Solutions of M3-4A16 Assessed Problems # 3

[#1] Exercises in exterior calculus operations

Vector notation for differential basis elements:
One denotes differential basis elements dx^i and $dS_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$, for $i,j,k = 1,2,3$, in vector notation as

$$
\begin{aligned}
d\mathbf{x} &:= (dx^1, dx^2, dx^3)\,, \\
d\mathbf{S} &= (dS_1, dS_2, dS_3) \\
&:= (dx^2 \wedge dx^3,\ dx^3 \wedge dx^1,\ dx^1 \wedge dx^2)\,, \\
dS_i &:= \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k\,, \\
d^{\,3}x &= d\mathrm{Vol} := dx^1 \wedge dx^2 \wedge dx^3\,.
\end{aligned}
$$

(1a) ***Vector algebra operations***

(i) *Show that contraction with the vector field $X = X^j\partial_j =: \mathbf{X}\cdot\nabla$ recovers the following familiar operations among vectors*

$$
\begin{aligned}
X \lrcorner\, d\mathbf{x} &= \mathbf{X}\,, \\
X \lrcorner\, d\mathbf{S} &= \mathbf{X} \times d\mathbf{x}\,, \\
(\textit{or,}\quad X \lrcorner\, dS_i &= \epsilon_{ijk}X^j dx^k) \\
Y \lrcorner\, X \lrcorner\, d\mathbf{S} &= \mathbf{X} \times \mathbf{Y}\,, \\
X \lrcorner\, d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S} = X^k dS_k\,, \\
Y \lrcorner\, X \lrcorner\, d^{\,3}x &= \mathbf{X}\times\mathbf{Y}\cdot d\mathbf{x} = \epsilon_{ijk}X^iY^j dx^k\,, \\
Z \lrcorner\, Y \lrcorner\, X \lrcorner\, d^{\,3}x &= \mathbf{X}\times\mathbf{Y}\cdot\mathbf{Z}\,.
\end{aligned}
$$

(ii) *Show that these are consistent with*

$$
X \lrcorner\, (\alpha\wedge\beta) = (X \lrcorner\, \alpha)\wedge\beta + (-1)^k\alpha\wedge(X \lrcorner\, \beta)\,,
$$

for a k-form α.

(iii) *Use (ii) to compute $Y \lrcorner\, X \lrcorner\, (\alpha\wedge\beta)$ and $Z \lrcorner\, Y \lrcorner\, X \lrcorner\, (\alpha\wedge\beta)$.*

(1b) ***Exterior derivative examples in vector notation***
Show that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation

$$
\begin{aligned}
df &= f_{,j}\,dx^j =: \nabla f\cdot d\mathbf{x} \\
0 = d^2 f &= f_{,jk}\,dx^k \wedge dx^j \\
df\wedge dg &= f_{,j}\,dx^j \wedge g_{,k}\,dx^k =: (\nabla f\times\nabla g)\cdot d\mathbf{S} \\
df\wedge dg\wedge dh &= f_{,j}\,dx^j\wedge g_{,k}\,dx^k\wedge h_{,l}\,dx^l =: (\nabla f\cdot\nabla g\times\nabla h)\,d^{\,3}x
\end{aligned}
$$

Likewise, show that

$$
\begin{aligned}
d(\mathbf{v}\cdot d\mathbf{x}) &= (\mathrm{curl}\,\mathbf{v})\cdot d\mathbf{S} \\
d(\mathbf{A}\cdot d\mathbf{S}) &= (\mathrm{div}\,\mathbf{A})\,d^{\,3}x\,.
\end{aligned}
$$

Verify the compatibility condition $d^2 = 0$ for these forms as

$$\begin{aligned} 0 = d^2 f = d(\nabla f \cdot d\mathbf{x}) &= (\operatorname{curl} \operatorname{grad} f) \cdot d\mathbf{S}\,, \\ 0 = d^2(\mathbf{v} \cdot d\mathbf{x}) = d\big((\operatorname{curl} \mathbf{v}) \cdot d\mathbf{S}\big) &= (\operatorname{div} \operatorname{curl} \mathbf{v})\, d^{\,3}x\,. \end{aligned}$$

