M3-4-5 A16 Notes for Geometric Mechanics: Rigid Body, mid Oct – mid Nov 2011

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Text for the course:

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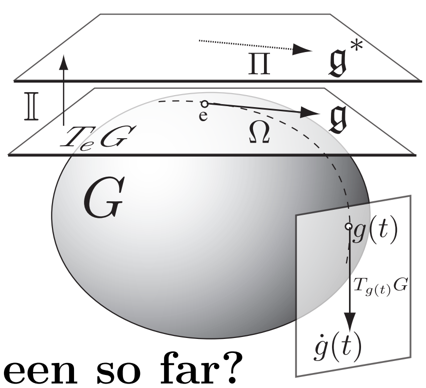
Where are we going in this course?

1. Euler–Poincaré equation
2. Rigid body
3. Spherical pendulum
4. Elastic spherical pendulum
5. Differential forms



Where have we been so far?

- Mathematical setting for geometric mechanics, first on manifolds, then on (matrix) Lie groups
  - Manifold $M \simeq_{loc} \mathbb{R}^n$ e.g., $n = 1$ (scalars), $n = m$ (m-vectors), $n = m \times m$ (matrices),

- Motion equation on TM: $\dot{q}(t) = f(q) \Longrightarrow$ transformation theory (pullbacks and all that)
- Hamilton's principle for Lagrangian $L : TM \to \mathbb{R}$ vector fields
  * Euler–Lagrange equations on T^*M
  * Hamilton's canonical equations on T^*M
  * Euler–Poincaré eqns on $T^*_eG \simeq \mathfrak{g}^*$ for reduced Lagrangian $\ell : \mathfrak{g} \to \mathbb{R}$, e.g., *rigid body.*

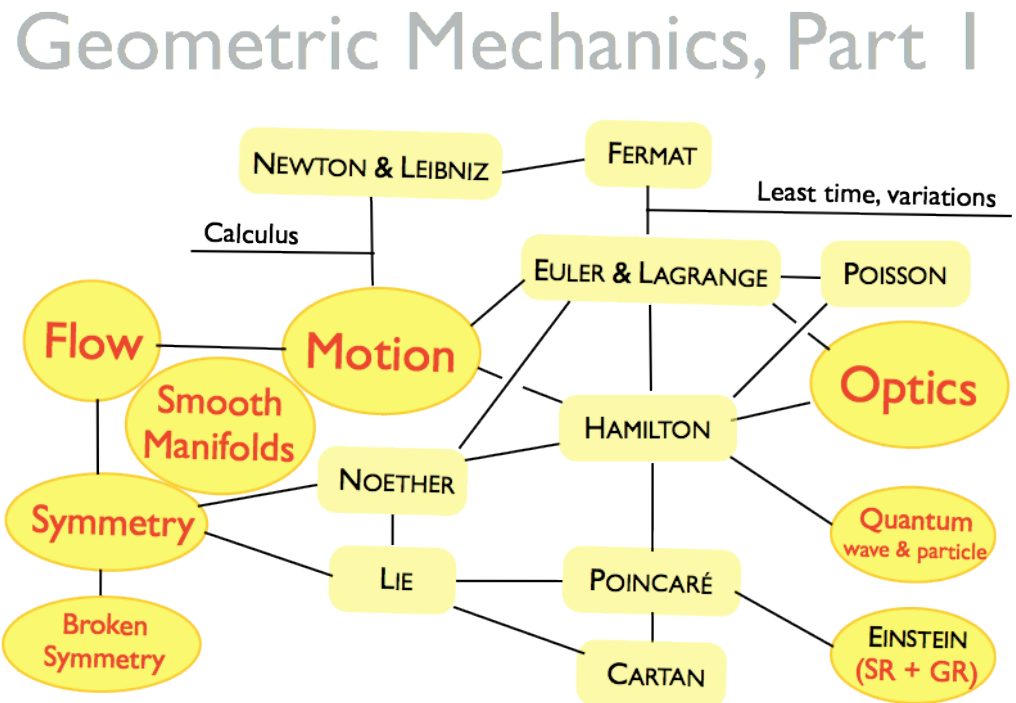


Figure 1: The fabric of geometric mechanics is woven by a network of fundamental contributions by at least a dozen people to the dual fields of optics and motion.

1 Euler–Poincaré Theorem

The definition of an invariant (or symmetric) function under a group action is as follows:

Definition 1.1. *Let G act on TG by left translation. A function $F : TG \to \mathbb{R}$ is called **left invariant** if and only if*

$$F(h(g, \dot{g})) = F(g, \dot{g}) \quad \textit{for all } (g, \dot{g}) \in TG\,,$$

where

$$h(g, \dot{g}) := (gh, T_eL_g(\dot{h}))\,.$$

If the Lagrangian is left invariant, then:

$$L(g,\dot g) = L(g^{-1}g, g^{-1}\dot g) = L(e, g^{-1}\dot g) = L(e,\xi) \quad \text{for all} \quad (g,\dot g)\in TG,$$

where $\xi := g^{-1}\dot g$. Note that in this case the Lagrangian satisfies

$$L(g,\dot g) = L(e,\xi),$$

so it is independent of g.

This equation can be re-expressed as

$$\frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}\,,$$

where l is defined to be the restriction of L to $\mathfrak{g}$:

$$l:\mathfrak{g}\to\mathbb{R}\,,\quad l(\xi) := L(e,\xi) \qquad \text{for all } g\in\xi\,.$$

The following theorem is now easily verified:

Theorem 1.1 (Euler–Poincaré reduction)**.** *Let G be a Lie group, $L : TG \to \mathbb{R}$ a **left-invariant** Lagrangian, and define the **reduced Lagrangian**,*

$$l:\mathfrak{g}\to\mathbb{R},\quad l(\xi) := L(e,\xi)\,,$$

as the restriction of L to $\mathfrak{g}$. For a curve $g(t)\in G$, let

$$\xi(t) = g(t)^{-1}\dot g(t) := T_{g(t)}L_{g(t)^{-1}}\dot g(t) \in \mathfrak{g}\,.$$

Then, the following four statements are equivalent:

(i) *The variational principle*

$$\delta\int_a^b L(g(t),\dot g(t))\mathrm{d}t = 0$$

holds, for variations among paths with fixed endpoints.

(ii) *$g(t)$ satisfies the Euler–Lagrange equations for Lagrangian L defined on G.*

(iii) *The variational principle*

$$\delta\int_a^b l(\xi(t))\mathrm{d}t = 0$$

holds on $\mathfrak{g}$, using variations of the form $\delta\xi = \dot\eta + [\xi,\eta]$, where $\eta(t)$ is an arbitrary path in $\mathfrak{g}$ that vanishes at the endpoints, i.e. $\eta(a) = 0 = \eta(b)$.

(iv) *The (left invariant) **Euler–Poincaré equations** hold:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}\,,$$

where $\langle \mathrm{ad}^*_\xi\mu,\,\eta\rangle := \langle\mu,\,\mathrm{ad}_\xi\eta\rangle$, for $\mu\in\mathfrak{g}^*$ and $\xi,\,\eta\in\mathfrak{g}$.

Remark 1.1. *A similar statement holds, with obvious changes for* ***right-invariant*** *Lagrangian systems on TG. In this case the Euler-Poincaré equations are given by:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta l}{\delta \xi} = -\,\mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}\,, \tag{1.1}$$

with the opposite sign.

Exercise. [***Components of*** $\mathrm{ad}^*_\xi \mu$]
If $\mu = \mu_i e^i$, $\xi = \xi^j e_j$ and $\eta = \eta^k e_k$, with $[e_j,\, e_k] = c^l_{jk} e_l$ and $\langle e^i,\, e_j\rangle = \delta^j_i$, show that

$$(\mathrm{ad}^*_\xi \mu)_k = \xi^j \mu_i c^i_{jk}\,.$$

★

Reconstruction

The reconstruction of the solution $g(t)$ of the Euler–Lagrange equations, with initial conditions $g(0) = g_0$ and $\dot{g}(0) = v_0$, is as follows: first, solve the initial value problem for the right invariant Euler–Poincaré equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} \quad \text{with} \quad \xi(0) = \xi_0 := g_0^{-1} v_0\,.$$

Second, using the solution $\xi(t)$ of the above, find the curve $g(t) \in G$ by solving the ***reconstruction equation***

$$\dot{g}(t) = g(t)\xi(t) \quad \text{with} \quad g(0) = g_0\,,$$

which is a differential equation with time-dependent coefficients.

Exercise. Prove the Euler–Poincaré reduction Theorem 1.1. ★

Exercise. Write out the proof of the Euler–Poincaré reduction theorem for right-invariant Lagrangians and describe the corresponding reconstruction procedure.
★

Exercise. [***Motion on*** $SO(4)$]
Write out the Euler–Poincaré equations in matrix form for a free rigid body fixed at its centre of mass in a 4-dimensional space. Use the analogue of the 'hat' map for $\mathfrak{so}(4)$ and write the $\mathbb{R}^6$ vector representation of the equations. ★

Exercise. Consider the following action of a Lie group G on a product space $G \times Y$, where Y is some manifold:

$$(g\,, (h, y)) \to (gh, y).$$

Let $L : T(G \times Y) \to \mathbb{R}$ be invariant with respect to this action. Define $l : \mathfrak{g} \times TY \to \mathbb{R}$ as the restriction of L, i.e.

$$l(\xi, y, \dot{y}) := L(e, \xi, y, \dot{y}).$$

Deduce the reduced Hamilton's principle for l and show that the equations of motion are given by

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}\,, \qquad \frac{d}{dt}\frac{\delta l}{\delta \dot y} = \frac{\delta l}{\delta y}\,.$$

★

What will we investigate about the rigid body?

