M3-4-5 A16 Notes for Geometric Mechanics: Oct–Nov 2011

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Text for the course:

Geometric Mechanics I: Dynamics and Symmetry, by Darryl D Holm

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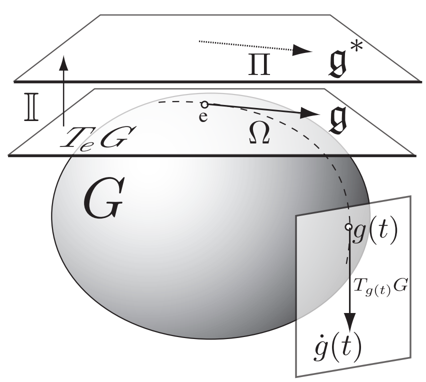
Where are we going in this course?

1. Euler–Poincaré equation
2. Rigid body
3. Spherical pendulum
4. Elastic spherical pendulum
5. Differential forms



These notes: Spherical pendulum

What will we study about the spherical pendulum?

1. Newtonian, Lagrangian and Hamiltonian formulations of motion equation.
2. S^1 symmetry and Noether's theorem
3. Lie symmetry reduction
4. Reduced Poisson bracket in $\mathbb{R}^3$
5. Reduced Poisson bracket in $\mathbb{R}^3$
6. Nambu Hamiltonian form
7. Geometric solution interpretation
8. Geometric phase

1 Spherical pendulum

A spherical pendulum of unit length swings from a fixed point of support under the constant acceleration of gravity g (Figure 2.7). This motion is equivalent to a particle of unit mass moving on the surface of the unit sphere S^2 under the influence of the gravitational (linear) potential $V(z)$ with $z = \hat{\mathbf{e}}_3 \cdot \mathbf{x}$. The only forces acting on the mass are the reaction from the sphere and gravity. This system is often treated by using spherical polar coordinates and the traditional methods of Newton, Lagrange and Hamilton. The present treatment of this problem is more geometrical and avoids polar coordinates.

In these notes, the equations of motion for the spherical pendulum will be derived according to the approaches of Lagrange and Hamilton on the tangent bundle TS^2 of $S^2 \in \mathbb{R}^3$:

$$TS^2 = \big\{(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3 \simeq \mathbb{R}^6 \;\big|\; 1 - |\mathbf{x}|^2 = 0,\, \mathbf{x} \cdot \dot{\mathbf{x}} = 0\big\}. \tag{1.1}$$

After the Legendre transformation to the Hamiltonian side, the canonical equations will be transformed to quadratic variables that are invariant under S^1 rotations about the vertical axis. This is the ***quotient map*** for the spherical pendulum.

Then the ***Nambu bracket*** in $\mathbb{R}^3$ will be found in these S^1 quadratic invariant variables and the equations will be reduced to the ***orbit manifold***, which is the zero level set of a distinguished function called the ***Casimir function*** for this bracket. On the intersections of the Hamiltonian with the orbit manifold, the reduced equations for the spherical pendulum will simplify to the equations of a quadratically nonlinear oscillator.

The solution for the motion of the spherical pendulum will be finished by finding expressions for its geometrical and dynamical phases.

The constrained Lagrangian We begin with the Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}}) : T\mathbb{R}^3 \to \mathbb{R}$ given by

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \tfrac{1}{2}|\dot{\mathbf{x}}|^2 - g\hat{\mathbf{e}}_3 \cdot \mathbf{x} - \tfrac{1}{2}\mu(1 - |\mathbf{x}|^2), \tag{1.2}$$