Verify the exterior derivatives of these contraction formulas for $X = \mathbf{X} \cdot \nabla$

(i) $d(X \,\lrcorner\, \mathbf{v} \cdot d\mathbf{x}) = d(\mathbf{X} \cdot \mathbf{v}) = \nabla(\mathbf{X} \cdot \mathbf{v}) \cdot d\mathbf{x}$

(ii) $d(X \,\lrcorner\, \boldsymbol{\omega} \cdot d\mathbf{S}) = d(\boldsymbol{\omega} \times \mathbf{X} \cdot d\mathbf{x}) = \operatorname{curl}(\boldsymbol{\omega} \times \mathbf{X}) \cdot d\mathbf{S}$

(iii) $d(X \,\lrcorner\, f\, d^{\,3}x) = d(f\mathbf{X} \cdot d\mathbf{S}) = \operatorname{div}(f\mathbf{X})\, d^{\,3}x$

(1c) *Use Cartan's formula,*

$$\pounds_X \alpha = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha)$$

for a $k-$form α, $k = 0, 1, 2, 3$ in $\mathbb{R}^3$ to verify the Lie derivative formulas:

(i) $\pounds_X f = X \,\lrcorner\, df = \mathbf{X} \cdot \nabla f$

(ii) $\pounds_X (\mathbf{v} \cdot d\mathbf{x}) = \big(-\, \mathbf{X} \times \operatorname{curl} \mathbf{v} + \nabla(\mathbf{X} \cdot \mathbf{v})\big) \cdot d\mathbf{x}$

(iii) $\pounds_X(\boldsymbol{\omega} \cdot d\mathbf{S}) = \big(\operatorname{curl}(\boldsymbol{\omega} \times \mathbf{X}) + \mathbf{X} \operatorname{div} \boldsymbol{\omega}\big) \cdot d\mathbf{S}$
$\qquad = \big(-\boldsymbol{\omega} \cdot \nabla\mathbf{X} + \mathbf{X} \cdot \nabla\boldsymbol{\omega} + \boldsymbol{\omega} \operatorname{div} \mathbf{X}\big) \cdot d\mathbf{S}$

(iv) $\pounds_X(f\, d^{\,3}x) = (\operatorname{div} f\mathbf{X})\, d^{\,3}x$

(v) *Derive these formulas from the dynamical definition of Lie derivative.*

(1d) *Verify the following Lie derivative identities both by using Cartan's formula and by using the dynamical definition of Lie derivative:*

(i) $\pounds_{fX}\alpha = f\pounds_X\alpha + df \wedge (X \,\lrcorner\, \alpha)$

(ii) $\pounds_X d\alpha = d\big(\pounds_X\alpha\big)$

(iii) $\pounds_X(X \,\lrcorner\, \alpha) = X \,\lrcorner\, \pounds_X\alpha$

(iv) $\pounds_X(Y \,\lrcorner\, \alpha) = (\pounds_X Y) \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X\alpha)$

(v) $\pounds_X(\alpha \wedge \beta) = (\pounds_X\alpha) \wedge \beta + \alpha \wedge \pounds_X\beta$

[#1] Exercises in exterior calculus operations

Answer

Problems **(1a)-(1c)** are easily verified by direct computation, as are parts **(i-iii)** in problem **(1d)**.