1. Euler–Poincaré equation in $\mathbb{R}^3$
2. Hamilton-Pontryagin matrix form
3. Noether theorem (coadjoint motion)
4. Manakov's matrix commutator form
5. Isospectral eigenvalue problem
6. Hamiltonian forms (both Lie-Poisson and Nambu)
7. Clebsch variational form (momentum map)

2 Lagrangian Euler–Poincaré form of rigid-body motion

In the absence of external torques, Euler's equations for rigid-body motion in principal axis coordinates are

$$\begin{aligned} I_1\dot\Omega_1 &= (I_2 - I_3)\Omega_2\Omega_3,\\ I_2\dot\Omega_2 &= (I_3 - I_1)\Omega_3\Omega_1,\\ I_3\dot\Omega_3 &= (I_1 - I_2)\Omega_1\Omega_2, \end{aligned} \tag{2.1}$$

or, equivalently,

$$\mathbb{I}\dot{\boldsymbol\Omega} = \mathbb{I}\boldsymbol\Omega \times \boldsymbol\Omega\,, \tag{2.2}$$

where $\boldsymbol\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the body angular velocity vector and $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$ is the moment of inertia tensor, which is diagonal in the principal axis frame of the rigid body. The moment of inertia $\mathbb{I}$ defines the following quadratic form $\mathbb{I}\mathbf{a}\cdot\mathbf{b}$ associated to the bilinear symmetric form for $\mathbb{R}^3$ vectors $\mathbf{a}$ and $\mathbf{b}$ in the body's principal axis frame,

$$(\mathbf{a},\, \mathbf{b}) := \int_{\mathcal{B}} \rho_0(\mathbf{X})(\mathbf{a}\times\mathbf{X})\cdot(\mathbf{b}\times\mathbf{X})d^3\mathbf{X} = \mathbb{I}\mathbf{a}\cdot\mathbf{b} = a^i\mathbb{I}_{ij}b^j\,. \tag{2.3}$$

Thus, the body's distribution of mass density $\rho_0(\mathbf{X})$ induces a Riemannian metric $\mathbb{I}$ for lowering indices of vectors in the body frame. That is, $\mathbb{I} : \mathbb{R}^3 \to \mathbb{R}^{3*} \simeq \mathbb{R}^3$. By the hat map then $\mathbb{I} : \mathfrak{so}(3) \to \mathfrak{so}(3)^* \simeq \mathbb{R}^3$.

We ask whether Equations (2.1) may be expressed using Hamilton's principle on $\mathbb{R}^3$. For this, we will need to define the variational derivative of a functional $S[(\boldsymbol\Omega]$.

Definition 2.1 (Variational derivative). *The variational derivative of a functional $S[(\mathbf{\Omega}]$ is defined as its linearisation in an arbitrary direction $\delta\mathbf{\Omega}$ in the vector space of body angular velocities. That is,*

$$\delta S[\mathbf{\Omega}] := \lim_{s\to 0} \frac{S[\mathbf{\Omega} + s\delta\mathbf{\Omega}] - S[\mathbf{\Omega}]}{s} = \frac{d}{ds}\Big|_{s=0} S[\mathbf{\Omega} + s\delta\mathbf{\Omega}] =: \Big\langle \frac{\delta S}{\delta \mathbf{\Omega}}\,, \delta\mathbf{\Omega} \Big\rangle\,,$$

where the new pairing, also denoted as $\langle\,\cdot\,,\,\cdot\,\rangle$, is between the space of body angular velocities and its dual, the space of body angular momenta.

Theorem 2.1 (Euler's rigid-body equations).
Euler's rigid-body equations are equivalent to Hamilton's principle

$$\delta S(\mathbf{\Omega}) = \delta \int_a^b l(\mathbf{\Omega})\, dt = 0, \tag{2.4}$$

*in which the Lagrangian $l(\mathbf{\Omega})$ appearing in the **action integral** $S(\mathbf{\Omega}) = \int_a^b l(\mathbf{\Omega})\, dt$ is given by the kinetic energy in principal axis coordinates,*

$$l(\mathbf{\Omega}) = \frac{1}{2}(\mathbf{\Omega}, \mathbf{\Omega}) := \frac{1}{2}\,\mathbb{I}\mathbf{\Omega}\cdot\mathbf{\Omega} = \frac{1}{2}\,(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)\,, \tag{2.5}$$

and variations of $\mathbf{\Omega}$ are restricted to be of the form

$$\delta\mathbf{\Omega} = \dot{\mathbf{\Xi}} + \mathbf{\Omega}\times\mathbf{\Xi}\,, \tag{2.6}$$

where $\mathbf{\Xi}(t)$ is a curve in $\mathbb{R}^3$ that vanishes at the endpoints in time.

Proof. Since $l(\mathbf{\Omega}) = \frac{1}{2}\langle \mathbb{I}\mathbf{\Omega}, \mathbf{\Omega}\rangle$, and $\mathbb{I}$ is symmetric, one obtains

$$\begin{aligned}
\delta \int_a^b l(\mathbf{\Omega})\, dt &= \int_a^b \Big\langle \mathbb{I}\mathbf{\Omega}, \delta\mathbf{\Omega}\Big\rangle\, dt \\
&= \int_a^b \Big\langle \mathbb{I}\mathbf{\Omega}, \dot{\mathbf{\Xi}} + \mathbf{\Omega}\times\mathbf{\Xi}\Big\rangle\, dt \\
&= \int_a^b \Big[\Big\langle -\frac{d}{dt}\mathbb{I}\mathbf{\Omega}, \mathbf{\Xi}\Big\rangle + \Big\langle \mathbb{I}\mathbf{\Omega}, \mathbf{\Omega}\times\mathbf{\Xi}\Big\rangle\Big]\, dt \\
&= \int_a^b \Big\langle -\frac{d}{dt}\mathbb{I}\mathbf{\Omega} + \mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega},\, \mathbf{\Xi}\Big\rangle\, dt + \Big\langle \mathbb{I}\mathbf{\Omega},\, \mathbf{\Xi}\Big\rangle\Big|_{t_a}^{t_b},
\end{aligned}$$

upon integrating by parts. The last term vanishes, because of the endpoint conditions,

$$\mathbf{\Xi}(a) = 0 = \mathbf{\Xi}(b)\,.$$

Since $\mathbf{\Xi}$ is otherwise arbitrary, (2.4) is equivalent to

$$-\frac{d}{dt}(\mathbb{I}\mathbf{\Omega}) + \mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega} = 0,$$

which recovers Euler's Equations (2.1) in vector form. □

Proposition 2.1 (Derivation of the restricted variation).
The restricted variation in (2.6) arises via the following steps:

(i) Vary the definition of the body angular velocity, $\widehat{\Omega} = O^{-1}\dot{O}$.

(ii) Take the time derivative of the variation, $\widehat{\Xi} = O^{-1}O\,'$.

(iii) Use the equality of cross derivatives, $O^{\cdot\,\prime} = d^2O/dtds = O\,'^{\,\cdot}$.

(iv) Apply the hat map.

Proof. One computes directly that

$$\begin{aligned}
\widehat{\Omega}\,' &= (O^{-1}\dot{O})\,' = -\,O^{-1}O\,'O^{-1}\dot{O} + O^{-1}O^{\cdot\,\prime} = -\,\widehat{\Xi}\,\widehat{\Omega} + O^{-1}O^{\cdot\,\prime}\,,\\
\widehat{\Xi}^{\,\cdot} &= (O^{-1}O\,')^{\cdot} = -\,O^{-1}\dot{O}O^{-1}O\,' + O^{-1}O\,'^{\,\cdot} = -\,\widehat{\Omega}\,\widehat{\Xi} + O^{-1}O\,'^{\,\cdot}\,.
\end{aligned}$$

On taking the difference, the cross derivatives cancel and one finds a variational formula equivalent to (2.6),

$$\widehat{\Omega}\,' - \widehat{\Xi}^{\,\cdot} = \big[\,\widehat{\Omega}\,,\,\widehat{\Xi}\,\big] \quad \text{with} \quad \big[\,\widehat{\Omega}\,,\,\widehat{\Xi}\,\big] := \widehat{\Omega}\,\widehat{\Xi} - \,\widehat{\Xi}\,\widehat{\Omega}\,. \tag{2.7}$$

Under the bracket relation

$$\big[\,\widehat{\Omega}\,,\,\widehat{\Xi}\,\big] = (\boldsymbol{\Omega}\times\boldsymbol{\Xi})\,\widehat{}$$

for the hat map, this equation recovers the vector relation (2.6) in the form

$$\boldsymbol{\Omega}\,' - \dot{\boldsymbol{\Xi}} = \boldsymbol{\Omega}\times\boldsymbol{\Xi}\,. \tag{2.8}$$

