}{ds}\right)\right]\cdot \delta\mathbf{r}\,d\tau$$

which agrees with the previous calculation upon reparameterising $\boxed{d\tau = nds}$.

The Hamiltonian arising by the Legendre transformation from the Lagrangian in the first form vanishes identically, while the Hamiltonian corresponding to the second form is given by $H = |\mathbf{p}|^2/(2n^2)$.

#1d $\boxed{\textbf{\textit{Fourth year students}}}$ Derive Snell's Law for refraction in a crystal at a planar interface.

The Lagrangian in the case of an anisotropic optical medium is given by

$$L(\mathbf{r}(s), \dot{\mathbf{r}}(s)) = n(\mathbf{r}(s), \widehat{\mathbf{s}}(s))\,|\dot{\mathbf{r}}(s)|\,.$$

Here the refractive index is modelled as a function of both position along the ray $\mathbf{r}(s)$ and the ray direction $\widehat{\mathbf{s}}(s)$, which is a unit vector at position s. The latter is defined when s is the arclength as

$$\widehat{\mathbf{s}} = \dot{\mathbf{r}}(s)/|\dot{\mathbf{r}}(s)| \quad \text{with} \quad |\dot{\mathbf{r}}| = 1\,.$$

Exercise. Show that the variation of the ray direction in this equation is related to the variation of the path $\delta\mathbf{r}(s)$ by

$$\delta\,\widehat{\mathbf{s}} = \frac{-1}{|\dot{\mathbf{r}}|}\,\widehat{\mathbf{s}}\times\Big(\widehat{\mathbf{s}}\times\delta\dot{\mathbf{r}}(s)\Big)\,.$$

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Fermat's principle with optical Lagrangian implies the following

3D eikonal equation for the vector $\mathbf{r}(s)$ defining the ray path,

$$\frac{d}{ds}\Big(n(\mathbf{r},\widehat{\mathbf{s}}\,)\widehat{\mathbf{s}}+\mathbf{A}(\mathbf{r},\widehat{\mathbf{s}}\,)\Big) = \frac{\partial n(\mathbf{r},\,\widehat{\mathbf{s}}\,)}{\partial \mathbf{r}} \qquad \textit{\textbf{(3D eikonal equation)}}$$

Here, the ***anisotropy vector*** $\mathbf{A}(\mathbf{r},\widehat{\mathbf{s}}\,)$ is defined as

$$\mathbf{A} := \frac{\partial n}{\partial \dot{\mathbf{r}}}\bigg|_{|\dot{\mathbf{r}}|=1} = -\,\widehat{\mathbf{s}}\times\Big(\widehat{\mathbf{s}}\times\frac{\partial n}{\partial \widehat{\mathbf{s}}}\Big)\,.$$

The anisotropy vector $\mathbf{A}$ is the projection of the vector $\partial n/\partial \dot{\mathbf{r}}$ onto the plane that is normal to $\dot{\mathbf{r}}$ and tangent to the direction sphere $|\dot{\mathbf{r}}| = 1$.

In the $\dot{\mathbf{r}}$ notation, the ***optical momentum*** is defined on recalling $|\dot{\mathbf{r}}| = 1$ as

$$\mathbf{p} := \frac{\partial L}{\partial \dot{\mathbf{r}}(s)} = n(\mathbf{r},\dot{\mathbf{r}}\,)\dot{\mathbf{r}} + \mathbf{A}(\mathbf{r},\dot{\mathbf{r}}) \qquad \textit{\textbf{(optical momentum)}}$$

Thus, the optical momentum $\mathbf{p}$ lies in the plane spanned by the vectors $\dot{\mathbf{r}}$ and $\partial n/\partial \dot{\mathbf{r}}$, which are orthogonal because of the constraint $|\dot{\mathbf{r}}| = 1$. The optical momentum is related to the tangent vector $\dot{\mathbf{r}}(s)$ along the ray path $\mathbf{r}(s)$ by

$$n(\mathbf{r},\dot{\mathbf{r}}\,)\dot{\mathbf{r}} = \mathbf{p} - \mathbf{A}(\mathbf{r},\dot{\mathbf{r}})\,,$$

whose norm is

$$|\mathbf{p} - \mathbf{A}(\mathbf{r},\dot{\mathbf{r}})| = n(\mathbf{r},\dot{\mathbf{r}}\,) \quad \text{since} \quad \dot{\mathbf{r}}\cdot\mathbf{A} = 0\,.$$