However, the linked parts **(iv & v)** in problem **(1d)** require a bit more thought, although both of them are easy from the dynamical viewpoint, by differentiating the properties of the pull-back ϕ_t^*, which commutes with exterior derivative, wedge product and contraction. That is, for $m \in M$,

$$\begin{aligned} d(\phi_t^*\alpha) &= \phi_t^* d\alpha\,, \\ \phi_t^*(\alpha \wedge \beta) &= \phi_t^*\alpha \wedge \phi_t^*\beta\,, \\ \phi_t^*(X(m) \,\lrcorner\, \alpha) &= X(\phi_t(m)) \,\lrcorner\, \phi_t^*\alpha\,. \end{aligned}$$

Setting the dynamical definition of Lie derivative equal to its geometrical definition by Cartan's formula yields

$$
\begin{aligned}
\pounds_X \alpha &= \frac{d}{dt}\Big|_{t=0} (\phi_t^* \alpha) \\
&= X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha)\,,
\end{aligned}
$$

where α is a k-form on a manifold M and X is a smooth vector field with flow ϕ_t on M. Informed by these identities and this equality, one may now derive

(1d) The general form of the relation required in part **(iv)** follows immediately from the product rule for the dynamical definition of the Lie derivative. Since pull-back commutes with contraction, insertion of a vector field into a k-form transforms under the flow ϕ_t of a smooth vector field Y as

$$\phi_t^*(Y(m) \,\lrcorner\, \alpha) = Y(\phi_t(m)) \,\lrcorner\, \phi_t^* \alpha\,.$$

A direct computation using the dynamical definition of the Lie derivative above

$$\pounds_Y \alpha = \frac{d}{dt}\Big|_{t=0} (\phi_t^* \alpha)\,,$$

then yields

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \phi_t^*\big(Y \,\lrcorner\, \alpha\big) &= \Big(\frac{d}{dt}\Big|_{t=0} Y(\phi_t(m))\Big) \,\lrcorner\, \alpha \\
&\quad + Y \,\lrcorner\, \Big(\frac{d}{dt}\Big|_{t=0} \phi_t^* \alpha\Big)\,.
\end{aligned}
$$

Hence, we recognise that the desired formula in part **(iv)** is the ***product rule***:

$$\pounds_X(Y \,\lrcorner\, \alpha) = (\pounds_X Y) \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X \alpha)\,.$$

Part **(v)** in problem **(1d)** is again simply a product rule, proved the same way.

▲

[#2] Operations among vector fields

The Lie derivative of one vector field by another is called the ***Jacobi-Lie bracket****, defined as*

$$\pounds_X Y := [X\,,\,Y] := \nabla Y \cdot X - \nabla X \cdot Y = -\,\pounds_Y X$$

In components, the Jacobi-Lie bracket is

$$[X\,,\,Y] = \Big[X^k \frac{\partial}{\partial x^k}\,,\,Y^l \frac{\partial}{\partial x^l}\Big] = \Big(X^k \frac{\partial Y^l}{\partial x^k} - Y^k \frac{\partial X^l}{\partial x^k}\Big) \frac{\partial}{\partial x^l}$$

The Jacobi-Lie bracket among vector fields satisfies the Jacobi identity,

$$[\,X\,,\,[Y\,,\,Z]\,] + [\,Y\,,\,[Z\,,\,X]\,] + [\,Z\,,\,[X\,,\,Y]\,] = 0$$

Verify the following formulas

(2a) $X \lrcorner (Y \lrcorner \alpha) = -Y \lrcorner (X \lrcorner \alpha)$

(2b) $[X\,,\,Y] \lrcorner \alpha = \pounds_X(Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha)$, *for zero-forms (functions) and one-forms.*

(2c) $\pounds_{[X\,,\,Y]}\alpha = \pounds_X \pounds_Y \alpha - \pounds_Y \pounds_X \alpha$, *as a result of* **(b)**. *Use* **2(c)** *to verify the Jacobi identity.*

(2d) *Verify formula 2(b) for arbitrary k−forms.*

[#2] Operations among vector fields

Answer

(2a) By direct substitution

$$
\begin{aligned}
X \lrcorner (Y \lrcorner \alpha) &= X^l Y^m \alpha_{mli_3\dots i_k} dx^{i_3} \wedge \cdots \wedge dx^{i_k} \\
&= -X^l Y^m \alpha_{lmi_3\dots i_k} dx^{i_3} \wedge \cdots \wedge dx^{i_k} \\
&= -Y \lrcorner (X \lrcorner \alpha)\,,
\end{aligned}
$$

by antisymmetry of $\alpha_{mli_3\dots i_k}$ in its first two indices.