Thus, Euler's equations for the rigid body in $T\mathbb{R}^3$,

$$\mathbb{I}\dot{\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega}\times\boldsymbol{\Omega}\,, \tag{2.9}$$

do follow from the variational principle (2.4) with variations of the form (2.6) derived from the definition of body angular velocity $\widehat{\Omega}$. $\square$

Remark 2.1. *The body angular velocity is expressed in terms of the spatial angular velocity by* $\boldsymbol{\Omega}(t) = O^{-1}(t)\boldsymbol{\omega}(t)$. *Consequently, the kinetic energy Lagrangian in (2.5) transforms as*

$$l(\boldsymbol{\Omega}) = \frac{1}{2}\,\boldsymbol{\Omega}\cdot\mathbb{I}\boldsymbol{\Omega} = \frac{1}{2}\,\boldsymbol{\omega}\cdot\mathbb{I}_{space}(t)\boldsymbol{\omega} =: l_{space}(\boldsymbol{\omega})\,,$$

where

$$\mathbb{I}_{space}(t) = O(t)\mathbb{I}O^{-1}(t)\,.$$

Exercise. Show that Hamilton's principle for the action

$$S(\boldsymbol{\omega}) = \int_a^b l_{space}(\boldsymbol{\omega})\,dt$$

yields conservation of spatial angular momentum

$$\boldsymbol{\pi} = \mathbb{I}_{space}(t)\boldsymbol{\omega}(t)\,.$$

Hint: First derive the formula $\delta\mathbb{I}_{space} = [\xi, \mathbb{I}_{space}]$ with right-invariant $\xi = \delta O O^{-1}$. ★

Exercise. (Noether's theorem for the rigid body) What conservation law does Noether's theorem imply for the rigid-body Equations (2.2)?

Hint: Transform the endpoint terms arising on integrating the variation δS by parts in the proof of Theorem 2.1 into the spatial representation by setting $\boldsymbol{\Xi} = O^{-1}(t)\boldsymbol{\Gamma}$ and $\boldsymbol{\Omega} = O^{-1}(t)\boldsymbol{\omega}$. ★

Remark 2.2 (Reconstruction of $O(t) \in SO(3)$)**.**
The Euler solution is expressed in terms of the time-dependent angular velocity vector in the body, $\boldsymbol{\Omega}$. The body angular velocity vector $\boldsymbol{\Omega}(t)$ yields the tangent vector $\dot{O}(t) \in T_{O(t)}SO(3)$ along the integral curve in the rotation group $O(t) \in SO(3)$ by the relation

$$\dot{O}(t) = O(t)\widehat{\Omega}(t)\,, \tag{2.10}$$

where the left-invariant skew-symmetric 3×3 matrix $\widehat{\Omega}$ is defined by the hat map

$$(O^{-1}\dot{O})_{jk} = \widehat{\Omega}_{jk} = -\,\Omega_i \epsilon_{ijk}\,. \tag{2.11}$$

*Equation (2.10) is the **reconstruction formula** for $O(t) \in SO(3)$.*

Once the time dependence of $\boldsymbol{\Omega}(t)$ and hence $\widehat{\Omega}(t)$ is determined from the Euler equations, solving formula (2.10) as a linear differential equation with time-dependent coefficients yields the integral curve $O(t) \in SO(3)$ for the orientation of the rigid body.

2.1 Hamilton–Pontryagin constrained variations

Formula (2.7) for the variation $\widehat{\Omega}$ of the skew-symmetric matrix

$$\widehat{\Omega} = O^{-1}\dot{O}$$

may be imposed as a constraint in Hamilton's principle and thereby provide a variational derivation of Euler's Equations (2.1) for rigid-body motion in principal axis coordinates. This constraint is incorporated into the matrix Euler equations, as follows.

Proposition 2.2 (Matrix Euler equations)**.** *Euler's rigid-body equation may be written in matrix form as*

$$\frac{d\Pi}{dt} = -\left[\,\widehat{\Omega}\,,\,\Pi\,\right] \quad \textit{with} \quad \Pi = \mathbb{I}\widehat{\Omega} = \frac{\delta l}{\delta\widehat{\Omega}}\,, \tag{2.12}$$

for the Lagrangian $l(\widehat{\Omega})$ given by

$$l = \frac{1}{2}\Big\langle\,\mathbb{I}\widehat{\Omega}\,,\,\widehat{\Omega}\,\Big\rangle\,. \tag{2.13}$$

Here, the bracket

$$\big[\,\widehat{\Omega}\,,\,\Pi\big] := \widehat{\Omega}\Pi - \Pi\widehat{\Omega} \tag{2.14}$$

*denotes the commutator and $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the **trace pairing**, e.g.,*

$$\Big\langle\,\Pi\,,\,\widehat{\Omega}\,\Big\rangle =: \frac{1}{2}\,\mathrm{trace}\left(\Pi^T\,\widehat{\Omega}\right). \tag{2.15}$$

Remark 2.3. *Note that the symmetric part of* Π *does not contribute in the pairing and if set equal to zero initially, it will remain zero.*

Proposition 2.3 (Constrained variational principle)**.**
The matrix Euler Equations (2.12) are equivalent to stationarity $\delta S = 0$ *of the following* ***constrained action****:*

$$
\begin{aligned}
S(\widehat{\Omega}, O, \dot{O}, \Pi) &= \int_a^b l(\widehat{\Omega}, O, \dot{O}, \Pi)\, dt \\
&= \int_a^b \Big[l(\widehat{\Omega}) + \big\langle\, \Pi\,,\, (O^{-1}\dot{O} - \widehat{\Omega}\,)\,\big\rangle \Big]\, dt\,.
\end{aligned}
\tag{2.16}
$$

Remark 2.4. *The integrand of the constrained action in (2.16) is similar to the formula for the Legendre transform, but its functional dependence is different. This variational approach is related to the classic* ***Hamilton–Pontryagin principle*** *for control theory. It has also be used to develop algorithms for geometric numerical integrations of rotating motion.*

Proof. The variations of S in formula (2.16) are given by

$$
\begin{aligned}
\delta S &= \int_a^b \bigg\{ \Big\langle \frac{\partial l}{\partial \widehat{\Omega}} - \Pi\,,\, \delta\widehat{\Omega} \Big\rangle \\
&\qquad + \Big\langle\, \delta\Pi\,,\, (O^{-1}\dot{O} - \Omega)\,\Big\rangle + \Big\langle\, \Pi\,,\, \delta(O^{-1}\dot{O})\,\Big\rangle \bigg\}\, dt\,,
\end{aligned}
$$

where

$$
\delta(O^{-1}\dot{O}) = \widehat{\Xi}^{\,\cdot} + \big[\,\widehat{\Omega}\,,\, \widehat{\Xi}\,\big]\,,
\tag{2.17}
$$

and $\widehat{\Xi} = (O^{-1}\delta O)$ from Equation (2.7).

Substituting for $\delta(O^{-1}\dot{O})$ into the last term of δS produces

$$
\begin{aligned}
\int_a^b \Big\langle\, \Pi\,,\, \delta(O^{-1}\dot{O})\,\Big\rangle\, dt &= \int_a^b \Big\langle\, \Pi\,,\, \widehat{\Xi}^{\,\cdot} + [\,\widehat{\Omega}\,,\, \widehat{\Xi}\,]\,\Big\rangle\, dt \\
&= \int_a^b \Big\langle\, -\,\Pi^{\,\cdot} - [\,\widehat{\Omega}\,,\, \Pi\,]\,,\, \widehat{\Xi}\,\Big\rangle\, dt \\
&\qquad + \Big\langle\, \Pi\,,\, \widehat{\Xi}\,\Big\rangle\Big|_a^b\,,
\end{aligned}
\tag{2.18}
$$

where one uses the cyclic properties of the trace operation for matrices,

$$
\mathrm{trace}\Big(\Pi^T\,\widehat{\Xi}\,\widehat{\Omega}\Big) = \mathrm{trace}\Big(\widehat{\Omega}\,\Pi^T\,\widehat{\Xi}\Big)\,.
\tag{2.19}
$$

Thus, stationarity of the Hamilton–Pontryagin variational principle for vanishing endpoint conditions $\widehat{\Xi}(a) = 0 = \widehat{\Xi}(b)$ implies the following set of equations:

$$
\frac{\partial l}{\partial \widehat{\Omega}} = \Pi\,, \quad O^{-1}\dot{O} = \widehat{\Omega}\,, \quad \Pi^{\,\cdot} = -[\,\widehat{\Omega}\,,\, \Pi\,]\,.
\tag{2.20}
$$

These are the Euler rigid body equations in matrix form on $SO(n)$. $\square$

Remark 2.5 (Interpreting the formulas in (2.20))**.**
The first formula in (2.20) defines the angular momentum matrix Π *as the* ***fibre derivative*** *of the Lagrangian with respect to the angular velocity matrix* $\widehat{\Omega}$. *The second formula is the reconstruction formula (2.10) for the solution curve* $O(t) \in SO(3)$*, given the solution* $\widehat{\Omega}(t) = O^{-1}\dot{O}$. *And the third formula is Euler's equation for rigid-body motion in matrix form.*

Exercise. Use the fibre derivative relation to compute the Hamiltonian $h(\Pi)$ via the Legendre transform,

$$h(\Pi) = \langle \Pi, \widehat{\Omega} \rangle - l(\widehat{\Omega}) \tag{2.21}$$

then express the matrix Euler rigid body equations in Hamiltonian form as a Poisson bracket relation. ★