Remark. The anisotropy vector is orthogonal to the desired ray direction and is a prescribed function of it and the position along the ray path. ■

May one solve for the ray direction $\dot{\mathbf{r}}$, given the optical momentum $\mathbf{p}$ and position $\mathbf{r}$? The optical momentum decomposes conveniently into components which are parallel and perpendicular to $\dot{\mathbf{r}}$, as

$$\mathbf{p} = n(\mathbf{r},\dot{\mathbf{r}}\,)\dot{\mathbf{r}} + \mathbf{A}(\mathbf{r},\dot{\mathbf{r}}) =: \mathbf{p}_{\|} + \mathbf{p}_{\perp}\,.$$

However, media for which these functional relations are nontrivial do not in general admit solutions for the tangent vector $\dot{\mathbf{r}}(s)$ as a function of $(\mathbf{r}(s),\ \mathbf{p}(s))$. Thus, the ray direction is not solvable in general from the optical momentum and ray path.[1] However, the 3D eikonal equation () still remains and so does its associated anisotropic Huygens wavefront description, now in the form

$$\frac{\partial S(\mathbf{r},\dot{\mathbf{r}})}{\partial \mathbf{r}} = n(\mathbf{r},\dot{\mathbf{r}}\,)\dot{\mathbf{r}} + \mathbf{A}(\mathbf{r},\dot{\mathbf{r}}) \qquad \textbf{\textit{(anisotropic Huygens wavefront)}}$$

whose norm yields the scalar Huygens equation for anisotropic media,

$$\left|\frac{\partial S}{\partial \mathbf{r}}\right|^2 = n^2(\mathbf{r},\dot{\mathbf{r}}\,) + \big|\mathbf{A}(\mathbf{r},\dot{\mathbf{r}})\big|^2 \qquad \textbf{\textit{(scalar Huygens equation)}}$$

Ibn Sahl-Snell Law for anisotropic media.
The statement of the Ibn Sahl-Snell Law relation at discontinuities of the refractive index in anisotropic media is rather more involved than for isotropic media. A break in the direction $\widehat{\mathbf{s}}$ of the ray vector is still expected at any finite discontinuity in the refractive index $n = |\mathbf{n}|$ encountered along the ray path $\mathbf{r}(s)$. According to the eikonal equation for anisotropic media () the jump (denoted by Δ) in *three dimensional* optical momentum across the discontinuity must satisfy the relation

$$\Delta \mathbf{p} \times \frac{\partial n}{\partial \mathbf{r}} = \Delta\bigg(n(\mathbf{r},\widehat{\mathbf{s}}\,)\widehat{\mathbf{s}} + \mathbf{A}(\mathbf{r},\widehat{\mathbf{s}})\bigg) \times \frac{\partial n}{\partial \mathbf{r}} = 0\,.$$

This means the transverse optical momenta $\mathbf{p}$ and $\mathbf{p}'$ will be invariant across the interface on which the discontinuity in refractive index occurs.

Thus, preservation of the vector of optical momentum tangential to the discontinuity in refractive index still holds, but a difficulty occurs because the ray direction and optical momentum are no longer co-linear. Instead, they differ by the anisotropy vector, which is orthogonal to the desired ray direction and also depends as a prescribed function of ray direction on either side of the discontinuity.

The geometry for determining the refracted ray direction in anisotropic media thus becomes considerably more involved than the simple Ibn Sahl-Snell Law of ray projection for isotropic media. There does exist a graphical construction, but its application in the Ibn Sahl-Snell Law for construction of the break in ray direction at a discontinuity in refractive index in an anisotropic medium is problematic, unless the prescribed dependence of the anisotropy vector on the ray direction is rather simple. ■

2. Phase plane for axisymmetric, translation invariant optical materials

#2a Write Hamilton's canonical equations for axial ray optics in an axisymmetric, translation invariant media.

Done in the notes.

#2b Solve these equations for the case of an optical fiber with radially varying index of refraction in the following form with $r^2 = |\mathbf{q}|^2$:

$$n^2(r) = \lambda^2 + (\mu - \nu r^2)^2\,, \quad \lambda,\, \mu,\, \nu = \text{constants},$$

by reducing the problem for axial ray optics to phase plane analysis.

#2c Show how the phase space portrait differs between $p_\phi = 0$ and $p_\phi \neq 0$. Explain what happens when ν changes sign.

Reduction to phase plane analysis is done in the paper distributed in the webnotes.