(2b)

For zero-forms (functions) all terms in the formula vanish identically. So that's easy enough.

For a 1-form $\alpha = \mathbf{v} \cdot d\mathbf{x}$ the formula

$$
[X\,,\,Y] \lrcorner \alpha = \pounds_X(Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha)\,,
$$

is seen to hold by comparing

$$
[X\,,\,Y] \lrcorner \alpha \;=\; (X^k Y^l_{,k} - Y^k X^l_{,k}) v_l\,,
$$

with

$$
\begin{aligned}
&\pounds_X(Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha) \\
&\qquad\qquad = X^k \partial_k (Y^l v_l) - Y^l (X^k v_{l,k} + v_j X^j_{,l})\,,
\end{aligned}
$$

(2c) Given $[X\,,\,Y] \lrcorner \alpha = \pounds_X(Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha)$ as verified in part **2(b)** for zero-forms (functions) and one-forms we use Cartan's formula to compute

$$
\begin{aligned}
\pounds_{[X\,,\,Y]}\alpha &= d([X\,,\,Y] \lrcorner \alpha) + [X\,,\,Y] \lrcorner d\alpha \\
&= d\big(\pounds_X(Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha)\big) \\
&\quad + \pounds_X(Y \lrcorner d\alpha) - Y \lrcorner (\pounds_X d\alpha) \\
&= \pounds_X d(Y \lrcorner \alpha) - d(Y \lrcorner (\pounds_X \alpha) \\
&\quad + \pounds_X(Y \lrcorner d\alpha) - Y \lrcorner d(\pounds_X \alpha) \\
&= \pounds_X(\pounds_Y \alpha) - \pounds_Y(\pounds_X \alpha)\,,
\end{aligned}
$$

as required. Thus, the product rule for Lie derivative of a contraction obtained in answering problem **2(b)** provides the key to solving **2(c)**. ■

Consequently,

$$\begin{aligned}\pounds_{[Z,[X,Y]]}\alpha &= \pounds_Z\pounds_X\pounds_Y\alpha - \pounds_Z\pounds_Y\pounds_X\alpha \\ &\quad - \pounds_X\pounds_Y\pounds_Z\alpha + \pounds_Y\pounds_X\pounds_Z\alpha\,,\end{aligned}$$

and summing over cyclic permutations immediately verifies that

$$\pounds_{[Z,[X,Y]]}\,\alpha + \pounds_{[X,[Y,Z]]}\,\alpha + \pounds_{[Y,[Z,X]]}\,\alpha = 0\,.$$

This is the ***Jacobi identity for the Lie derivative.***

(2d) Part **(iv)** of problem **(1d)** has already solved this part.

▲

[#3] A steady Euler fluid flow

A steady Euler fluid flow in a rotating frame satisfies

$$\pounds_u(\mathbf{v}\cdot d\mathbf{x}) = -\,d(p + \tfrac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v})\,,$$

where $\pounds_u$ is Lie derivative with respect to the divergenceless vector field $u = \mathbf{u}\cdot\nabla$, with $\nabla\cdot\mathbf{u} = 0$, and $\mathbf{v} = \mathbf{u} + \mathbf{R}$, with Coriolis parameter $\mathrm{curl}\,\mathbf{R} = 2\boldsymbol{\Omega}$.