Answer. The Hamiltonian $h(\Pi)$ satisfies

$$dh(\Pi) = \left\langle d\Pi, \frac{\partial h}{\partial \Pi} \right\rangle = \left\langle d\Pi, \widehat{\Omega} \right\rangle - \left\langle \Pi - \frac{\partial l}{\partial \widehat{\Omega}}, d\widehat{\Omega} \right\rangle$$

so that

$$\Pi = \frac{\partial l}{\partial \widehat{\Omega}}, \quad \frac{\partial h}{\partial \Pi} = \widehat{\Omega}$$

The matrix Euler rigid body equations (2.20) are then expressed as

$$\frac{d\Pi}{dt} = -\left[\frac{\partial h}{\partial \Pi}, \Pi \right] \tag{2.22}$$

and a function $f(\Pi)$ has time derivative

$$\begin{aligned} \frac{d}{dt} f(\Pi) &= -\left\langle \frac{\partial f}{\partial \Pi}, \left[\frac{\partial h}{\partial \Pi}, \Pi \right] \right\rangle \\ &= -\left\langle \Pi, \left[\frac{\partial f}{\partial \Pi}, \frac{\partial h}{\partial \Pi} \right] \right\rangle =: \Big\{ f, h \Big\}(\Pi) . \end{aligned} \tag{2.23}$$

The last expression defines the ***Lie-Poisson bracket***, which inherits the Jacobi property from the matrix commutator. ▲

Exercise. Use equation (2.21) to rewrite the Hamilton–Pontryagin variational principle (2.16) as $\delta S = 0$ for the action

$$S(O^{-1}\dot{O}, \Pi) = \int_a^b \Big(\big\langle \Pi , O^{-1}\dot{O} \big\rangle - h(\Pi) \Big) \, dt . \tag{2.24}$$

Take the variations using (2.17) and recover the Hamiltonian form of the matrix Euler rigid body equations (2.20). How does this compare with the results for $\delta S = 0$ with $S = \int_a^b \langle p, \dot{q} \rangle - H(q,p)\, dt$? ★

Exercise. Write the Lie-Poisson bracket in (2.23) in three dimensions for $\mathfrak{so}(3)^*$ in $\mathbb{R}^3$ vector form by using the hat map. Thereby, discover the Nambu bracket form of the rigid body equations. ★

Answer. In $\mathbb{R}^3$ vector form the Lie-Poisson bracket in (2.23) becomes

$$\frac{d}{dt}f(\mathbf{\Pi}) = -\frac{\partial f}{\partial \mathbf{\Pi}} \cdot \frac{\partial h}{\partial \mathbf{\Pi}} \times \mathbf{\Pi} = -\mathbf{\Pi} \cdot \frac{\partial f}{\partial \mathbf{\Pi}} \times \frac{\partial h}{\partial \mathbf{\Pi}} =: \Big\{f,\, h\Big\}(\mathbf{\Pi})\,. \tag{2.25}$$

Euler's equations are recovered by setting $f(\mathbf{\Pi}) = \mathbf{\Pi}$

$$\frac{d\mathbf{\Pi}}{dt} = -\frac{\partial h}{\partial \mathbf{\Pi}} \times \mathbf{\Pi} =: \Big\{\mathbf{\Pi},\, h\Big\}\,. \tag{2.26}$$

If we write $c(\mathbf{\Pi}) = \frac{1}{2}\|\mathbf{\Pi}\|^2$, then the Lie-Poisson bracket in (2.25) may be expressed in ***Nambu bracket*** form,

$$\frac{d}{dt}f(\mathbf{\Pi}) = -\frac{\partial c}{\partial \mathbf{\Pi}} \cdot \frac{\partial f}{\partial \mathbf{\Pi}} \times \frac{\partial h}{\partial \mathbf{\Pi}} =: \Big\{c,\, f,\, h\Big\}(\mathbf{\Pi})\,, \tag{2.27}$$

which is the triple scalar product of gradients in $\mathbf{\Pi}$.[1] ▲

Remark 2.6 (Interpreting the endpoint terms in (2.18))**.**
We transform the endpoint terms in (2.18), arising on integrating the variation δS by parts in the proof of Theorem 2.1 into the spatial representation by setting $\widehat{\Xi}(t) =: O(t)\,\widehat{\xi}\,O^{-1}(t)$ and $\widehat{\Pi}(t) =: O(t)\widehat{\pi}(t)O^{-1}(t)$, as follows:

$$\Big\langle\, \Pi\,,\, \widehat{\Xi}\,\Big\rangle = \mathrm{trace}\Big(\Pi^T\,\widehat{\Xi}\Big) = \mathrm{trace}\Big(\pi^T\,\widehat{\xi}\Big) = \Big\langle\, \pi\,,\, \widehat{\xi}\,\Big\rangle\,. \tag{2.28}$$

Thus, the vanishing of both endpoints for a constant *infinitesimal spatial rotation $\widehat{\xi} = (\delta O O^{-1}) = const$ implies*

$$\pi(a) = \pi(b)\,. \tag{2.29}$$

*This is **Noether's theorem** for the rigid body.*

Theorem 2.2 (Noether's theorem for the rigid body)**.**
Invariance of the constrained Hamilton–Pontryagin action under spatial rotations implies conservation of spatial angular momentum,

$$\pi = O^{-1}(t)\Pi(t)O(t) =: \mathrm{Ad}^*_{O^{-1}(t)}\Pi(t). \tag{2.30}$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\Big\langle\, \pi\,,\, \widehat{\xi}\,\Big\rangle &= \frac{d}{dt}\Big\langle\, O^{-1}\Pi O\,,\, \widehat{\xi}\,\Big\rangle = \frac{d}{dt}\,\mathrm{trace}\Big(\Pi^T\, O^{-1}\widehat{\xi}O\Big) \\
&= \Big\langle\, \frac{d}{dt}\Pi + \big[\,\widehat{\Omega}\,,\, \Pi\,\big]\,,\, O^{-1}\widehat{\xi}O\,\Big\rangle = 0 \\
&=: \Big\langle\, \frac{d}{dt}\Pi - \mathrm{ad}^*_{\widehat{\Omega}}\Pi\,,\, \mathrm{Ad}_{O^{-1}}\widehat{\xi}\,\Big\rangle, \\
\frac{d}{dt}\Big\langle\, \mathrm{Ad}^*_{O^{-1}}\Pi\,,\, \widehat{\xi}\,\Big\rangle &= \Big\langle\, \mathrm{Ad}^*_{O^{-1}}\Big(\frac{d}{dt}\Pi - \mathrm{ad}^*_{\widehat{\Omega}}\Pi\Big)\,,\, \widehat{\xi}\,\Big\rangle.
\end{aligned} \tag{2.31}$$

[1] The Lie-Poisson and Nambu brackets introduced by discovery in these two exercises will be discussed further below.

The proof of Noether's theorem for the rigid body is already on the second line. However, the last line gives a general result. □

Remark 2.7. *The proof of Noether's theorem for the rigid body when the constrained Hamilton–Pontryagin action is invariant under spatial rotations also proves a general result in Equation (2.31), with $\widehat{\Omega} = O^{-1}\dot{O}$ for a Lie group O, that*

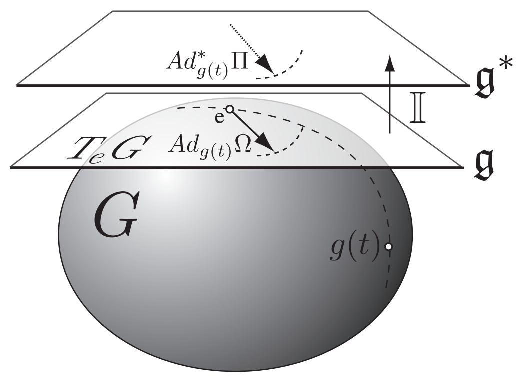
$$\frac{d}{dt}\Big(\mathrm{Ad}^*_{O^{-1}}\Pi \Big) = \mathrm{Ad}^*_{O^{-1}}\Big(\frac{d}{dt}\Pi - \mathrm{ad}^*_{\widehat{\Omega}}\Pi \Big). \tag{2.32}$$

This equation will be useful in the remainder of the text. In particular, it provides the solution of a differential equation defined on the dual of a Lie algebra. Namely, for a Lie group O with Lie algebra $\mathfrak{o}$, the equation for $\Pi \in \mathfrak{o}^*$ and $\widehat{\Omega} = O^{-1}\dot{O} \in \mathfrak{o}$

$$\frac{d}{dt}\Pi - \mathrm{ad}^*_{\widehat{\Omega}}\Pi = 0 \quad \textit{has solution} \quad \Pi(t) = \mathrm{Ad}^*_{O(t)}\pi\,, \tag{2.33}$$

in which the constant $\pi \in \mathfrak{o}^*$ is obtained from the initial conditions.

$$\begin{tikzcd}
T^*G \arrow[r, "\Phi_{g(t)}"] \arrow[d, "J(0)"'] \arrow[dr, phantom, "\text{Equivariant Momentum Map}"] & T^*G \arrow[d, "J(t)"] \\
\mathfrak{g}^* \arrow[r, "\mathrm{Ad}^*_{g(t)^{-1}}"] & \mathfrak{g}^* \simeq T^*G/G
\end{tikzcd}$$



2.2 Manakov's formulation of the $SO(n)$ rigid body

Proposition 2.4 (Manakov [Man1976])**.** *Euler's equations for a rigid body on $SO(n)$ take the matrix commutator form,*

$$\frac{dM}{dt} = \big[\, M\,,\, \Omega \,\big] \quad \textit{with} \quad M = \mathbb{A}\Omega + \Omega\mathbb{A}\,, \tag{2.34}$$

where the $n \times n$ matrices M, Ω are skew-symmetric (forgoing superfluous hats) and $\mathbb{A}$ is symmetric.

