1 M3-4-5 A16 Enriched Coursework Due 11am 10 Jan 2012

Please budget your time: Some of the more interesting problems may become time consuming. So work steadily through them, don't wait until the last minute. Do *four* of these five problems.

Exercise 1.1 ***Nahm's equation for*** $\mathfrak{su}(3)$

Consider the dynamical system for the $\mathfrak{su}(3)$ Nahm equation

$$\frac{dT_i}{dt} = \frac{1}{2}\epsilon_{ijk}\big[T_j,\, T_k\big]\,,$$

with 3×3 skew-Hermitian matrices $T_k = -\overline{T}_k \in \mathfrak{su}(3)$, $k = 1, 2, 3$, given by

$$T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -u_1 \\ 0 & \overline{u}_1 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & \overline{u}_2 \\ 0 & 0 & 0 \\ -u_2 & 0 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & -u_3 & 0 \\ \overline{u}_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $u_1, u_2, u_3 \in \mathbb{C}^3$ and overline in $\overline{u}_k$ denotes the complex conjugate of u_k.

Problem statement:

(a) Write the corresponding equations for the variables $u_1, u_2, u_3 \in \mathbb{C}^3$.

(b) Show that these equations may also be written in Lax form

$$\frac{dL}{dt} = [L, M]$$

with

$$L(\zeta) = \sum_{k=1}^{3} \omega^k\, T_k - \zeta J_0^2$$

$$M(\zeta) = -\frac{1}{\omega(\omega-1)}\left(\sum_{k=1}^{3} T_k + \zeta J_0\right)$$

where $\omega = e^{2i\pi/3}$ is the cube-root of unity and $J_0 = diag(\omega, \omega^2, 1)$.

(c) Show that the real and imaginary parts of the characteristic polynomial (CP) of $L(\zeta)$,

$$\det L(\zeta) = -\,\zeta^3 - \zeta(\,\omega|u_1|^2 + \omega^2|u_2|^2 + |u_3|^2\,) + (\overline{u}_1\overline{u}_2\overline{u}_3 - u_1u_2u_3),$$

are constants of motion. Explain why the CP of $L(\zeta)$ implies their preservation.

(d) From among these constants of motion, identify the Hamiltonian for the system and explain why the other constants of motion generate symmetries of this Hamiltonian.

(e) Is this system completely integrable? That is, are there enough constants of motion in involution to either reduce it to a Hamiltonian system on the plane, or put it into action angle form?

Hint: Transform variables to

$$u_1 = ze^{-i(\phi+\theta)}\,, \qquad u_2 = |u_2|e^{i\phi}\,, \qquad u_3 = |u_3|e^{i\theta}\,, \quad \text{with} \quad z = |z|e^{i\zeta} \in \mathbb{C}.$$

(f) Will the analysis here generalise to n degrees of freedom? Is the corresponding system on $\mathbb{C}^n$ completely integrable? Write this system explicitly and justify your answer.

Exercise 1.2 ***Poisson bracket relations for $n\!:\!m$ resonance***

(a) Using the canonical Poisson bracket relations,

$$\{a_j, a_k^*\} = -\,2i\delta_{jk} \quad \text{for} \quad j,k = 1,2\,,$$

explicitly compute expressions for the following

(i) $\big\{|a_1|^2\,,\,a_1^m\big\}$

(ii) $\big\{|a_1|^2\,,\,a_1^{*\,m}\big\}$

(iii) $\{a_1^{*m}, a_1^n\}$

(iv) Compute $\dot a_1 = \{a_1, H\}$ and $\dot a_2 = \{a_2, H\}$ for the Hamiltonian

$$H = \frac{n}{2}\,|a_1|^2 - \frac{m}{2}\,|a_2|^2 + \mathrm{Im}\,(a_1^m a_2^{*n})$$

(b) Show that the transformation

$$a_1 = |a_1|e^{in\phi}, \qquad a_2 = ze^{mi\phi}, \qquad z = |z|e^{i\zeta}$$

is canonical. Write the transformed equations in the new canonical variables and explain how to solve them by quadratures.