(3a) *Write out this Lie-derivative relation in Cartesian coordinates.*

(3b) *By taking the exterior derivative, show that this relation implies that the exact two-form*

$$\mathrm{curl}v \,\lrcorner\, d^{\,3}x = \mathrm{curl}\mathbf{v}\cdot\nabla \,\lrcorner\, d^{\,3}x = \mathrm{curl}\mathbf{v}\cdot d\mathbf{S} = d(\mathbf{v}\cdot d\mathbf{x}) =: d\Xi\wedge d\Pi$$

is invariant under the flow of the divergenceless vector field u.

(3c) *Show that Cartan's formula for the Lie derivative in the steady Euler flow condition implies that*

$$u \,\lrcorner\, \Big(\mathrm{curl}v \,\lrcorner\, d^{\,3}x\Big) = dH(\Xi,\Pi)$$

and identify the function H.

(3d) *Use the result of (3c) to write $\pounds_u\Xi = \mathbf{u}\cdot\nabla\Xi$ and $\pounds_u\Pi = \mathbf{u}\cdot\nabla\Xi$ in terms of the partial derivatives of H.*

(3e) *What do the results of (3d) mean geometrically? Hint: Is a symplectic form involved?*

[#3] A steady Euler fluid flow

Answer

(3a) With $\pi := p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}$, the steady flow satisfies

$$\begin{aligned}0 &= \pounds_u(\mathbf{v}\cdot d\mathbf{x}) + d\pi \\ &= \Big(\mathbf{u}\cdot\nabla\mathbf{v} + (\nabla\mathbf{u})^T\cdot\mathbf{v} + \nabla\pi\Big)\cdot d\mathbf{x} \\ &= \Big(\underbrace{-\mathbf{u}\times\mathrm{curl}\,\mathbf{v}}_{\text{Lamb vector}} + \underbrace{\nabla\big(p + \tfrac{1}{2}|\mathbf{u}|^2\big)}_{\text{Pressure head}}\Big)\cdot d\mathbf{x}\end{aligned}$$

upon
(1) expanding the Lie derivative and
(2) using a vector identity & the definition of π.

(3b) Taking the exterior derivative of this relation using $d^2 f = 0$ for any smooth function f yields

$$
\begin{aligned}
0 &= d\pounds_u(\mathbf{v}\cdot d\mathbf{x}) + d^2(p + \tfrac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}) \\
\text{setting } d^2 = 0 \text{ \& commuting } d \text{ and } \pounds_u \quad &= \pounds_u d(\mathbf{v}\cdot d\mathbf{x}) \\
\text{expanding the exterior derivative} \quad &= \pounds_u(\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}) \\
\text{inserting the definition of } d\mathbf{S} \quad &= \pounds_u\Big((\mathrm{curl}\,\mathbf{v}\cdot\nabla) \,\lrcorner\, d^3x\Big) \\
\text{inserting the definition of curl} v \quad &= \pounds_u\Big(\mathrm{curl}v \,\lrcorner\, d^3x\Big)
\end{aligned}
$$

where $d\mathbf{S} = *d\mathbf{x} = \nabla \,\lrcorner\, d^3x$.

The relation $\pounds_u(\mathrm{curl}v \,\lrcorner\, d^3x) = 0$ is the condition for the 2-form $\mathrm{curl}v \,\lrcorner\, d^3x$ to be invariant under the flow of the vector field u.

If we now substitute the Clebsch relation $\mathbf{v}\cdot d\mathbf{x} = \Xi d\Pi + d\Psi$ we find

$$
\begin{aligned}
0 &= d\pounds_u(\mathbf{v}\cdot d\mathbf{x}) \\
&= \pounds_u(d\Xi\wedge d\Pi)
\end{aligned}
$$

so the exact 2-form $d\Xi\wedge d\Pi$ is invariant under the flow of the divergenceless vector field u. (This should remind us of Poincaré's theorem.)