Proof. Manakov's commutator form of the $SO(n)$ rigid-body Equations (2.34) follows as the Euler–Lagrange equations for Hamilton's principle $\delta S = 0$ with $S = \int l\,dt$ for the Lagrangian

$$l = -\frac{1}{2}\mathrm{tr}(\Omega\mathbb{A}\Omega)\,,$$

where $\Omega = O^{-1}\dot{O} \in so(n)$ and the $n \times n$ matrix $\mathbb{A}$ is symmetric. Taking matrix variations in Hamilton's principle yields

$$\delta S = -\frac{1}{2}\int_a^b \mathrm{tr}\big(\delta\Omega\,(\mathbb{A}\Omega + \Omega\mathbb{A})\big)\,dt = -\frac{1}{2}\int_a^b \mathrm{tr}\big(\delta\Omega\, M\big)\,dt\,,$$

after cyclically permuting the order of matrix multiplication under the trace and substituting $M := \mathbb{A}\Omega + \Omega\mathbb{A}$. Using the variational formula (2.17) for $\delta\Omega$ now leads to

$$\delta S \;=\; -\frac{1}{2}\int_a^b \mathrm{tr}\big((\Xi^{\cdot} + \Omega\Xi - \Xi\Omega)M\big)\,dt\,.$$

Integrating by parts and permuting under the trace then yields the equation

$$\delta S \;=\; \frac{1}{2}\int_a^b \mathrm{tr}\big(\Xi\,(\,\dot{M} + \Omega M - M\Omega\,)\big)\,dt\,.$$

Finally, invoking stationarity for arbitrary Ξ implies the commutator form (2.34). □

2.3 Matrix Euler–Poincaré equations

Manakov's commutator form of the rigid-body equations recalls much earlier work by Poincaré [Po1901], who also noticed that the matrix commutator form of Euler's rigid-body equations suggests an additional mathematical structure going back to Lie's theory of groups of transformations depending continuously on parameters. In particular, Poincaré [Po1901] remarked that the commutator form of Euler's rigid-body equations would make sense for any Lie algebra, not just for $so(3)$. The proof of Manakov's commutator form (2.34) by Hamilton's principle is essentially the same as Poincaré's proof in [Po1901], which is translated into English and discussed thoroughly in [JKLOR2011].

Theorem 2.3 (Matrix Euler–Poincaré equations)**.**
The Euler–Lagrange equations for Hamilton's principle $\delta S = 0$ with $S = \int l(\Omega)\,dt$ may be expressed in matrix commutator form,

$$\frac{dM}{dt} = \big[\,M\,,\,\Omega\,\big] \quad \textit{with} \quad M = \frac{\delta l}{\delta\Omega}\,, \tag{2.35}$$

for any Lagrangian $l(\Omega)$, where $\Omega = g^{-1}\dot{g} \in \mathfrak{g}$ and $\mathfrak{g}$ is the matrix Lie algebra of any matrix Lie group G.

Proof. The proof here is the same as the proof of Manakov's commutator formula via Hamilton's principle, modulo replacing $O^{-1}\dot{O} \in so(n)$ with $g^{-1}\dot{g} \in \mathfrak{g}$. □

Remark 2.8. *Poincaré's observation leading to the matrix Euler–Poincaré Equation (2.35) was reported in two pages with no references [Po1901]. The proof above shows that the matrix Euler–Poincaré equations possess a natural variational principle. Note that if $\Omega = g^{-1}\dot{g} \in \mathfrak{g}$, then $M = \delta l/\delta\Omega \in \mathfrak{g}^*$, where the dual is defined in terms of the matrix trace pairing.*

Exercise. Retrace the proof of the variational principle for the Euler–Poincaré equation, replacing the left-invariant quantity $g^{-1}\dot{g}$ with the right-invariant quantity $\dot{g}g^{-1}$. ★

2.4 An isospectral eigenvalue problem for the $SO(n)$ rigid body

The solution of the $SO(n)$ rigid-body dynamics

$$\frac{dM}{dt} = [\,M\,,\,\Omega\,] \quad \text{with} \quad M = \mathbb{A}\Omega + \Omega\mathbb{A}\,, \tag{2.36}$$

for the evolution of the $n \times n$ skew-symmetric matrices M, Ω, with constant symmetric $\mathbb{A}$, is given by a similarity transformation (later to be identified as coadjoint motion),

$$M(t) = O(t)^{-1}M(0)O(t) =: \mathrm{Ad}^*_{O(t)}M(0)\,,$$

with $O(t) \in SO(n)$ and $\Omega := O^{-1}\dot{O}(t)$. Consequently, the evolution of $M(t)$ is ***isospectral***. This means that

- The initial eigenvalues of the matrix $M(0)$ are preserved by the motion; that is, $d\lambda/dt = 0$ in
$$M(t)\psi(t) = \lambda\psi(t)\,,$$
provided its eigenvectors $\psi \in \mathbb{R}^n$ evolve according to
$$\psi(t) = O(t)^{-1}\psi(0)\,.$$
The proof of this statement follows from the corresponding property of similarity transformations.

- Its matrix invariants are preserved:
$$\frac{d}{dt}\mathrm{tr}(M - \lambda\mathrm{Id})^K = 0\,,$$
for every non-negative integer power K.

  This is clear because the invariants of the matrix M may be expressed in terms of its eigenvalues; but these are invariant under a similarity transformation.

Proposition 2.5. *Isospectrality allows the quadratic rigid-body dynamics (2.36) on $SO(n)$ to be rephrased as a system of two* coupled linear equations*: the eigenvalue problem for M and an evolution equation for its eigenvectors ψ, as follows:*

$$M\psi = \lambda\psi \quad \textit{and} \quad \dot{\psi} = -\,\Omega\psi\,, \quad \textit{with} \quad \Omega = O^{-1}\dot{O}(t)\,.$$

Proof. Applying isospectrality in the time derivative of the first equation yields

$$(\,\dot{M} + [\,\Omega, M\,]\,)\psi + (M - \lambda\mathrm{Id})(\dot{\psi} + \Omega\psi) = 0\,.$$

Now substitute the second equation to recover the $SO(n)$ rigid-body dynamics (2.36). □

3 Hamiltonian form of rigid-body motion

The ***Legendre transform*** of the Lagrangian (2.5) in the variational principle (2.4) for Euler's rigid-body dynamics (2.9) on $\mathbb{R}^3$ will reveal its well-known Hamiltonian formulation.

Definition 3.1 (Legendre transformation).
The Legendre transformation $\mathbb{F}l : \mathbb{R}^3 \to \mathbb{R}^{3*} \simeq \mathbb{R}^3$ is defined by the ***fibre derivative****,*

$$\mathbb{F}l(\Omega) = \frac{\delta l}{\delta \Omega} = \Pi\,.$$

The Legendre transformation defines the ***body angular momentum*** by the variations of the rigid body's reduced Lagrangian with respect to the body angular velocity. For the Lagrangian in (2.4), the $\mathbb{R}^3$ components of the body angular momentum are

$$\Pi_i = I_i \Omega_i = \frac{\partial l}{\partial \Omega_i}\,, \quad i = 1, 2, 3. \tag{3.1}$$

3.1 Hamiltonian form and Poisson bracket

Definition 3.2 (Dynamical systems in Hamiltonian form).
A dynamical system on a manifold M

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x})\,, \quad \mathbf{x} \in M\,,$$

is said to be in ***Hamiltonian form****, if it can be expressed as*

$$\dot{\mathbf{x}}(t) = \{\mathbf{x}, H\}\,, \quad \textit{for} \quad H : M \mapsto \mathbb{R}\,,$$

in terms of a Poisson bracket operation $\{\cdot\,,\,\cdot\}$ among smooth real functions $\mathcal{F}(M) :\, M \mapsto \mathbb{R}$ on the manifold M,

$$\{\cdot\,,\,\cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \mapsto \mathcal{F}(M)\,,$$

so that $\dot{F} = \{F\,,\,H\}$ for any $F \in \mathcal{F}(M)$.

Definition 3.3 (Poisson bracket).
A ***Poisson bracket operation*** *$\{\cdot\,,\,\cdot\}$ is defined as possessing the following properties:*

- *It is* ***bilinear****.*
- *It is* ***skew-symmetric****, $\{F\,,\,H\} = -\,\{H\,,\,F\}$.*
- *It satisfies the* ***Leibniz rule*** *(product rule),*

  $$\{FG\,,\,H\} = \{F\,,\,H\}G + F\{G\,,\,H\}\,,$$

  for the product of any two functions F and G on M.