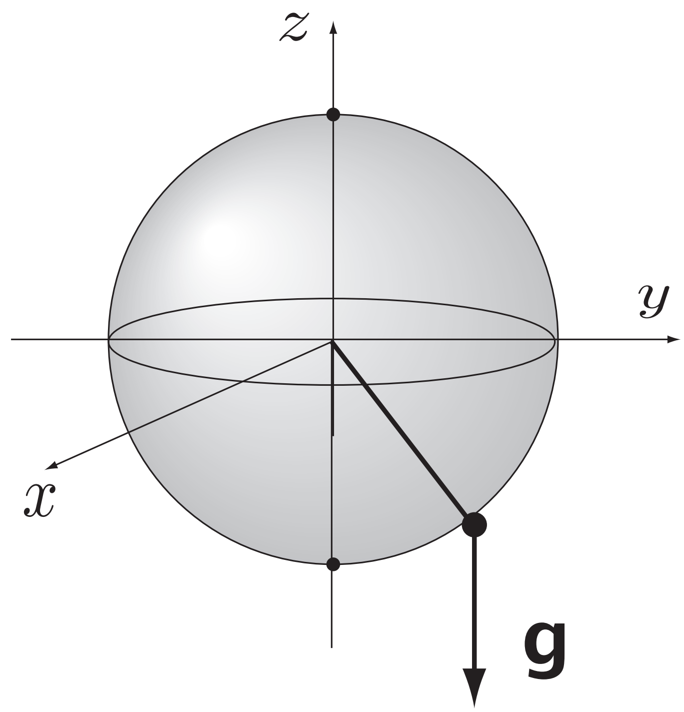


Figure 1: Spherical pendulum moving under gravity on TS^2 in $\mathbb{R}^3$.

in which the Lagrange multiplier μ constrains the motion to remain on the sphere S^2 by enforcing $(1 - |\mathbf{x}|^2) = 0$ when it is varied in Hamilton's principle. The corresponding Euler–Lagrange equation is

$$\ddot{\mathbf{x}} = -g\hat{\mathbf{e}}_3 + \mu\mathbf{x}\,. \tag{1.3}$$

This equation preserves both of the TS^2 relations $1 - |\mathbf{x}|^2 = 0$ and $\mathbf{x} \cdot \dot{\mathbf{x}} = 0$, provided the Lagrange multiplier is given by

$$\mu = g\hat{\mathbf{e}}_3 \cdot \mathbf{x} - |\dot{\mathbf{x}}|^2\,. \tag{1.4}$$

Remark 1.1. *In Newtonian mechanics, the motion equation obtained by substituting (1.4) into (1.3) may be interpreted as*

$$\ddot{\mathbf{x}} = \mathbf{F} \cdot (\mathrm{Id} - \mathbf{x} \otimes \mathbf{x}) - |\dot{\mathbf{x}}|^2\mathbf{x}\,,$$

where $\mathbf{F} = -g\hat{\mathbf{e}}_3$ *is the force exerted by gravity on the particle,*

$$\mathbf{T} = \mathbf{F} \cdot (\mathrm{Id} - \mathbf{x} \otimes \mathbf{x})$$

is its component tangential to the sphere and, finally, $-|\dot{\mathbf{x}}|^2\mathbf{x}$ *is the centripetal force for the motion to remain on the sphere.*

S^1 **symmetry and Noether's theorem** The Lagrangian in (1.2) is invariant under S^1 rotations about the vertical axis, whose infinitesimal generator is $\delta\mathbf{x} = \hat{\mathbf{e}}_3 \times \mathbf{x}$. Consequently Noether's theorem, that each smooth symmetry of the Lagrangian in which an action principle implies a conservation law for its Euler–Lagrange equations, implies in this case that Equation (1.3) conserves

$$J_3(\mathbf{x}, \dot{\mathbf{x}}) = \dot{\mathbf{x}} \cdot \delta\mathbf{x} = \mathbf{x} \times \dot{\mathbf{x}} \cdot \hat{\mathbf{e}}_3\,, \tag{1.5}$$

which is the angular momentum about the vertical axis.