(3c) Cartan's formula for the Lie derivative

$$
\pounds_u(\mathbf{v}\cdot d\mathbf{x}) = u \,\lrcorner\, d(\mathbf{v}\cdot d\mathbf{x}) + d(u \,\lrcorner\, \mathbf{v}\cdot d\mathbf{x}),
$$

when inserted into the steady Euler flow condition yields

$$
\begin{aligned}
0 &= \pounds_u(\mathbf{v}\cdot d\mathbf{x}) + d(p + \tfrac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}) \\
\text{by Cartan's formula} \quad &= u \,\lrcorner\, \Big(\mathrm{curl}v \,\lrcorner\, d^3x\Big) + d(p + \tfrac{1}{2}|\mathbf{u}|^2).
\end{aligned}
$$

Hence, $H = -\,(p + \frac{1}{2}|\mathbf{u}|^2)$, up to a constant. That is,

$$
u \,\lrcorner\, \mathrm{curl}v \,\lrcorner\, d^3x = dH = -\,d(p + \tfrac{1}{2}|\mathbf{u}|^2).
$$

(3d) Since the 2-form $\mathrm{curl}v \,\lrcorner\, d^3x = d(\mathbf{v}\cdot d\mathbf{x})$ is exact, it may be written as

$$
\mathrm{curl}\mathbf{v}\cdot d\mathbf{S} = \mathrm{curl}v \,\lrcorner\, d^3x = d\Xi\wedge d\Pi
$$

The result of **3(c)** then implies

$$
\begin{aligned}
dH(\Xi,\,\Pi) &= u \,\lrcorner\, (\mathrm{curl}\mathbf{v}\cdot d\mathbf{S}) \\
&= u \,\lrcorner\, (d\Xi\wedge d\Pi) \\
&= (\mathbf{u}\cdot\nabla\Xi)\, d\Pi - (\mathbf{u}\cdot\nabla\Pi)\, d\Xi \\
&= \frac{\partial H}{\partial \Pi}\, d\Pi + \frac{\partial H}{\partial \Xi}\, d\Xi\,.
\end{aligned}
$$

Upon identifying corresponding terms, the steady flow of the fluid velocity $\mathbf{u}$ is found to imply the canonical Hamiltonian equations,

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\Xi) = \pounds_u \Xi &= \frac{\partial H}{\partial \Pi}\,,\\
(\mathbf{u}\cdot\nabla\Pi) = \pounds_u \Pi &= -\frac{\partial H}{\partial \Xi}\,.
\end{aligned}
$$

(3e) The results of **3(d)** may be written as

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\Xi) &= \big\{\Xi,\, H\big\}\,,\\
(\mathbf{u}\cdot\nabla\Pi) &= \big\{\Pi,\, H\big\}\,,
\end{aligned}
$$

where $\{\,\cdot\,,\,\cdot\,\}$ is the canonical Poisson bracket for the symplectic form $d\Xi\wedge d\Pi$.

This means geometrically that the steady Euler flow is symplectic on level sets of $H(\Xi,\Pi)$.

▲

[#4] Maxwell form of Euler's fluid equations

Euler's equations for the incompressible motion of an ideal flow of a fluid of unit density and velocity $\mathbf{u}$ *satisfying* $\mathrm{div}\mathbf{u}=0$ *in a rotating frame with tme-independent Coriolis parameter* $\mathrm{curl}\mathbf{R}(\mathbf{x})=2\boldsymbol{\Omega}$ *are given in the form of Newton's Law of Force by*

$$
\underbrace{\partial_t\,\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}}_{Acceleration} = \underbrace{\mathbf{u}\times 2\boldsymbol{\Omega}}_{Coriolis} - \underbrace{\nabla p}_{Pressure}\,. \tag{1}
$$

(4a) *Show that this Newton's Law equation for Euler fluid motion in a rotating frame may be expressed as,*

$$
\partial_t\,\mathbf{v}-\mathbf{u}\times\boldsymbol{\omega}+\nabla\Big(p+\tfrac12|\mathbf{u}|^2\Big)=0\,,\quad \textit{with}\quad \nabla\cdot\mathbf{u}=0\,, \tag{2}
$$

where we denote,

$$
\mathbf{v}\equiv\mathbf{u}+\mathbf{R}\,,\quad \boldsymbol{\omega}=\mathrm{curl}\mathbf{v}=\mathrm{curl}\mathbf{u}+2\boldsymbol{\Omega}\,.
$$