- *It satisfies the **Jacobi identity**,*

$$\{F\,,\,\{G\,,\,H\}\}+\{G\,,\,\{H\,,\,F\}\}+\{H\,,\,\{F\,,\,G\}\}=0\,, \tag{3.2}$$

for any three functions F, G and H on M.

Remark 3.1. *This definition of a Poisson bracket does not require it to be the standard canonical bracket in position q and conjugate momentum p, although it does include that case as well.*

3.2 Lie–Poisson Hamiltonian rigid-body dynamics

Let

$$h(\mathbf{\Pi}) := \mathbf{\Pi}\cdot\mathbf{\Omega} - l(\mathbf{\Omega})\,, \tag{3.3}$$

in terms of the vector dot product on $\mathbb{R}^3$. Hence, one finds the expected expression for the rigid-body Hamiltonian

$$h = \frac{1}{2}\,\mathbf{\Pi}\cdot\mathbb{I}^{-1}\mathbf{\Pi} := \frac{\Pi_1^2}{2I_1}+\frac{\Pi_2^2}{2I_2}+\frac{\Pi_3^2}{2I_3}\,. \tag{3.4}$$

The Legendre transform $\mathbb{F}l$ for this case is a diffeomorphism, so one may solve for the body angular velocity as the derivative of the reduced Hamiltonian with respect to the body angular momentum, namely,

$$\frac{\partial h}{\partial\mathbf{\Pi}} = \mathbb{I}^{-1}\mathbf{\Pi} = \mathbf{\Omega}\,. \tag{3.5}$$

Hence, the reduced Euler–Lagrange equations for l may be expressed equivalently in angular momentum vector components in $\mathbb{R}^3$ and Hamiltonian h as

$$\frac{d}{dt}(\mathbb{I}\mathbf{\Omega}) = \mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega} \Longleftrightarrow \dot{\mathbf{\Pi}} = \mathbf{\Pi}\times\frac{\partial h}{\partial\mathbf{\Pi}} := \{\mathbf{\Pi}, h\}\,.$$

This expression suggests we introduce the following ***rigid-body Poisson bracket*** on functions of the $\mathbf{\Pi}$'s:

$$\{f,h\}(\mathbf{\Pi}) := -\,\mathbf{\Pi}\cdot\left(\frac{\partial f}{\partial\mathbf{\Pi}}\times\frac{\partial h}{\partial\mathbf{\Pi}}\right). \tag{3.6}$$

For the Hamiltonian (3.4), one checks that the Euler equations in terms of the rigid-body angular momenta,

$$\begin{aligned}
\dot{\Pi}_1 &= \left(\frac{1}{I_3}-\frac{1}{I_2}\right)\Pi_2\Pi_3\,,\\
\dot{\Pi}_2 &= \left(\frac{1}{I_1}-\frac{1}{I_3}\right)\Pi_3\Pi_1\,,\\
\dot{\Pi}_3 &= \left(\frac{1}{I_2}-\frac{1}{I_1}\right)\Pi_1\Pi_2\,,
\end{aligned} \tag{3.7}$$

that is, the equations

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbb{I}^{-1}\mathbf{\Pi}\,, \tag{3.8}$$

are equivalent to

$$\dot{f} = \{f, h\}\,, \quad \text{with} \quad f = \mathbf{\Pi}\,.$$

3.3 Lie–Poisson bracket

The Poisson bracket proposed in (3.6) is an example of a ***Lie–Poisson bracket***.

It satisfies the defining relations of a Poisson bracket for a number of reasons, not least because it is the hat map to $\mathbb{R}^3$ of the following bracket defined by the general form in Equation (2.31) in terms of the $so(3)^* \times so(3)$ pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ in Equation (2.18). Namely,

$$\begin{aligned} \frac{dF}{dt} &= \left\langle \frac{d}{dt}\Pi,\, \frac{\partial F}{\partial \Pi} \right\rangle = \left\langle \mathrm{ad}^*_{\widehat{\Omega}}\Pi,\, \frac{\partial F}{\partial \Pi} \right\rangle \\ &= \left\langle \Pi,\, \mathrm{ad}_{\widehat{\Omega}}\frac{\partial F}{\partial \Pi} \right\rangle = \left\langle \Pi,\, \left[\widehat{\Omega},\, \frac{\partial F}{\partial \Pi}\right] \right\rangle \\ &= -\left\langle \Pi,\, \left[\frac{\partial F}{\partial \Pi},\, \frac{\partial H}{\partial \Pi}\right] \right\rangle, \end{aligned} \tag{3.9}$$

where we have used the equation corresponding to (3.5) under the inverse of the hat map

$$\widehat{\Omega} = \frac{\partial H}{\partial \Pi}$$

and applied antisymmetry of the matrix commutator. Writing Equation (3.9) as

$$\frac{dF}{dt} = -\left\langle \Pi,\, \left[\frac{\partial F}{\partial \Pi},\, \frac{\partial H}{\partial \Pi}\right] \right\rangle =: \Big\{F,\, H\Big\} \tag{3.10}$$

defines the ***Lie–Poisson bracket*** $\{\,\cdot\,,\,\cdot\,\}$ on smooth functions $(F,H) : so(3)^* \to \mathbb{R}$. This bracket satisfies the defining relations of a Poisson bracket because it is a linear functional of the commutator product of skew-symmetric matrices, which is bilinear, skew-symmetric, satisfies the Leibniz rule (because of the partial derivatives) and also satisfies the Jacobi identity.

These Lie–Poisson brackets may be written in tabular form as

$$\{\Pi_i,\,\Pi_j\} = \begin{array}{|c|ccc|} \hline \{\,\cdot\,,\,\cdot\,\} & \Pi_1 & \Pi_2 & \Pi_3 \\ \hline \Pi_1 & 0 & -\Pi_3 & \Pi_2 \\ \Pi_2 & \Pi_3 & 0 & -\Pi_1 \\ \Pi_3 & -\Pi_2 & \Pi_1 & 0 \\ \hline \end{array} \tag{3.11}$$

or, in index notation,

$$\{\Pi_i\,,\,\Pi_j\} = -\epsilon_{ijk}\Pi_k = \widehat{\Pi}_{ij}\,. \tag{3.12}$$

Remark 3.2. *The Lie–Poisson bracket in the form (3.10) would apply to any Lie algebra. This Lie–Poisson Hamiltonian form of the rigid-body dynamics substantiates Poincaré's observation in [Po1901] that the corresponding equations could have been written on the dual of any Lie algebra by using the* ad^* *operation for that Lie algebra. See [JKLOR2011] for more discussion.*

The corresponding Poisson bracket in (3.6) in $\mathbb{R}^3$-vector form also satisfies the defining relations of a Poisson bracket because it is an example of a ***Nambu bracket***, to be discussed next.

3.4 Nambu's $\mathbb{R}^3$ Poisson bracket

The rigid-body Poisson bracket (3.6) is a special case of the Poisson bracket for functions of $\mathbf{x} \in \mathbb{R}^3$,

$$\{f, h\} = -\,\nabla c \cdot \nabla f \times \nabla h\,. \tag{3.13}$$

This bracket generates the motion

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\} = \nabla c \times \nabla h\,. \tag{3.14}$$

For this bracket the motion takes place along the intersections of level surfaces of the functions c and h in $\mathbb{R}^3$. In particular, for the rigid body, the motion takes place along intersections of angular momentum spheres $c = |\mathbf{x}|^2/2$ and energy ellipsoids $h = \mathbf{x} \cdot \mathbb{I}\mathbf{x}$. (See the cover illustration of [MaRa1994].)

Exercise. Consider the Nambu $\mathbb{R}^3$ bracket

$$\{f, h\} = -\,\nabla c \cdot \nabla f \times \nabla h\,. \tag{3.15}$$

Let $c = \mathbf{x}^T \cdot \mathbb{C}\mathbf{x}/2$ be a quadratic form on $\mathbb{R}^3$, and let $\mathbb{C}$ be the associated symmetric 3×3 matrix. Show by direct computation that this Nambu bracket satisfies the Jacobi identity. ★

Exercise. Find the general conditions on the function $\mathbf{c}(\mathbf{x})$ so that the $\mathbb{R}^3$ bracket

$$\{f, h\} = -\,\nabla c \cdot \nabla f \times \nabla h$$

satisfies the defining properties of a Poisson bracket. Is this $\mathbb{R}^3$ bracket also a derivation satisfying the Leibniz relation for a product of functions on $\mathbb{R}^3$? If so, why? ★

Answer.
The bilinear skew-symmetric Nambu $\mathbb{R}^3$ bracket yields the divergenceless vector field

$$X_{c,h} = \{\,\cdot\,,\,h\} = (\nabla c \times \nabla h) \cdot \nabla \quad \text{with} \quad \mathrm{div}(\nabla c \times \nabla h) = 0\,.$$

Divergenceless vector fields are derivative operators that satisfy the Leibniz product rule. They also satisfy the Jacobi identity for any choice of C^2 functions c and h. Hence, the Nambu $\mathbb{R}^3$ bracket is a bilinear skew-symmetric operation satisfying the defining properties of a Poisson bracket. ▲

Theorem 3.1 (Jacobi identity)**.** *The Nambu $\mathbb{R}^3$ bracket (3.15) satisfies the Jacobi identity.*

Proof. The isomorphism $X_H = \{\,\cdot\,,\,H\}$ between the Lie algebra of divergenceless vector fields and functions under the $\mathbb{R}^3$ bracket is the key to proving this theorem. The Lie derivative among vector fields is identified with the Nambu bracket by

$$\mathcal{L}_{X_G} X_H = [X_G, X_H] = -X_{\{G,H\}}\,.$$

Repeating the Lie derivative produces

$$\mathcal{L}_{X_F}(\mathcal{L}_{X_G} X_H) = [X_F, [X_G, X_H]\,] = X_{\{F,\{G,H\}\}}\,.$$

The result follows because both the left- and right-hand sides in this equation satisfy the Jacobi identity. □

Exercise. How is the $\mathbb{R}^3$ bracket related to the canonical Poisson bracket?