Legendre transform and canonical equations The fibre derivative of the Lagrangian L in (1.2) is

$$\mathbf{y} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \dot{\mathbf{x}}\,. \tag{1.6}$$

The variable $\mathbf{y}$ will be the momentum canonically conjugate to the radial position $\mathbf{x}$, after the Legendre transform to the corresponding Hamiltonian,

$$H(\mathbf{x},\mathbf{y}) = \tfrac{1}{2}|\mathbf{y}|^2 + g\hat{\mathbf{e}}_3\cdot\mathbf{x} + \tfrac{1}{2}(g\hat{\mathbf{e}}_3\cdot\mathbf{x} - |\mathbf{y}|^2)(1-|\mathbf{x}|^2)\,, \tag{1.7}$$

whose canonical equations on $(1-|\mathbf{x}|^2)=0$ are

$$\dot{\mathbf{x}} = \mathbf{y} \quad \text{and} \quad \dot{\mathbf{y}} = -g\hat{\mathbf{e}}_3 + (g\hat{\mathbf{e}}_3\cdot\mathbf{x} - |\mathbf{y}|^2)\mathbf{x}\,. \tag{1.8}$$

This Hamiltonian system on $T^*\mathbb{R}^3$ admits TS^2 as an invariant manifold, provided the initial conditions satisfy the defining relations for TS^2 in (1.1). On TS^2, Equations (1.8) conserve the energy

$$E(\mathbf{x},\mathbf{y}) = \tfrac{1}{2}|\mathbf{y}|^2 + g\hat{\mathbf{e}}_3\cdot\mathbf{x} \tag{1.9}$$

and the vertical angular momentum

$$J_3(\mathbf{x},\mathbf{y}) = \mathbf{x}\times\mathbf{y}\cdot\hat{\mathbf{e}}_3\,.$$

Under the $(\mathbf{x},\mathbf{y})$ canonical Poisson bracket, the angular momentum component J_3 generates the Hamiltonian vector field

$$\begin{aligned} X_{J_3} = \{\,\cdot\,,\,J_3\} &= \frac{\partial J_3}{\partial \mathbf{y}}\cdot\frac{\partial}{\partial \mathbf{x}} - \frac{\partial J_3}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{y}} \\ &= \hat{\mathbf{e}}_3\times\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{x}} + \hat{\mathbf{e}}_3\times\mathbf{y}\cdot\frac{\partial}{\partial \mathbf{y}}\,, \end{aligned} \tag{1.10}$$

for infinitesimal rotations about the vertical axis $\hat{\mathbf{e}}_3$. Because of the S^1 symmetry of the Hamiltonian in (1.7) under these rotations, we have the conservation law,

$$\dot{J}_3 = \{J_3, H\} = X_{J_3}H = 0\,.$$

1.1 Lie symmetry reduction

Algebra of invariants To take advantage of the S^1 symmetry of the spherical pendulum, we transform to S^1-invariant quantities. A convenient choice of basis for the algebra of polynomials in $(\mathbf{x},\mathbf{y})$ that are S^1-invariant under rotations about the third axis is given by

$$\begin{array}{llllll} \sigma_1 &= x_3 & \sigma_3 &= y_1^2+y_2^2+y_3^2 & \sigma_5 &= x_1y_1+x_2y_2 \\ \sigma_2 &= y_3 & \sigma_4 &= x_1^2+x_2^2 & \sigma_6 &= x_1y_2-x_2y_1 \end{array}\,.$$

Quotient map The transformation defined by

$$\pi : \ (\mathbf{x}, \mathbf{y}) \to \{\sigma_j(\mathbf{x}, \mathbf{y}), \ j = 1, \dots, 6\} \tag{1.11}$$

is the ***quotient map*** $T\mathbb{R}^3 \to \mathbb{R}^6$ for the spherical pendulum. Each of the fibres of the quotient map π is an S^1 orbit generated by the Hamiltonian vector field X_{J_3} in (1.10).