(c) Show that the following variables are invariant under the S^1 transformation for $n\!:\!m$ resonance, $(a_1, a_2) \to (a_1e^{in\phi}, a_2e^{im\phi})$

$$\begin{aligned} R &= \frac{n}{2}\,|a_1|^2 + \frac{m}{2}\,|a_2|^2\,, \\ Z &= \frac{n}{2}\,|a_1|^2 - \frac{m}{2}\,|a_2|^2\,, \\ X - iY &= 2a_1^m a_2^{*n}\,. \end{aligned}$$

(d) Compute the Hamiltonian vector fields for R, Z, X, Y under the canonical Poisson bracket above, and determine the transformations of (a_1, a_2) that are generated under their Hamiltonian flows.

(e) Determine whether the quantities R, Z, X, Y are functionally independent.

(f) Compute the Poisson bracket relations among the quantities R, Z, X, Y and make a table of your results.

(g) Use the Poisson brackets in to write the Poisson bracket between two functions F and H of (X, Y, Z) as the triple vector product of gradients

$$\{F,\,H\} = -\,\nabla C\cdot\nabla F\times\nabla H\,, \quad \text{so that} \quad \{X,\,Y\} = -\,\partial C/\partial Z\,, \quad \text{etc.}$$

Hint: use $C(X, Y, Z, R) = 0$.

(h) Prove that the brackets among the $n\!:\!m$ invariants satisfy the Jacobi identity.

(i) Explain the geometric meaning of the equation of motion for this Poisson bracket. In particular, what is the orbit in $(X, Y, Z) \in \mathbb{R}^3$ when the Hamiltonian is chosen to be $H = Z - Y/2$ for a given value of R?

Exercise 1.3 ***The C. Neumann problem (1859)***

For the origin of this problem see [Ne1859] and for some recent progress on it see [De1978, Ra1980].

(a) Derive the equations of motion

$$\ddot{\mathbf{x}} = -A\mathbf{x} + (A\mathbf{x}\cdot\mathbf{x} - \|\dot{\mathbf{x}}\|^2)\mathbf{x}$$

of a particle of unit mass moving on the sphere S^{n-1} under the influence of a quadratic potential

$$V(\mathbf{x}) = \tfrac{1}{2}A\mathbf{x}\cdot\mathbf{x} = \tfrac{1}{2}a_1x_1^2 + \tfrac{1}{2}a_2x_2^2 + \cdots + \tfrac{1}{2}a_nx_n^2\,,$$

for $\mathbf{x}\in\mathbb{R}^n$, where $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ is a fixed $n\times n$ diagonal matrix. Here $V(\mathbf{x})$ is a harmonic oscillator with spring constants that are taken to be fully anisotropic, with $a_1 < a_2 < \cdots < a_n$.

Hint: These are the Euler–Lagrange equations obtained when a Lagrange multiplier μ is used to restrict the motion to a sphere by adding a term,

$$\mathcal{L}(\mathbf{x},\dot{\mathbf{x}}) = \tfrac{1}{2}\|\dot{\mathbf{x}}\|^2 - \tfrac{1}{2}A\mathbf{x}\cdot\mathbf{x} - \mu(1-\|\mathbf{x}\|^2)\,,$$

on the tangent bundle

$$TS^{n-1} = \{(\mathbf{x},\dot{\mathbf{x}})\in\mathbb{R}^n\times\mathbb{R}^n \,|\, \|\mathbf{x}\|^2 = 1,\ \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}.$$

(b) Form the matrices

$$Q = (x^ix^j) \quad\text{and}\quad L = (x^i\dot{x}^j - x^j\dot{x}^i)\,,$$

and show that the Euler–Lagrange equations for the Lagrangian in (a) are equivalent to

$$\dot{Q} = [L, Q] \quad\text{and}\quad \dot{L} = [Q, A]\,.$$

Show further that for a constant parameter λ these Euler–Lagrange equations imply

$$\frac{d}{dt}(-Q + L\lambda + A\lambda^2) = [-Q + L\lambda + A\lambda^2, -L - A\lambda]\,.$$

Explain why this formula is important from the viewpoint of conservation laws.

(c) Verify that the energy

$$E(Q, L) = -\frac{1}{4}\mathrm{trace}(L^2) + \frac{1}{2}\mathrm{trace}(AQ)$$

is conserved for this system.