Hint: Restrict to level surfaces of the function $c(\mathbf{x})$. ★

Exercise. (Casimirs of the $\mathbb{R}^3$ bracket) The Casimirs (or distinguished functions, as Lie called them) of a Poisson bracket satisfy

$$\big\{c, h\big\}(\mathbf{x}) = 0\,, \quad \text{for all } h(\mathbf{x})\,.$$

Suppose the function $c(\mathbf{x})$ is chosen so that the $\mathbb{R}^3$ bracket (3.13) defines a proper Poisson bracket. What are the Casimirs for the $\mathbb{R}^3$ bracket (3.13)? Why? ★

Exercise. (Geometric interpretation of Nambu motion)

- Show that the Nambu motion equation (3.14)

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\} = \nabla c \times \nabla h$$

  for the $\mathbb{R}^3$ bracket (3.13) is invariant under a certain linear combination of the functions c and h. Interpret this invariance geometrically.

- Show that the rigid-body equations (3.7) for

$$\mathbb{I} = \mathrm{diag}(1,\ 1/2,\ 1/3)$$

  may be interpreted as intersections in $\mathbb{R}^3$ of the spheres $x_1^2 + x_2^2 + x_3^2 =$ constant and the hyperbolic cylinders $x_1^2 - x_3^2 =$ constant, as in Fig. 3.4.

- Show that the rigid-body equations (3.7) may be written as

$$\dot{x}_1 = -\,a_1 a_3 x_2 x_3\,,\ \dot{x}_2 = -\,a_2 a_3 x_3 x_1\,,\ \dot{x}_3 = a_1 a_2 x_1 x_2\,, \tag{3.16}$$

  with nonzero constants a_1, a_2 and a_3 that satisfy $1/a_1 + 1/a_2 = 1/a_3$. Write these equations as a Nambu motion equation on $\mathbb{R}^3$ of the form (3.14). Interpret the solutions of Equations (3.16) geometrically as intersections of orthogonal cylinders (elliptic or hyperbolic) for various values and signs of a_1, a_2 and a_3, as in Fig. 3.4. ★

Answer. $\dot{\mathbf{x}} := (\dot{x}_1, \dot{x}_2, \dot{x}_3)^T = \frac{1}{4}\nabla(a_1x_1^2 + a_3x_3^2) \times \nabla(a_2x_2^2 + a_3x_3^2)$, where (a_1, a_2, a_3) may be written in terms of (I_1, I_2, I_3), when they satisfy $1/a_1 + 1/a_2 = 1/a_3$. ▲

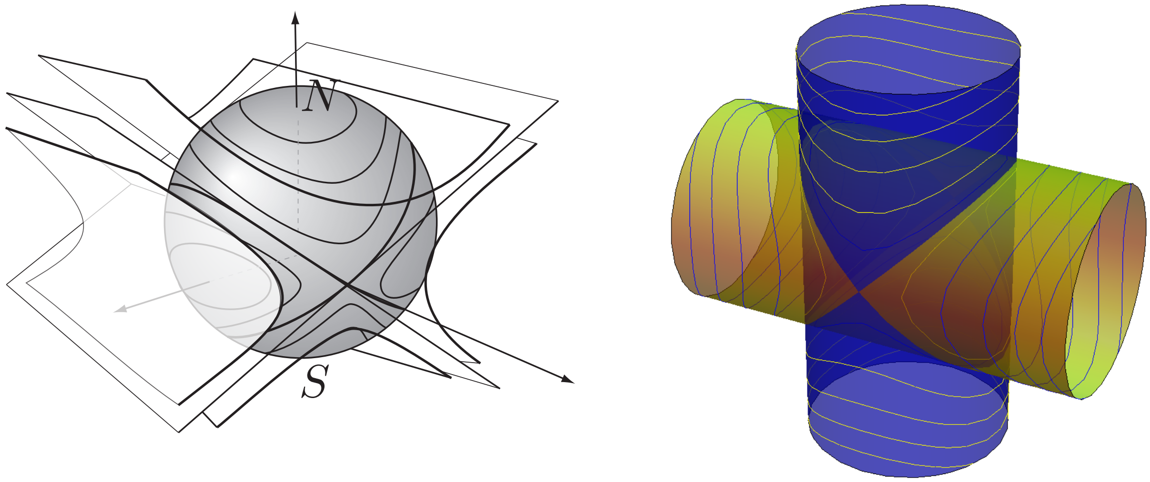


Figure 2: Left: Rigid body motions, seen as intersections in $\mathbb{R}^3$ of the sphere $x_1^2+x_2^2+x_3^2=$ constant and the hyperbolic cylinders $x_1^2-x_3^2=$ constant. Right: The same rigid body motions, seen as intersections in $\mathbb{R}^3$ of orthogonal elliptic cylinders.

3.5 Clebsch variational principle for the rigid body

Proposition 3.1 (Clebsch variational principle)**.**
The Euler rigid-body Equations (2.2) on $T\mathbb{R}^3$ are equivalent to the ***constrained variational principle****,*

$$\delta S(\mathbf{\Omega},\mathbf{Q},\dot{\mathbf{Q}};\mathbf{P}) = \delta\int_a^b l(\mathbf{\Omega},\mathbf{Q},\dot{\mathbf{Q}};\mathbf{P})\,dt = 0, \tag{3.17}$$

for a ***constrained action integral***

$$\begin{aligned} S(\mathbf{\Omega},\mathbf{Q},\dot{\mathbf{Q}}) &= \int_a^b l(\mathbf{\Omega},\mathbf{Q},\dot{\mathbf{Q}})\,dt \qquad (3.18)\\ &= \int_a^b \frac{1}{2}\mathbf{\Omega}\cdot\mathbb{I}\mathbf{\Omega} + \mathbf{P}\cdot\Big(\dot{\mathbf{Q}}+\mathbf{\Omega}\times\mathbf{Q}\Big)\,dt\,. \end{aligned}$$

Remark 3.3 (Reconstruction as constraint)**.**

- *The first term in the Lagrangian (3.18),*

$$l(\mathbf{\Omega}) = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2) = \frac{1}{2}\mathbf{\Omega}^T\mathbb{I}\mathbf{\Omega}\,, \tag{3.19}$$

  is again the (rotational) kinetic energy of the rigid body.

- *The second term in the Lagrangian (3.18) introduces the Lagrange multiplier* $\mathbf{P}$ *which imposes the constraint*

$$\dot{\mathbf{Q}} + \mathbf{\Omega}\times\mathbf{Q} = 0\,.$$

This ***reconstruction formula*** *has the solution*

$$\mathbf{Q}(t) = O^{-1}(t)\mathbf{Q}(0)\,,$$

which satisfies

$$\begin{aligned}\dot{\mathbf{Q}}(t) &= -(O^{-1}\dot{O})O^{-1}(t)\mathbf{Q}(0)\\ &= -\hat{\Omega}(t)\mathbf{Q}(t) = -\boldsymbol{\Omega}(t)\times\mathbf{Q}(t)\,.\end{aligned}\tag{3.20}$$

Proof. The variations of S are given by

$$\begin{aligned}\delta S &= \int_a^b \Big(\frac{\delta l}{\delta\boldsymbol{\Omega}}\cdot\delta\boldsymbol{\Omega} + \frac{\delta l}{\delta\mathbf{P}}\cdot\delta\mathbf{P} + \frac{\delta l}{\delta\mathbf{Q}}\cdot\delta\mathbf{Q}\Big)\,dt\\ &= \int_a^b \Big[\Big(\frac{\delta l}{\delta\boldsymbol{\Omega}} - \mathbf{P}\times\mathbf{Q}\Big)\cdot\delta\boldsymbol{\Omega}\\ &\qquad + \delta\mathbf{P}\cdot\Big(\dot{\mathbf{Q}} + \boldsymbol{\Omega}\times\mathbf{Q}\Big) - \delta\mathbf{Q}\cdot\Big(\dot{\mathbf{P}} + \boldsymbol{\Omega}\times\mathbf{P}\Big)\Big]\,dt\,.\end{aligned}$$

Thus, stationarity of this ***implicit variational principle*** implies the following set of equations:

$$\boldsymbol{\Pi} := \frac{\delta l}{\delta\boldsymbol{\Omega}} = \mathbf{P}\times\mathbf{Q}\,,\quad \dot{\mathbf{Q}} = -\,\boldsymbol{\Omega}\times\mathbf{Q}\,,\quad \dot{\mathbf{P}} = -\,\boldsymbol{\Omega}\times\mathbf{P}\,.\tag{3.21}$$