(4b) *[Kelvin's circulation theorem]*

Show that the Euler equations (2) preserve the circulation integral $I(t)$ *defined by*

$$
I(t)=\oint_{c(\mathbf{u})}\mathbf{v}\cdot d\mathbf{x}\,,
$$

where $c(\mathbf{u})$ *is a closed circuit moving with the fluid at velocity* $\mathbf{u}$*.*

(4c) *[Stokes theorem for vorticity of a rotating fluid]*

Show that the Euler equations (2) satisfy

$$
\frac{d}{dt}\iint_{S(\mathbf{u})}\mathrm{curl}\mathbf{v}\cdot d\mathbf{S}=0\,,
$$

where the surface $S(\mathbf{u})$ *is bounded by an arbitrary circuit* $\partial S=c(\mathbf{u})$ *moving with the fluid.*

(4d) *The* ***Lamb vector****,*

$$\boldsymbol{\ell} := -\,\mathbf{u}\times\boldsymbol{\omega}\,,$$

represents the nonlinearity in Euler's fluid equation (2).
Show that by making the following identifications

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\omega} + \operatorname{curl}\mathbf{A}_0 \\ \mathbf{E} &= \boldsymbol{\ell} + \nabla\big(p + \tfrac{1}{2}|\mathbf{u}|^2\big) + \big(\nabla\phi - \partial_t\mathbf{A}_0\big) \\ \mathbf{D} &= \boldsymbol{\ell} \\ \mathbf{H} &= \nabla\psi\,, \end{aligned} \tag{3}$$

the Euler fluid equations (2) imply the ***Maxwell form***

$$\begin{aligned} \partial_t\mathbf{B} &= -\operatorname{curl}\mathbf{E} \\ \partial_t\mathbf{D} &= \operatorname{curl}\mathbf{H} + \mathbf{J} \\ \operatorname{div}\mathbf{B} &= 0 \\ \operatorname{div}\mathbf{E} &= 0 \\ \operatorname{div}\mathbf{D} &= \rho \;= -\,\Delta\big(p + \tfrac{1}{2}|\mathbf{u}|^2\big) \\ \mathbf{J} &= \mathbf{E}\times\mathbf{B} + (\operatorname{curl}^{-1}\mathbf{E})\times\operatorname{curl}\mathbf{B}\,, \end{aligned} \tag{4}$$

provided the (smooth) gauge functions ϕ and $\mathbf{A}_0$ satisfy $\Delta\phi - \partial_t\operatorname{div}\mathbf{A}_0 = 0$ with $\partial_n\phi = \hat{\mathbf{n}}\cdot\partial_t\mathbf{A}_0$ at the boundary and ψ may be arbitrary, because $\mathbf{H}$ plays no further role.

(4e) *Show that Euler's fluid equations (2) imply the following two elegant relations,*

$$dF = 0 \quad \textit{and} \quad dG = J\,,$$

where the 2-forms F, G and the 3-form J are given as

$$\begin{aligned} F &= \boldsymbol{\ell}\cdot d\mathbf{x}\wedge dt + \boldsymbol{\omega}\cdot d\mathbf{S}, \\ G &= \boldsymbol{\ell}\cdot d\mathbf{S}, \\ J &= \mathbf{J}\cdot d\mathbf{S}\wedge dt + \rho\, d^3x, \end{aligned}$$

and ρ and $\mathbf{J}$ are defined as in equations (4).