The six S^1 invariants that define the quotient map in (1.11) for the spherical pendulum satisfy the cubic algebraic relation

$$\sigma_5^2 + \sigma_6^2 = \sigma_4(\sigma_3 - \sigma_2^2)\,. \tag{1.12}$$

They also satisfy the positivity conditions

$$\sigma_4 \geq 0, \quad \sigma_3 \geq \sigma_2^2. \tag{1.13}$$

In these variables, the defining relations (1.1) for TS^2 become

$$\sigma_4 + \sigma_1^2 = 1 \quad \text{and} \quad \sigma_5 + \sigma_1\sigma_2 = 0\,. \tag{1.14}$$

Perhaps not unexpectedly, since TS^2 is invariant under the S^1 rotations, it is also expressible in terms of S^1 invariants. The three relations in Equations (1.12)–(1.14) will define the orbit manifold for the spherical pendulum in $\mathbb{R}^6$.

Reduced space and orbit manifold in $\mathbb{R}^3$ On TS^2, the variables $\sigma_j(\mathbf{x}, \mathbf{y}), \ j = 1, \dots, 6$ satisfying (1.14) allow the elimination of σ_4 and σ_5 to satisfy the algebraic relation

$$\sigma_1^2\sigma_2^2 + \sigma_6^2 = (\sigma_3 - \sigma_2^2)(1 - \sigma_1^2)\,,$$

which on expansion simplifies to

$$\sigma_2^2 + \sigma_6^2 = \sigma_3(1 - \sigma_1^2)\,, \tag{1.15}$$

where $\sigma_3 \geq 0$ and $(1 - \sigma_1^2) \geq 0$. Restoring $\sigma_6 = J_3$, we may write the previous equation as

$$C(\sigma_1, \sigma_2, \sigma_3; J_3^2) = \sigma_3(1 - \sigma_1^2) - \sigma_2^2 - J_3^2 = 0\,. \tag{1.16}$$

This is the ***orbit manifold*** for the spherical pendulum in $\mathbb{R}^3$. The motion takes place on the following family of surfaces depending on $(\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$ and parameterised by the conserved value of J_3^2,

$$\sigma_3 = \frac{\sigma_2^2 + J_3^2}{1 - \sigma_1^2}\,. \tag{1.17}$$

The orbit manifold for the spherical pendulum is a graph of σ_3 over $(\sigma_1, \sigma_2) \in \mathbb{R}^2$, provided $1 - \sigma_1^2 \neq 0$. The two solutions of $1 - \sigma_1^2 = 0$ correspond to the north and south poles of the sphere. In the case $J_3^2 = 0$, the spherical pendulum is restricted to the *planar* pendulum.