Euler's form of the rigid-body equations emerges from these ***symmetric equations***, upon elimination of $\mathbf{Q}$ and $\mathbf{P}$, as

$$\begin{aligned}\dot{\boldsymbol{\Pi}} &= \dot{\mathbf{P}}\times\mathbf{Q} + \mathbf{P}\times\dot{\mathbf{Q}}\\ &= \mathbf{Q}\times(\boldsymbol{\Omega}\times\mathbf{P}) + \mathbf{P}\times(\mathbf{Q}\times\boldsymbol{\Omega})\\ &= -\,\boldsymbol{\Omega}\times(\mathbf{P}\times\mathbf{Q}) = -\,\boldsymbol{\Omega}\times\boldsymbol{\Pi}\,,\end{aligned}$$

which are Euler's equations for the rigid body in $T\mathbb{R}^3$ when $\boldsymbol{\Pi} = \mathbb{I}\boldsymbol{\Omega}$. □

Remark 3.4. *The Clebsch variational principle for the rigid body is a natural approach in developing geometric algorithms for numerical integrations of rotating motion.*

Remark 3.5. *The Clebsch approach is also a natural path across to the Hamiltonian formulation of the rigid-body equations. This becomes clear in the course of the following exercise.*

Exercise. Given that the canonical Poisson brackets in Hamilton's approach are

$$\big\{Q_i, P_j\big\} = \delta_{ij} \quad\text{and}\quad \big\{Q_i, Q_j\big\} = 0 = \big\{P_i, P_j\big\}\,,$$

what are the Poisson brackets for $\boldsymbol{\Pi}=\mathbf{P}\times\mathbf{Q}\in\mathbb{R}^3$ in (3.21)? Show these Poisson brackets recover the rigid-body Poisson bracket (3.6). ★

Answer. The components of the angular momentum $\mathbf{\Pi} = \mathbb{I}\mathbf{\Omega}$ in (3.21) are

$$\Pi_a = \epsilon_{abc} P_b Q_c \,,$$

and their canonical Poisson brackets are (noting the similarity with the hat map)

$$\big\{\Pi_a, \Pi_i\big\} = \big\{\epsilon_{abc} P_b Q_c \,,\, \epsilon_{ijk} P_j Q_k\big\} = -\,\epsilon_{ail}\Pi_l \,.$$

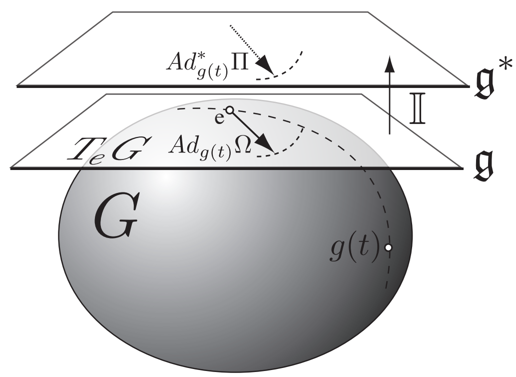
Consequently, the derivative property of the canonical Poisson bracket yields

$$\big\{f, h\big\}(\mathbf{\Pi}) = \frac{\partial f}{\partial \Pi_a}\big\{\Pi_a, \Pi_i\big\}\frac{\partial h}{\partial \Pi_b} = -\,\epsilon_{abc}\Pi_c \frac{\partial f}{\partial \Pi_a}\frac{\partial h}{\partial \Pi_b}\,, \tag{3.22}$$

which is indeed the Lie–Poisson bracket in (3.6) on functions of the $\mathbf{\Pi}$'s. The correspondence with the hat map noted above shows that this Poisson bracket satisfies the Jacobi identity as a result of the Jacobi identity for the vector cross product on $\mathbb{R}^3$. ▲

Remark 3.6. *This exercise proves that the map $T^*\mathbb{R}^3 \to \mathbb{R}^3$ given by $\mathbf{\Pi} = \mathbf{P} \times \mathbf{Q} \in \mathbb{R}^3$ in (3.21) is Poisson. That is, the map takes Poisson brackets on one manifold into Poisson brackets on another manifold. This is one of the properties of a **momentum map**.*

$$\begin{tikzcd}
T^*G \arrow[r, "\Phi_{g(t)}"] \arrow[d, "J(0)"'] \arrow[dr, phantom, "\text{Equivariant Momentum Map}"] & T^*G \arrow[d, "J(t)"] \\
\mathfrak{g}^* \arrow[r, "\mathrm{Ad}^*_{g(t)^{-1}}"] & \mathfrak{g}^* \simeq T^*G/G
\end{tikzcd}$$



Definition: Cotangent lift (CL) momentum map The CL momentum map

$$J \,:\, T^*M \mapsto \mathfrak{g}^*$$

is defined for the Lie algebra action $\xi_M(q)$ of $\xi \in \mathfrak{g}$ on q in manifold M by the pairings

$$\begin{aligned} J^\xi(p, q) \;&:=\; \Big\langle J(p,q),\, \xi \Big\rangle_{\mathfrak{g}^*\times\mathfrak{g}} \\ &=\; \Big\langle\!\Big\langle p_q,\, \xi_M(q) \Big\rangle\!\Big\rangle_{T^*M\times TM} \end{aligned}$$

where $p_q \in T_q^*M$ is the momentum at position $q \in M$ and $\xi_M(q)$ is the vector field tangent to the flow of $g(t) \in G$ at q.

Proposition. $J^\xi(p,q)$ is the *Hamiltonian* for the infinitesimal action $\xi_M(q)$ and its cotangent lift.

Proof.

$$\dot{q} = \{q, J^\xi\} = \xi_M(q) \quad \text{and} \quad \dot{p} = \{p, J^\xi\} = -\frac{d\xi_M}{dq}^T \cdot p$$

□

Example: Body angular momentum, $G = SO(3)$ **and** $M = \mathbb{R}^3$

$$J(q,p) = p \times q, \quad \big(\text{Body angular momentum } J \in so(3)^* \simeq \mathbb{R}^3\big)\,.$$

The Hamiltonian $J^\xi(q,p) = p \times q \cdot \xi$ generates the infinitesimal $SO(3)$ rotations,

$$q'(t) = \{q, J^\xi(q,p)\} = -\,\xi \times q(t), \quad p'(t) = \{p, J^\xi(q,p)\} = -\,\xi \times p(t),$$

for the canonical Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$. These imply the Euler-Poincaré (EP) equation for $J(q,p) = p \times q \in so(3)^* \simeq \mathbb{R}^3$

$$J'(t) = -\,\xi \times J(t) = \mathrm{ad}^*_\xi J \quad \text{for } \xi \in so(3) \text{ and } J \in so(3)^*\,.$$

Proof.

$$\begin{aligned} J'(t) &= p'(t) \times q + p \times q'(t) \\ &= -(\xi \times p) \times q - p \times (\xi \times q) \\ &= -q \times (p \times \xi) - p \times (\xi \times q) \\ \text{By Jacobi identity} &= \xi \times (q \times p) \\ &= -\,\xi \times J \\ &= \mathrm{ad}^*_\xi J \end{aligned}$$

□

This calculation also illustrates the following.

Theorem. The CL momentum map $J(p,q)$ is infinitesimally equivariant.

That is, the CL momentum map $J(p,q)$ satisfies the EP equation, when (p,q) satisfy the canonical equations for the Hamiltonian $J^\xi(p,q) = \langle p_q, \Phi_M(q)\rangle$. Consequently, (p,q) satisfy the equations of motion for the canonical transformation $\Phi_{g(t)}$ of T^*M and the momentum map satisfies $J'(t) = \mathrm{ad}^*_\xi J$, which is the infinitesimal (linearised) version of $J(t) = \mathrm{Ad}^*_{g(t)} J(0)$. To remind ourselves of the latter fact, we recall equation (2.32) in the present notation, as

$$\frac{d}{dt}\Big(\mathrm{Ad}^*_{g^{-1}(t)} J\Big) = \mathrm{Ad}^*_{g^{-1}(t)}\Big(\frac{d}{dt}J - \mathrm{ad}^*_\xi J\Big) = 0.$$

Exercise. The Euler–Lagrange equations in matrix commutator form of Manakov's formulation of the rigid body on $SO(n)$ are

$$\frac{dM}{dt} = \big[\, M \,,\, \Omega \,\big] \,,$$

where the $n \times n$ matrices M, Ω are skew-symmetric. Show that these equations may be derived from Hamilton's principle $\delta S = 0$ with constrained action integral

$$S(\Omega, Q, P) = \int_a^b l(\Omega) + \mathrm{tr}\Big(P^T \left(\dot{Q} - Q\Omega\right)\Big)\, dt \,,$$

for which M is the cotangent lift momentum map

$$M = \frac{\partial l}{\partial \Omega} = \frac{1}{2}(P^T Q - Q^T P)$$

and $Q, P \in SO(n)$ satisfy the following symmetric equations reminiscent of those in (3.21),

$$\dot{Q} = Q\Omega \quad \text{and} \quad \dot{P} = P\Omega \,, \tag{3.23}$$

as a result of the constraints.

Show that M satisfies the Euler-Poincaré equation

$$\frac{dM}{dt} = \mathrm{ad}^*_\Omega M = -\,\big[\, \Omega,\, M \,\big] \,,$$

as it should, since it is a cotangent lift momentum map and those are equivariant.

★

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