Sur une forme nouvelle des équations de la méchanique *

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Having had the opportunity to work on the rotational motion of hollow solid bodies filled with liquid, I have been led to cast the equations of mechanics into a new form that could be interesting to know. Assume there are n degrees of freedom and let $\{x^1, ..., x^n\}$ be the variables describing the state of the system. Let T and U be the kinetic and potential energy of the system.
Consider any continuous, transitive group (that is, its action covers the entire manifold). Let $X_i(f)$ be any infinitesimal transformation of this group such that [1]

$$X_i(f) = \sum_{\mu=1}^{n} X_i(x^\mu)\frac{\partial f}{\partial x^\mu} = X_i^1\frac{\partial f}{\partial x^1} + X_i^2\frac{\partial f}{\partial x^2} + ... + X_i^n\frac{\partial f}{\partial x^n}\,.$$

Since these transformations form a group, we must have

$$X_iX_k - X_kX_i = \sum_{s=1}^{r} c_{ik}{}^{s}X_s\,.$$

Since the group is transitive we can write

$$\dot{x}^\mu(t) = \frac{dx^\mu}{dt} = \sum_{i=1}^{r}\eta^i(t)X_i(x^\mu) = \eta^1(t)X_1^\mu + \eta^2(t)X_2^\mu + ... + \eta^r(t)X_r^\mu\,,$$

in such a way that we can go from the state $(x^1, \ldots, x^n)$ of the system to a state $(x^1 + \dot{x}^1\mathrm{d}t, \ldots, x^n + \dot{x}^n\mathrm{d}t)$ by using the infinitesimal transformation of the group, $\sum_{i=1}^{r}\eta^iX_i(f)$.

T instead of being expressed as a function of the x and $\dot{x}$ can be written as a function of the η and x. If we increase the η and x by virtual displacements $\delta\eta$ and δx, respectively, there will be resulting increases in T and U

$$\delta T = \sum\frac{\delta T}{\delta\eta}\delta\eta + \sum\frac{\delta T}{\delta x}\delta x \quad \text{and} \quad \delta U = \sum\frac{\delta U}{\delta x}\delta x\,.$$

Since the group is transitive I will be able to write

$$\delta x^\mu = \xi^1X_1^\mu + \xi^2X_2^\mu + ... + \xi^rX_r^\mu\,,$$

in such a way that we can go from the state x^μ of the system to the state $x^\mu + \delta x^\mu$ by using the infinitesimal transformation of the group $\delta x^\mu = \sum_{i=1}^{r}\xi^iX_i(x^\mu)$.
I will then write [2]

$$\delta T - \delta U = \sum_{i=1}^{r}\frac{\delta T}{\delta\eta^i}\delta\eta^i + \sum_{\mu=1}^{n}\left(\frac{\delta T}{\delta x^\mu} - \frac{\delta U}{\delta x^\mu}\right)\delta x^\mu = \sum_{i=1}^{r}\frac{\delta T}{\delta\eta^i}\delta\eta^i + \sum_{i=1}^{r}\Xi_i\xi^i\,.$$

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[1] For the finite dimensional case considered here, the $\{X_i\}$ may be regarded as a set of r constant $n\times n$ matrices that act linearly on the set of states $\{x\}$. Then, for example, $X_i(x^\mu) = \sum_{\nu=1}^{n}[X_i]^\mu_\nu x^\nu$. In the case of the action of rotations in $SO(3)$ on vectors $x\in\mathbb{R}^3$, for example, this would mean $X_i(x^\mu) = \sum_{\nu=1}^{3}\varepsilon_{i\nu}{}^{\mu}x^\nu$, so that $[X_i]^\mu_\nu = \varepsilon_{i\nu}{}^{\mu}$. (Translator's note)

[2] Here Poincaré's formula reveals that $\Xi_i = \sum_{\mu,\nu=1}^{n}\frac{\delta L}{\delta x^\mu}[X_i]^\mu_\nu x^\nu$ with $L = T - U$, or equivalently $\Xi = \frac{\partial L}{\partial x}\diamond x$ in the notation of Chapter **??**. In the case of the Lie algebra action of infinitesimal rotations $\mathfrak{so}(3)$ on vectors $\mathbf{x}\in\mathbb{R}^3$, for example, this means $\Xi_i = \sum_{\mu,\nu=1}^{3}\frac{\delta L}{\delta x^\mu}\varepsilon^\mu_{i\nu}x^\nu$, or in vector notation, $\boldsymbol{\Xi} = \frac{\delta L}{\delta\mathbf{x}}\times\mathbf{x}$. (Translator's note)

Next, let the Hamilton integral be

$$J = \int (T - U)\,\mathrm{d}t\,,$$

so we will have

$$\delta J = \int \left(\sum \frac{\delta T}{\delta \eta^i} \delta \eta^i + \sum \Xi_i \xi^i \right) \mathrm{d}t\,,$$

and can easily find

$$\delta \eta^i = \frac{d\xi^i}{dt} + \sum_{s,k=1}^{r} c_{sk}{}^i\, \eta^k \xi^s\,.$$

The principle of stationary action then gives

$$\frac{d}{dt} \frac{\delta T}{\delta \eta^s} = \sum c_{sk}{}^i\, \frac{\delta T}{\delta \eta^i} \eta^k + \Xi_s\,.$$

These equations encompass some particular cases:

1. The Lagrange equations, when the group is reduced to the transformations, all commuting amongst each other, which each shift one of the variables x by an infinitesimally small constant.

2. Also, Euler's equations for solid body rotations emerge, in which the role of the η_i is played by the components p, q, r of the rotations and the role of Ξ by the coupled external forces.

These equations will be of special interest where the potential U is zero and the kinetic energy T only depends on the η in which case Ξ vanishes.