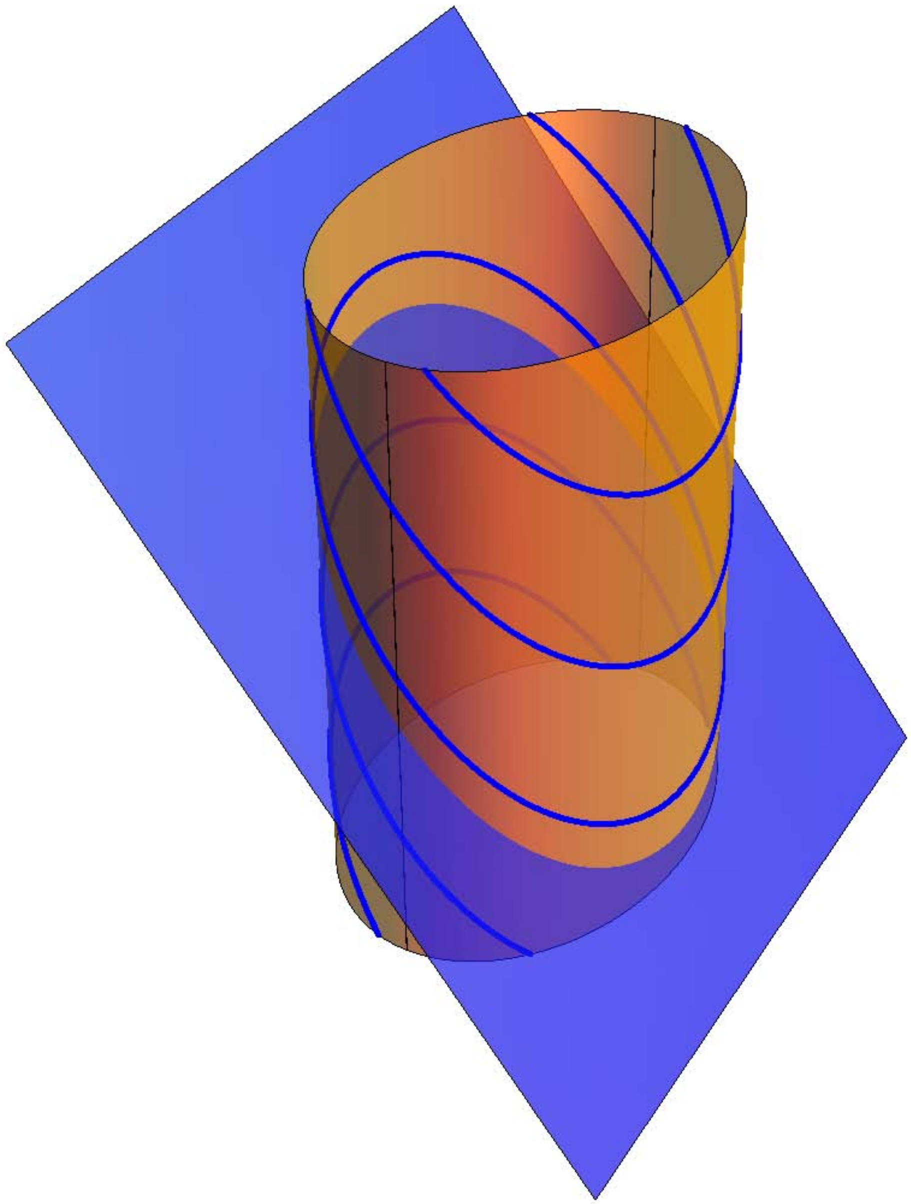


Figure 2: The dynamics of the spherical pendulum in the space of S^1 invariants $(\sigma_1, \sigma_2, \sigma_3)$ is recovered by taking the union in $\mathbb{R}^3$ of the intersections of level sets of two families of surfaces. These surfaces are the roughly cylindrical level sets of angular momentum about the vertical axis given in (1.17) and the (planar) level sets of the Hamiltonian in (1.18). (Only one member of each family is shown in the figure here, although the curves show a few of the other intersections.) On each *planar* level set of the Hamiltonian, the dynamics reduces to that of a quadratically nonlinear oscillator for the vertical coordinate (σ_1) given in Equation (1.24).

Reduced Poisson bracket in $\mathbb{R}^3$ When evaluated on TS^2, the Hamiltonian for the spherical pendulum is expressed in these S^1-invariant variables by the linear relation

$$H = \tfrac{1}{2}\sigma_3 + g\sigma_1 \,, \tag{1.18}$$

whose level surfaces are planes in $\mathbb{R}^3$. The motion in $\mathbb{R}^3$ takes place on the intersections of these Hamiltonian planes with the level sets of J_3^2 given by $C = 0$ in Equation (1.16). Consequently, in $\mathbb{R}^3$-vector form, the motion is governed by the cross-product formula

$$\dot{\boldsymbol{\sigma}} = \frac{\partial C}{\partial \boldsymbol{\sigma}} \times \frac{\partial H}{\partial \boldsymbol{\sigma}} \,. \tag{1.19}$$

In components, this evolution is expressed as

$$\dot{\sigma}_i = \{\sigma_i, H\} = \epsilon_{ijk}\frac{\partial C}{\partial \sigma_j}\frac{\partial H}{\partial \sigma_k} \quad \text{with} \quad i,j,k = 1,2,3. \tag{1.20}$$