Holm, D. D. and G. Kovacic [1991] Homoclinic Chaos for Ray Optics in a Fiber. *Physica D* **51**, 177–188.

#2d $\boxed{\textbf{\textit{Fourth year students}}}$ Derive the formula for reconstructing the angle canonically conjugate to p_ϕ for this dynamics.

See

Holm, D. D. and G. Kovacic [1991] Homoclinic Chaos for Ray Optics in a Fiber. *Physica D* **51**, 177–188.

3. $\mathbb{R}^3-$reduction for axisymmetric, translation invariant optical media

#3a Compute by chain rule that

$$\frac{dF}{dt} = \{F, H\} \;=\; \nabla F \cdot \nabla S^2 \times \nabla H = \frac{\partial F}{\partial X_k}\,\epsilon_{klm}\,\frac{\partial S^2}{\partial X_l}\,\frac{\partial H}{\partial X_m}\,,$$

for

$$X_1 = |\mathbf{q}|^2 \ge 0\,, \quad X_2 = |\mathbf{p}|^2 \ge 0\,, \quad X_3 = \mathbf{p}\cdot\mathbf{q}\,.$$

Done in notes. That $S^2 = \mathbf{p}\times\mathbf{q} \ge 0$ is evident.

#3b Show that the Poisson bracket $\{F, H\} = \nabla F \cdot \nabla S^2 \times \nabla H$, with definition $S^2 = X_1 X_2 - X_3^2$ satisfies the Jacobi identity.

This Poisson bracket may be written equivalently as

$$\{F, H\} = X_k\, c^k_{ij}\,\frac{\partial F}{\partial X_i}\frac{\partial H}{\partial X_j}$$

From the table

$$\{X_i,\,X_j\} = \begin{array}{|c|ccc|}\hline \{\,\cdot\,,\,\cdot\,\} & X_1 & X_2 & X_3 \\ \hline X_1 & 0 & 4X_3 & 2X_1 \\ X_2 & -4X_3 & 0 & -2X_2 \\ X_3 & -2X_1 & 2X_2 & 0 \\ \hline \end{array}$$

one identifies c^k_{ij}

$$\{X_i,\,X_j\} = c^k_{ij}X_k\,.$$

We also have

$$\{X_l,\,\{X_i,\,X_j\}\} = c^k_{ij}\{X_l,\,X_k\} = c^k_{ij}c^m_{lk}X_m\,.$$

Hence, the Jacobi identity is satisfied as a consequence of

$$\begin{aligned}&\{X_l,\,\{X_i,\,X_j\}\} + \{X_i,\,\{X_j,\,X_l\}\} + \{X_j,\,\{X_l,\,X_i\}\}\\ &= c^k_{ij}\{X_l,\,X_k\} + c^k_{jl}\{X_i,\,X_k\} + c^k_{li}\{X_j,\,X_k\}\\ &= \Big(c^k_{ij}c^m_{lk} + c^k_{jl}c^m_{ik} + c^k_{li}c^m_{jk}\Big)X_m = 0\,,\end{aligned}$$

This is the condition required for the Jacobi identity to hold in terms of the structure constants.

Remark. This calculation provides an independent proof of the Jacobi identity for the $\mathbb{R}^3$ bracket in the case of quadratic distinguished functions. ■

#3c Consider the Hamiltonian

$$H = aY_1 + bY_2 + cY_3 \tag{1}$$

with the linear combinations

$$Y_1 = \frac{1}{2}(X_1 + X_2)\,,\quad Y_2 = \frac{1}{2}(X_2 - X_1)\,,\quad Y_3 = X_3\,,$$

and constant values of (a, b, c). Compute the canonical dynamics generated by Hamiltonian (1) on level sets of $S^2 > 0$ and $S^2 = 0$.

This amounts to computing the intersections of the planes

$$H = aY_1 + bY_2 + cY_3 = constant$$

with the hyperboloids of revolution about the Y_1-axis,

$$S^2 = Y_1^2 - Y_2^2 - Y_3^2 = constant\,.$$

One may solve this problem graphically, algebraically by choosing special cases for the orientations of the family of planes, or most generally by restricting the equations to a level set of either $S^2 = constant$, or $H = constant$.

Restricting to $S^2 = constant$ hyperboloids of revolution.