[#4] Maxwell form of Euler's fluid equations

Answer

(4a) We begin by comparing the dynamical and Cartan forms of the Lie derivative of the following 1-form

$$\begin{aligned} \pounds_{\mathbf{u}}(\mathbf{v}\cdot d\mathbf{x}) &= \big(\mathbf{u}\cdot\nabla\mathbf{v} + v_j\nabla u^j\big)\cdot d\mathbf{x} \\ &= u \,\lrcorner\, d(\mathbf{v}\cdot d\mathbf{x}) + d(u \,\lrcorner\, \mathbf{v}\cdot d\mathbf{x}) \\ &= u \,\lrcorner\, d(\operatorname{curl}\mathbf{v}\cdot d\mathbf{S}) + d(\mathbf{u}\cdot\mathbf{v}) \\ &= \big(-\,\mathbf{u}\times\operatorname{curl}\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\big)\cdot d\mathbf{x}\,. \end{aligned}$$

This proves the fundamental vector identity of fluid mechanics,

$$\mathbf{u}\cdot\nabla\mathbf{u} = -\,\mathbf{u}\times\operatorname{curl}\mathbf{u} + \tfrac{1}{2}\nabla|\mathbf{u}|^2\,,$$

and tells us that equation (2) results from (1), since $\partial_t\mathbf{R}=0$.

(4b) The dynamical definition of Lie derivative yields the following for the time rate of change of this circulation integral,

$$\begin{aligned}
\frac{d}{dt}I(t) = \frac{d}{dt}\oint_{c(\mathbf{u})}\mathbf{v}\cdot d\mathbf{x} \;&=\; \oint_{c(\mathbf{u})}\left(\frac{\partial}{\partial t}+\pounds_{\mathbf{u}}\right)(\mathbf{v}\cdot d\mathbf{x})\\
&= \oint_{c(\mathbf{u})}\left(\frac{\partial \mathbf{v}}{\partial t}+\frac{\partial \mathbf{v}}{\partial x^j}u^j + v_j\frac{\partial u^j}{\partial \mathbf{x}}\right)\cdot d\mathbf{x}\\
&= -\oint_{c(\mathbf{u})}\nabla\Big(p+\frac{1}{2}|\mathbf{u}|^2-\mathbf{u}\cdot\mathbf{v}\Big)\cdot d\mathbf{x}\\
&= -\oint_{c(\mathbf{u})} d\,\Big(p+\frac{1}{2}|\mathbf{u}|^2-\mathbf{u}\cdot\mathbf{v}\Big) = 0\,.
\end{aligned}\tag{5}$$

(4c) By Stokes theorem

$$I(t)=\oint_{c(\mathbf{u})}\mathbf{v}\cdot d\mathbf{x} = \iint_{S(\mathbf{u})}\operatorname{curl}\mathbf{v}\cdot d\mathbf{S}$$

where the surface $S(\mathbf{u})$ is bounded by an arbitrary circuit $\partial S = c(\mathbf{u})$ moving with the fluid. Consequently, since the Euler equations (2) imply $dI/dt=0$, we also have

$$\frac{d}{dt}\iint_{S(\mathbf{u})}\operatorname{curl}\mathbf{v}\cdot d\mathbf{S}=0\,.$$

(4d) The formulas in (4) result from manipulating (2) to make the desired variables appear and imposing the gauge constraints.

(4e) For $dF=0$ it helps to notice that

$$F = d\Big(\mathbf{v}\cdot d\mathbf{x} - (p+\tfrac{1}{2}u^2)dt\Big)\,,$$

and keep in mind that the exterior derivative involves both space and time on an equal footing $(\mathbf{x},t)\in\mathbb{R}^4$ for these differential forms.

The equation $dG=J$ follows simply from $G:=\boldsymbol{\ell}\cdot d\mathbf{S}$ and

$$dG = \partial_t\boldsymbol{\ell}\cdot d\mathbf{S}\wedge dt + \operatorname{div}\boldsymbol{\ell}\, d^3x$$

after identifying corresponding terms from equations (3) and (4).

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