The motion may be expressed in Hamiltonian form by introducing the following bracket operation, defined for a function F of the S^1-invariant vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$,

$$\{F, H\} = -\,\frac{\partial C}{\partial \boldsymbol{\sigma}} \cdot \frac{\partial F}{\partial \boldsymbol{\sigma}} \times \frac{\partial H}{\partial \boldsymbol{\sigma}} = -\,\epsilon_{ijk}\frac{\partial C}{\partial \sigma_i}\frac{\partial F}{\partial \sigma_j}\frac{\partial H}{\partial \sigma_k} \,. \tag{1.21}$$

This is another example of the Nambu $\mathbb{R}^3$ bracket, which we learned earlier satisfies the defining relations to be a Poisson bracket. In this case, the distinguished function $C(\sigma_1, \sigma_2, \sigma_3; J_3^2)$ in (1.16) defines a level set of the squared vertical angular momentum J_3^2 in $\mathbb{R}^3$ given by $C = 0$. The distinguished function C is a ***Casimir function*** for the Nambu bracket in $\mathbb{R}^3$. That is, the Nambu bracket in (1.21) with C obeys $\{C, H\} = 0$ for *any* Hamiltonian $H(\sigma_1, \sigma_2, \sigma_3) : \mathbb{R}^3 \to \mathbb{R}$. Consequently, the motion governed by this $\mathbb{R}^3$ bracket takes place on level sets of J_3^2 given by $C = 0$.

Poisson map Introducing the Nambu bracket in (1.21) ensures that the quotient map for the spherical pendulum $\pi : T\mathbb{R}^3 \to \mathbb{R}^6$ in (1.11) is a ***Poisson map***. That is, the subspace obtained by using the relations (1.14) to restrict to the invariant manifold TS^2 produces a set of Poisson brackets $\{\sigma_i, \sigma_j\}$ for $i, j = 1, 2, 3$ that close amongst themselves. Namely,

$$\{\sigma_i, \sigma_j\} = \epsilon_{ijk}\frac{\partial C}{\partial \sigma_k} \,, \tag{1.22}$$

with C given in (1.16). These brackets may be expressed in tabular form, as

$\{\,\cdot\,,\,\cdot\,\}$	σ_1	σ_2	σ_3
σ_1	0	$1 - \sigma_1^2$	$2\sigma_2$
σ_2	$-1 + \sigma_1^2$	0	$-\,2\sigma_1\sigma_3$
σ_3	$-\,2\sigma_2$	$2\sigma_1\sigma_3$	0

.

In addition, $\{\sigma_i, \sigma_6\} = 0$ for $i = 1, 2, 3$, since $\sigma_6 = J_3$ and the $\{\sigma_i \big| \, i = 1, 2, 3\}$ are all S^1-invariant under X_{J_3} in (1.10).

Reduced motion: Restriction in $\mathbb{R}^3$ to Hamiltonian planes The individual components of the equations of motion may be obtained from (1.20) as

$$\dot\sigma_1 = -\sigma_2\,,\qquad \dot\sigma_2 = \sigma_1\sigma_3 + g(1-\sigma_1^2)\,,\qquad \dot\sigma_3 = 2g\sigma_2\,. \tag{1.23}$$

Substituting $\sigma_3 = 2(H - g\sigma_1)$ from Equation (1.18) and setting the acceleration of gravity to be unity $g = 1$ yields

$$\ddot\sigma_1 = 3\sigma_1^2 - 2H\sigma_1 - 1 \tag{1.24}$$

which has equilibria at $\sigma_1^\pm = \frac{1}{3}(H \pm \sqrt{H^2+3})$ and conserves the energy integral

$$\tfrac{1}{2}\dot\sigma_1^2 + V(\sigma_1) = E \tag{1.25}$$

with the potential $V(\sigma_1)$ parameterised by H in (1.18) and given by

$$V(\sigma_1) = -\,\sigma_1^3 + H\sigma_1^2 + \sigma_1\,. \tag{1.26}$$

Equation (1.25) is an energy equation for a particle of unit mass, with position σ_1 and energy E, moving in a cubic potential field $V(\sigma_1)$. For $H = 0$, its equilibria in the $(\sigma_1, \dot\sigma_1)$ phase plane are at $(\sigma_1, \dot\sigma_1) = (\pm\sqrt{3}/3, 0)$, as sketched in Figure 3.