Each of the family of hyperboloids of revolution $S^2 = constant$ comprises a layer in the "hyperbolic onion" preserved by axisymmetric ray optics. We use hyperbolic polar coordinates on these layers in analogy to spherical coordinates,

$$Y_1 = S\cosh u\,, \quad Y_2 = S\sinh u\cos\psi\,, \quad Y_3 = S\sinh u\sin\psi\,.$$

The $\mathbb{R}^3-$bracket thereby transforms into hyperbolic coordinates as

$$\{F,\, H\}\, dY_1\wedge dY_2\wedge dY_3 = -\,\{F,\, H\}_{hyperb}\, S^2\, dS\wedge d\psi\wedge d\cosh u\,.$$

Note that the oriented quantity

$$S^2 d\cosh u\wedge d\psi = -S^2 d\psi\wedge d\cosh u\,,$$

is the ***area element on the hyperboloid*** corresponding to the constant S^2.

On a constant level surface of S^2 the function $\{F, H\}_{hyperb}$ only depends on $(\cosh u, \psi)$ so the Poisson bracket for optical motion on any *particular* hyperboloid is then

$$\begin{aligned}\{F,\, H\}\, d^3Y &= -\,S^2 dS\wedge dF\wedge dH = -\,S^2 dS\wedge\{F,\, H\}_{hyperb}\, d\psi\wedge d\cosh u\\ &= -\,S^2 dS\wedge\left(\frac{\partial F}{\partial\psi}\frac{\partial H}{\partial\cosh u} - \frac{\partial H}{\partial\cosh u}\frac{\partial F}{\partial\psi}\right) d\psi\wedge d\cosh u\,.\end{aligned}$$

Being a constant of the motion, the value of S^2 may be absorbed by a choice of units for any given initial condition and the Poisson bracket for the optical motion thereby becomes ***canonical on each hyperboloid***,

$$\frac{d\psi}{dz} = \{\psi,\, H\}_{hyperb} = \frac{\partial H}{\partial \cosh u}\,, \qquad \frac{d\cosh u}{dz} = \{\cosh u,\, H\}_{hyperb} = -\,\frac{\partial H}{\partial \psi}\,.$$

In the Cartesian variables $(Y_1, Y_2, Y_3) \in \mathbb{R}^3$, one has $\cosh u = Y_1/S$ and $\psi = \tan^{-1}(Y_3/Y_2)$. The Hamiltonian $H = aY_1 + bY_2 + cY_3$ becomes

$$H = aS\cosh u + bS\sinh u\cos\psi + cS\sinh u\sin\psi\,,$$

with

$$\sinh u = \sqrt{\cosh^2 u - 1} \quad \text{and} \quad \sin\psi = \sqrt{1-\cos^2\psi}$$

so that

$$\frac{\partial \sinh u}{\partial \cosh u} = \coth u\,,$$

and

$$\begin{aligned} \frac{1}{S}\frac{d\psi}{dz} = \frac{1}{S}\frac{\partial H}{\partial \cosh u} &= a + b\coth u\cos\psi + c\coth u\sin\psi\,, \\ \frac{1}{S}\frac{d\cosh u}{dz} = \frac{1}{S}\frac{\partial H}{\partial \psi} &= -b\sinh u\sin\psi + c\sinh u\cos\psi\,. \end{aligned}$$

Restricting to the conical surface $S^2 = 0$

To restrict to the conical surface $S^2 = Y_1^2 - Y_2^2 - Y_3^2 = 0$ one chooses coordinates

$$Y_1 = Z\,, \quad Y_2 = Z\cos\psi\,, \quad Y_3 = Z\sin\psi$$

The Poisson bracket for the optical motion thereby becomes ***canonical on the cone***,

$$\frac{d\psi}{dz} = \{\psi,\, H\}_{cone} = \frac{\partial H}{\partial Z}\,, \qquad \frac{dZ}{dz} = \{Z,\, H\}_{cone} = -\,\frac{\partial H}{\partial \psi}\,.$$

and

$$
\begin{aligned}
\frac{1}{S}\frac{d\psi}{dz} = \frac{1}{S}\frac{\partial H}{\partial Z} &= a + b\cos\psi + c\sin\psi\,,\\
\frac{1}{S}\frac{dZ}{dz} = \frac{1}{S}\frac{\partial H}{\partial \psi} &= -bZ\sin\psi + cZ\cos\psi\,.
\end{aligned}
$$

The equations on the hyperboloids and the cone are a bit complicated. It turns out that a very simple solution is possible on the level sets of the Hamiltonian planes.

Restricting to $H = constant$ planes.

One may also restrict the equations to a planar level set of $H = constant$. This tactic is very insightful and particularly simple, because only linear equations arise.