(d) Prove that the following $(n-1)$ quantities for $j = 1, 2, \ldots, n-1$ are also conserved:

$$\Phi_j = \dot{x}_j^2 + \frac{1}{2}\sum_{i\neq j}\frac{(x^i\dot{x}^j - x^j\dot{x}^i)^2}{a_j - a_i}\,,$$

where $(\mathbf{x},\dot{\mathbf{x}}) = (x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) \in TS^{n-1}$ and the a_j are the eigenvalues of the diagonal matrix A.

References

[De1978] Devaney, R. L. [1978] Transversal homoclinic orbits in an integrable system. *Am. J. Math.* **100**, 631–642.

[Ne1859] Neumann, C. [1859] De problemate quodam mechanica, quod ad primam integralium ultra-ellipticorum classem revocatur. *J. Reine Angew. Math.* **56**, 54–66.

[Ra1980] Ratiu, T. [1980] The C. Neumann problem as a completely integrable system on an adjoint orbit. *Trans. Amer. Math. Soc.* **264**, 321–329.

Exercise 1.4 ***Peakon dynamics***

Consider the Clebsch Lagrangian $L(u,\{q\},\{\dot q\}) : TDiff(\mathbb{R})\times(T\mathbb{R})^N\to\mathbb{R}$

$$L(u,\{q\},\{\dot q\}) := \ell(u) + \sum_{a=1}^{N} p_a(t)\big(\dot q_a(t) - u(q_a(t),t)\big) \quad\text{where}\quad \ell(u) = \int_{-\infty}^{\infty}\frac12(u^2+u_x^2)\,dx\,.$$

The variables are positions $q_a\in\mathbb{R}^N$, Lagrange multipliers $p_a\in\mathbb{R}^N$ and flow velocity $u(x,t)\in T\mathrm{Diff}(\mathbb{R})\cong\mathfrak{X}$ with asymptotic behaviour $\lim_{|x|\to\infty}u(x)=0$, so that u and its spatial derivative $u_x=\partial u/\partial x$ both vanish sufficiently rapidly at infinity and are smooth enough for the integral to exist.

Use Hamilton's principle to address the following tasks.

(a) Derive ***Hamilton's canonical equations*** for the parameters $p_a(t)$ and $q_a(t)$. Namely,

$$\dot q_a(t) = \frac{\partial H_N}{\partial p_a} \quad\text{and}\quad \dot p_a(t) = -\frac{\partial H_N}{\partial q_a}\,, \tag{1}$$

for $a=1,2,\dots,N$, with Hamiltonian given by,

$$H_N(\{p\},\{q\}) = \frac14\sum_{a,b=1}^{N} p_a\,p_b\,\mathrm{e}^{-|q_a-q_b|}\,. \tag{2}$$

Approach the problem by generalising the Clebsch treatment in class of the rigid body on $T^*SO(3)\cong\mathfrak{so}(3)^*$ to the case of $T^*\mathrm{Diff}(\mathbb{R})\cong\mathfrak{X}^*(\mathbb{R})$.

Hint: Verify that the kernel $K(x)=\frac12 e^{-|x|}$ is the Green's function for the Helmholtz operator $(1-\partial_x^2)$ on the real line.

(b) Solve these Hamilton's equations for $N=2$ and discuss the solution behaviour as a type of scattering of particles.

(c) Show that equations (1) and (2) imply that the quantity $m(x,t)=\delta\ell/\delta u$ satisfies the partial differential equation

$$m_t + (mu)_x + mu_x = 0\,. \tag{3}$$

(d) What type of function is m? In particular, what is its support?

(e) Describe how the solutions of (3) for $u(x,t)$ would develop in time t from an initially smooth confined positive distribution of velocity $u_0=u(x,0)>0$ in x, say, a Gaussian profile? In particular, describe the solution for the velocity distribution that emerges asymptotically in time by answering the following questions.

  (i) Would the total area $\int_{-\infty}^{\infty}u\,dx$ be preserved by the evolution under EPDiff-eqn? Prove it.

  (ii) Would the form of the time-asymptotic velocity distribution depend on the height and width of its initial Gaussian profile? Prove it.

  (iii) Would the slope u_x remain everywhere finite for an initially Gaussian profile in $u_0=u(x,0)$? Prove it by considering the evolution under (3) of the slope at an inflection point.

(f) What is the geometric meaning of the partial differential equation in (3)? Hint: On what space is the flow defined and does it preserve a norm?