Each curve in the lower panel of Figure 3 represents the intersection in the reduced phase space with S^1-invariant coordinates $(\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$ of one of the Hamiltonian planes (1.18) with a level set of J_3^2 given by $C = 0$ in Equation (1.16). The critical points of the potential are relative equilibria, corresponding to S^1-periodic solutions. The case $H = 0$ includes the homoclinic trajectory, for which the level set $E = 0$ in (1.25) starts and ends with zero velocity at the north pole of the unit sphere.

1.2 Geometric phase for the spherical pendulum

We write the Nambu bracket (1.21) for the spherical pendulum as a differential form in $\mathbb{R}^3$,

$$\{F, H\}\, d^3\sigma = dC \wedge dF \wedge dH\,, \tag{1.27}$$

with oriented volume element $d^3\sigma = d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3$. Hence, on a level set of H we have the canonical Poisson bracket

$$\{f, h\} d\sigma_1 \wedge d\sigma_2 = df \wedge dh = \left(\frac{\partial f}{\partial \sigma_1}\frac{\partial h}{\partial \sigma_2} - \frac{\partial f}{\partial \sigma_2}\frac{\partial h}{\partial \sigma_1}\right) d\sigma_1 \wedge d\sigma_2 \tag{1.28}$$

and we recover Equation (1.24) in canonical form with Hamiltonian

$$h(\sigma_1, \sigma_2) = -\Big(\tfrac{1}{2}\sigma_2^2 -\, \sigma_1^3 + H\sigma_1^2 + \sigma_1\Big) = -\Big(\tfrac{1}{2}\sigma_2^2 + V(\sigma_1)\Big), \tag{1.29}$$

which, not unexpectedly, is also the conserved energy integral in (1.25) for motion on level sets of H.