The latter approach summons the linear transformation with constant coefficients,

$$
\begin{pmatrix} dY_1 \\ dY_2 \\ dY_3 \end{pmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \begin{pmatrix} dH \\ dx \\ dy \end{pmatrix}
$$

One finds the constant Jacobian from $d^3Y = (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\,dH\wedge dx\wedge dy$. Then one transforms to level sets of H by writing

$$
\begin{aligned}
\frac{dF}{dz} = \{F,\,H\}d^3Y &= -dC\wedge dF\wedge dH = dH\wedge dF\wedge dC\\
&= (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})dH\wedge\{F,\,C\}_{xy}\,dx\wedge dy
\end{aligned}
$$

Thus, on one of the planes in the family of level sets of H, one finds

$$
\begin{aligned}
\frac{dx}{dz} &= \frac{\partial C}{\partial y}\\
\frac{dy}{dz} &= -\frac{\partial C}{\partial x}
\end{aligned}
$$

Because C is quadratic and the transformation from (Y_1, Y_2, Y_3) to (H, x, y) is linear, these canonical equations on any of the planar level sets of H are linear. That is,

$$\begin{pmatrix} dx/dz \\ dy/dz \end{pmatrix} = \begin{bmatrix} \alpha & \beta \\ \bar{\alpha} & \bar{\beta} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \gamma \\ \bar{\gamma} \end{pmatrix}$$

with constants $(\alpha, \beta, \bar{\alpha}, \bar{\beta}, \gamma, \bar{\gamma})$. These are linear equations.

Direct solution in canonical variables. For the Hamiltonian

$$H = aY_1 + bY_2 + cY_3 = \frac{a-b}{2}|\mathbf{q}|^2 + \frac{a+b}{2}|\mathbf{p}|^2 + c\,\mathbf{p}\cdot\mathbf{q}$$

one has

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{\partial H}{\partial \mathbf{p}} = (a+b)\mathbf{p} + c\mathbf{q} \\ \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{q}} = (b-a)\mathbf{q} - c\mathbf{p} \end{aligned}$$

or

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} c & a+b \\ b-a & -c \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}$$

As expected, this is a linear symplectic transformation and the matrix is given by

$$\begin{pmatrix} c & a+b \\ b-a & -c \end{pmatrix} = \frac{a-b}{2} m_1 + \frac{a+b}{2} m_2 + c\, m_3\,.$$

#3d $\boxed{\textbf{\textit{Fourth year students}}}$

Derive the formula for reconstructing the angle canonically conjugate to S for the canonical dynamics generated by the planar Hamiltonian (1).

The volume elements corresponding to the Poisson brackets are

$$d^3Y =: dY_1 \wedge dY_2 \wedge dY_3 \;=\; d\frac{S^3}{3} \wedge d\cosh u \wedge d\psi$$

On a level set of $S = p_\phi$ this implies canonical variables $(\cosh u,\, \psi)$ with symplectic form,

$$d\cosh u \wedge d\psi\,,$$

and since $(S = p_\phi, \phi)$ are also canonically conjugate, one has

$$dp_j \wedge dq_j = dS \wedge d\phi + \, d\cosh u \wedge d\psi\,.$$

One recalls Stokes Theorem on phase space

$$\iint_A dp_j \wedge dq_j = \oint_{\partial A} p_j dq_j\,,$$

where the boundary of the phase space area ∂A is taken around a loop on a closed orbit. On an invariant hyperboloid S this loop integral becomes

$$\oint \mathbf{p}\cdot d\mathbf{q} := \oint p_j dq_j = \oint \Big(S d\phi + \cosh u \, d\psi \Big)\,.$$

Thus we may compute the total phase change around a closed periodic orbit on the level set of hyperboloid S from

$$\oint S d\phi \;=\; S\Delta\phi = \underbrace{-\oint \cosh u \, d\psi}_{\text{Geometric } \Delta\phi} + \underbrace{\oint \mathbf{p}\cdot d\mathbf{q}}_{\text{Dynamic } \Delta\phi} \qquad (2)$$

Evidently, one may denote the total change in phase as the sum

$$\Delta\phi = \Delta\phi_{geom} + \Delta\phi_{dyn}\,,$$

by identifying the corresponding terms in the previous formula. By Stokes theorem, one sees that the geometric phase associated with a periodic motion on a particular hyperboloid is given by the hyperbolic solid angle enclosed by the orbit. Thus, the name: ***geometric phase***.