For the S^1 reduction considered in the present case, the canonical one-form is

$$p_i dq_i = \sigma_2\, d\sigma_1 + H d\psi\,, \tag{1.30}$$

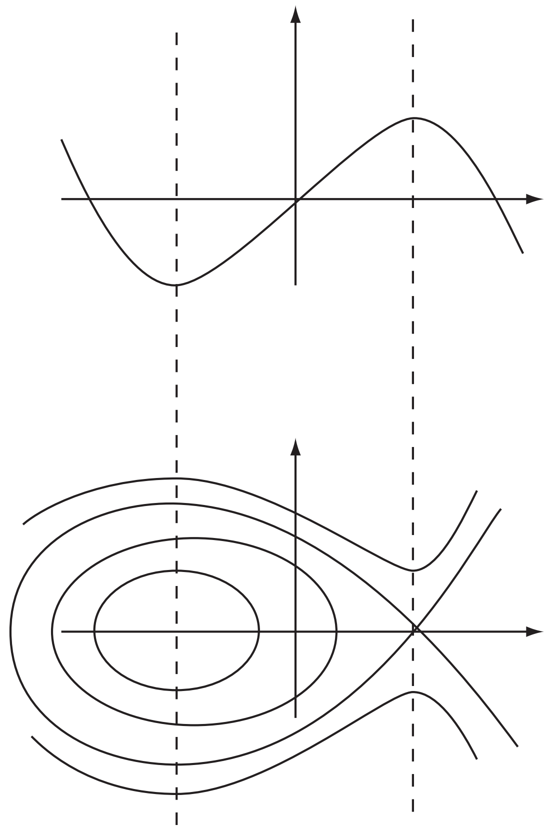


Figure 3: The upper panel shows a sketch of the cubic potential $V(\sigma_1)$ in Equation (1.26) for the case $H = 0$. For $H = 0$, the potential has three zeros located at $\sigma_1 = 0, \pm 1$ and two critical points (relative equilibria) at $\sigma_1 = -\sqrt{3}/3$ (centre) and $\sigma_1 = +\sqrt{3}/3$ (saddle). The lower panel shows a sketch of its fish-shaped saddle-centre configuration in the $(\sigma_1, \dot{\sigma}_1)$ phase plane, comprising several level sets of $E(\sigma_1, \dot{\sigma}_1)$ from Equation (1.25) for $H = 0$.

where σ_1 and σ_2 are the symplectic coordinates for the level surface of H on which the reduced motion takes place and $\psi \in S^1$ is canonically conjugate to H.

Our goal is to finish the solution for the spherical pendulum motion by reconstructing the phase $\psi \in S^1$ from the symmetry-reduced motion in $(\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$ on a level set of H. Rearranging Equation (1.30) gives

$$H d\psi = -\,\sigma_2\, d\sigma_1 + p_i dq_i\,. \tag{1.31}$$

Thus, the phase change around a closed periodic orbit on a level set of H in the $(\sigma_1, \sigma_2, \psi, H)$ phase space decomposes into the sum of the following two parts:

$$\oint H\, d\psi = H\,\Delta\psi = \underbrace{-\oint \sigma_2\, d\sigma_1}_{\text{geometric}} + \underbrace{\oint p_i dq_i}_{\text{dynamic}}\,. \tag{1.32}$$

On writing this decomposition of the phase as

$$\Delta\psi = \Delta\psi_{geom} + \Delta\psi_{dyn}\,, \tag{1.33}$$

one sees from (1.23) that

$$H\Delta\psi_{geom} = \oint \sigma_2^2\, dt = \iint d\sigma_1 \wedge d\sigma_2 \tag{1.34}$$

is the area enclosed by the periodic orbit on a level set of H. Thus the name ***geometric phase*** for $\Delta\psi_{geom}$, because this part of the phase equals the geometric area of the periodic orbit. The rest of the phase is given by

$$H\Delta\psi_{dyn} = \oint p_i\, dq_i = \int_0^T (\sigma_2\dot{\sigma}_1 + H\dot{\psi})\, dt\,. \tag{1.35}$$

Hence, from the canonical equations $\dot{\sigma}_1 = \partial h/\partial\sigma_2$ and $\dot{\psi} = \partial h/\partial H$ with Hamiltonian h in (1.29), we have

$$\begin{aligned}
\Delta\psi_{dyn} &= \frac{1}{H}\int_0^T \left(\sigma_2\frac{\partial h}{\partial\sigma_2} + H\frac{\partial h}{\partial H}\right) dt \\
&= \frac{2T}{H}\Big(h + \big\langle V(\sigma_1)\big\rangle - \tfrac{1}{2}H\big\langle\sigma_1^2\big\rangle\Big) \\
&= \frac{2T}{H}\Big(h + \big\langle V(\sigma_1)\big\rangle\Big) - T\big\langle\sigma_1^2\big\rangle\,,
\end{aligned} \tag{1.36}$$

where T is the period of the orbit around which the integration is performed and the angle brackets $\langle\,\cdot\,\rangle$ denote time average.

The second summand $\Delta\psi_{dyn}$ in (1.33) depends on the Hamiltonian $h = E$, the orbital period T, the value of the level set H and the time averages of the potential energy and σ_1^2 over the orbit. Thus, $\Delta\psi_{dyn}$ deserves the name ***dynamic phase***, since it depends on several aspects of the dynamics along the orbit, not just its area.

This finishes the solution for the periodic motion of the spherical pendulum up to quadratures for the phase. In addition there is a homoclinic trajectory corresponding to the stable and unstable manifolds of the upward vertical equilibrium, which is determined easily by a quadrature.

Next: the elastic spherical pendulum

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